

Solutions to Problem Set 13

I. Problems to be graded on completion.

1. a.

$$\begin{aligned}\frac{dy}{dx} &= y \sin(2x + 3) \\ \frac{dy}{y} &= \sin(2x + 3) dx \\ \int \frac{dy}{y} &= \int \sin(2x + 3) dx \\ \log y &= -\frac{1}{2} \cos(2x + 3) + C \\ y &= e^{-\frac{1}{2} \cos(2x+3)+C} = e^C e^{-\frac{1}{2} \cos(2x+3)}.\end{aligned}$$

Using the initial condition,

$$\begin{aligned}5 &= e^C e^{-\frac{1}{2} \cos 3} \\ 5e^{\frac{1}{2} \cos 3} &= e^C \\ y &= 5e^{\frac{1}{2} \cos 3} e^{-\frac{1}{2} \cos(2x+3)} = 5e^{\frac{1}{2}(\cos 3 - \cos(2x+3))}.\end{aligned}$$

This satisfies the original equation:

$$y' = 5e^{\frac{1}{2}(\cos 3 - \cos(2x+3))} \left(\frac{1}{2} \sin(2x + 3) \right) (2) = y \sin(2x + 3).$$

b.

$$\begin{aligned}\frac{dy}{dx} &= \frac{y^2 + 1}{y(1+x)} \\ \frac{y dy}{y^2 + 1} &= \frac{dx}{1+x} \\ \int \frac{y dy}{y^2 + 1} &= \int \frac{dx}{1+x} \\ \frac{1}{2} \log(y^2 + 1) &= \log(1+x) + C\end{aligned}$$

where we substituted $u = y^2 + 1$ on the left-hand side. Using the initial condition, $0 = \log 3 + C$, so $C = -\log 3$.

$$\begin{aligned}\log(y^2 + 1) &= 2(\log(1+x) - \log 3) = \log \left(\frac{1+x}{3} \right)^2 \\ y^2 + 1 &= \left(\frac{1+x}{3} \right)^2 \\ y &= \sqrt{\left(\frac{1+x}{3} \right)^2 - 1}.\end{aligned}$$

This satisfies the original equation:

$$y' = \frac{1}{2} \left[\left(\frac{1+x}{3} \right)^2 - 1 \right]^{-1/2} 2 \left(\frac{1+x}{3} \right) \frac{1}{3} = \frac{1}{9} \frac{1+x}{\sqrt{\left(\frac{1+x}{3} \right)^2 - 1}} = \frac{1+x}{9y}$$

and

$$\frac{y^2 + 1}{y(1+x)} = \frac{\left[\left(\frac{1+x}{3} \right)^2 - 1 \right] + 1}{y(1+x)} = \frac{\frac{(1+x)^2}{9}}{y(1+x)} = \frac{1+x}{9y}.$$

c.

$$\begin{aligned} \frac{dy}{dx} &= \frac{e^x e^{-y}}{1+e^x} \\ e^y dy &= \frac{e^x dx}{1+e^x} \\ \int e^y dy &= \int \frac{e^x dx}{1+e^x} \\ e^y &= \log(1+e^x) + C \end{aligned}$$

where we substituted $u = 1 + e^x$ on the right-hand side. Using the initial condition, $1 = \log(1+e) + C$, so $C = 1 - \log(1+e)$.

$$\begin{aligned} e^y &= \log(1+e^x) + 1 - \log(1+e) \\ y &= \log[\log(1+e^x) - \log(1+e) + 1]. \end{aligned}$$

This satisfies the original equation:

$$y' = \frac{1}{\log(1+e^x) - \log(1+e) + 1} \frac{1}{1+e^x} e^x = \frac{1}{e^y} \frac{1}{1+e^x} e^x = \frac{e^{x-y}}{1+e^x}.$$

2. a. The integrating factor is $e^{\int 1 dx} = e^x$.

$$\begin{aligned} y' + y &= x \\ e^x y' + e^x y &= x e^x \\ (e^x y)' &= x e^x \\ e^x y &= \int x e^x dx = (x-1)e^x + C \\ y &= x - 1 + C e^{-x}. \end{aligned}$$

Using the initial condition, $1 = -1 + C$, so $C = 2$, so $y = x - 1 + 2e^{-x}$. This satisfies the original equation:

$$y' + y = (1 - 2e^{-x}) + (x - 1 + 2e^{-x}) = x.$$

b. First we divide through by x to get $y' - \frac{2}{x}y = x^2$. The integrating factor is $e^{\int -\frac{2}{x} dx} = e^{-2 \log x} = x^{-2}$.

$$\begin{aligned} y' - \frac{2}{x}y &= x^2 \\ x^{-2}y' - 2x^{-3}y &= 1 \\ (x^{-2}y)' &= 1 \\ x^{-2}y &= \int 1 dx = x + C \\ y &= x^3 + Cx^2. \end{aligned}$$

Using the initial condition, $0 = 1 + C$, so $C = -1$, so $y = x^3 - x^2$. This satisfies the original equation:

$$xy' - 2y = x(3x^2 - 2x) - 2(x^3 - x^2) = x^3.$$

c. The integrating factor is $e^{\int \tan x \, dx} = e^{\log \sec x} = \sec x$.

$$\begin{aligned} y' + y \tan x &= \sec x \\ (\sec x)y' + (\sec x \tan x)y &= (\sec x)^2 \\ [(\sec x)y]' &= (\sec x)^2 \\ (\sec x)y &= \int (\sec x)^2 \, dx = \tan x + C \\ y &= \tan x \cos x + C \cos x = \sin x + C \cos x \end{aligned}$$

