Solutions to Problem Set 3

I. Problems to be graded on completion.

1. $b' = x$. $c' = v$. $e' = u$. $g' = t$. $h' = a$. $j' = 0$. $l' = q$.
   $p' = h$. $r' = n$. $s' = f$. $t' = d$. $u' = m$. $x' = k$.

2.

$$\begin{aligned}
\lim_{u\to x}\frac{u^{1/3}-x^{1/3}}{u-x} &= \lim_{u\to x}\frac{(u^{1/3}-x^{1/3})(u^{2/3}+u^{1/3}x^{1/3}+x^{2/3})}{(u-x)(u^{2/3}+u^{1/3}x^{1/3}+x^{2/3})}\\
&= \lim_{u\to x}\frac{(u+u^{2/3}x^{1/3}+u^{1/3}x^{2/3})-(u^{2/3}x^{1/3}+u^{1/3}x^{2/3}+x)}{(u-x)(u^{2/3}+u^{1/3}x^{1/3}+x^{2/3})}\\
&= \lim_{u\to x}\frac{u-x}{(u-x)(u^{2/3}+u^{1/3}x^{1/3}+x^{2/3})}\\
&= \lim_{u\to x}\frac{1}{u^{2/3}+u^{1/3}x^{1/3}+x^{2/3}}\\
&= \frac{1}{x^{2/3}+x^{1/3}x^{1/3}+x^{2/3}}\\
&= \frac{1}{3x^{2/3}} = \frac{1}{3}x^{-2/3}
\end{aligned}$$

3. a. $f(x) = \sqrt{x} = x^{1/2}$. $f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$. $f'(4) = \frac{1}{4}$.
   b. $f(x) = x^3$. $f'(x) = 3x^2$. $f'(-2) = 12$.
   c. $f(x) = \sin x$. $f'(x) = \cos x$. $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$.
4. a. $3x^2 - 6x + \frac{1}{2}$
   b. $-3x^{-4} + 6x^{-3} = \frac{-3}{x^4} + \frac{6}{x^3}$
   c. $\frac{8}{5}x^{-1/5} - 9x^{2/7} = \frac{8}{5\sqrt[5]{x}} - 9\sqrt[7]{x^2}$
   d. $4x^3 - 4x^{-5} + \frac{1}{4}x^{-3/4} - \frac{1}{4}x^{-5/4}$
5. a. $2x(x-3) + (x^2-1) = 3x^2 - 6x - 1$
   b. $6x(x^{-3} - x) + 3x^2(-3x^{-4} - 1) = -3x^{-2} - 9x^2$
   c. $2\log x + (2x+3)\frac{1}{x} = 2\log x + 2 + \frac{3}{x}$
   d. $e^x \sin x + e^x \cos x - (\sec x)^2$
6. a. $2(x^2+1)(2x) = 4x(x^2+1)$
   b. $-(x+1)^{-2} = \frac{-1}{(x+1)^2}$
   c. $\cos(e^x)\cdot e^x$
   d. $\frac{1}{4}\frac{1}{x^2+1}2x = \frac{x}{2x^2+2}$
   e. $-\sin(x^2)\cdot 2x - 2\cos x