Using the initial condition, $2 = 0 + C$, so $y = \sin x + 2 \cos x$. This satisfies the original equation:

$$\begin{aligned} y' - y \tan x &= (\cos x - 2 \sin x) + (\sin x + 2 \cos x) \frac{\sin x}{\cos x} = \cos x - 2 \sin x + \frac{(\sin x)^2}{\cos x} + 2 \sin x \\ &= \cos x + \frac{(\sin x)^2}{\cos x} = \frac{(\cos x)^2 + (\sin x)^2}{\cos x} = \frac{1}{\cos x} = \sec x. \end{aligned}$$

d. The integrating factor is $e^{\int -\tan x \, dx} = e^{\log \cos x} = \cos x$.

$$\begin{aligned} y' - y \tan x &= \sec x \\ (\cos x)y' - (\cos x \tan x)y &= \cos x \sec x \\ (\cos x)y' - (\sin x)y &= 1 \\ [(\cos x)y]' &= 1 \\ (\cos x)y &= \int 1 \, dx = x + C \\ y &= (x + C) \sec x \end{aligned}$$

Using the initial condition, $2 = C$, so $y = (x + 2) \sec x$, which is radically different from the previous answer, just because of a minus sign. This satisfies the original equation:

$$y' - y \tan x = [(x + 2) \sec x \tan x + \sec x] - (x + 2) \sec x \tan x = \sec x.$$

3. The acceleration, i.e. the derivative of velocity, is proportional to velocity, so $\frac{dv}{dt} = kv$ for some constant k , so $\frac{dv}{v} = k \, dt$, so $\log v = kt + C$, so $v = e^{kt+C} = e^C e^{kt}$. When $t = 0$, $v = 4$, so $e^C = 4$, so $v = 4e^{kt}$. When $t = \frac{1}{60}$, $v = 2$, so $2 = 4e^{k/60}$, so $k = 60 \log \frac{1}{2} = -60 \log 2$, so $v = 4e^{-60 \log 2 \cdot t} = 4(\frac{1}{2})^{60t}$. Velocity is the derivative of position, so change in position is the integral of velocity, so the boat drifts

$$\begin{aligned} \int_0^{1/60} 4e^{-60 \log 2 \cdot t} \, dt &= \int_0^{-\log 2} \frac{4}{-60 \log 2} e^u \, du = \int_{-\log 2}^0 \frac{1}{15 \log 2} e^u \, du \\ &= \frac{1}{15 \log 2} e^u \Big|_{-\log 2}^0 = \frac{1}{30 \log 2} \approx .048 \text{ miles} \approx 254 \text{ feet} \end{aligned}$$

where we substituted $u = -60 \log 2 \cdot t$.

4. Let P be the population in billions and t be the time in years since 1980. $\frac{dP}{dt} = kP$ for some constant k , so as in the previous problem, $P = e^C e^{kt}$. When $t = 0$, $P = 4.5$, so $e^C = 4.5$, so $P = 4.5e^{kt}$. When $t = 10$, $P = 5.3$, so $5.3 = 4.5e^{10k}$, so $k = \frac{1}{10} \log \frac{5.3}{4.5}$, so $P = 4.5e^{\frac{1}{10} \log \frac{5.3}{4.5} t} = 4.5(\frac{5.3}{4.5})^{t/10}$. In 2006, $t = 26$ and we predict that $P = 4.5(\frac{5.3}{4.5})^{2.6} \approx 6.9$, which is slightly higher than the actual current population. So the population of the world has grown more slowly since 1990 than it did between 1980 and 1990. Which is nice.

5. Let x be the amount of Colbalt-60 in grams and t be the time in years since the beginning of our experiment. $\frac{dx}{dt} = kx$ for some constant k , so again, $x = e^C e^{kt}$. When $t = 0$, $x = 100$, so $e^C = 100$, so $x = 100e^{kt}$. When $t = 1$, $x = 88$, so $88 = 100e^k$, so $k = \log \frac{88}{100}$, so $x = 100e^{\log \frac{88}{100} \cdot t} = 100\left(\frac{88}{100}\right)^t$. We want to find t when $x = 50$, so $50 = 100\left(\frac{88}{100}\right)^t$, so $\frac{50}{100} = \left(\frac{88}{100}\right)^t$, so $\log \frac{50}{100} = t \log \frac{88}{100}$, so $t = \frac{\log 50 - \log 100}{\log 88 - \log 100} \approx 5.42$ years.
6. The rate at which the coffee cools is proportional to the difference between its temperature and that of the room, so $\frac{dT}{dt} = k(T - 65)$, so $\frac{dT}{T-65} = k dt$, so $\log(T - 65) = kt + C$. When $t = 0$, $T = 185$, so $C = \log 120$. When $t = 2$, $T = 155$, so $\log 90 = 2k + \log 120$, so $k = \frac{\log 90 - \log 120}{2}$. When $x = 105$, $\log 40 = \frac{\log 90 - \log 120}{2}t + \log 120$, so $t = 2\frac{\log 40 - \log 120}{\log 90 - \log 120} \approx 7.6$ minutes, which is about $5\frac{1}{2}$ minutes later.
7. $\frac{dP}{dt} = k(M - P)$, so $\frac{dP}{M-P} = k dt$, so $-\log(M - P) = kt + C$. When $t = 0$, $P = 0$, so $C = -\log M$, so $kt = \log M - \log M - P = \log \frac{M-P}{M}$. When $t = 10$, $P = .3M$, so $10k = \log .7$, so $k = \frac{1}{10} \log .7$. When $P = .9M$, $\frac{1}{10} \log .7 \cdot t = \log .1$, so $t = 10\frac{\log .1}{\log .7} \approx 64.6$ days, which is about two months.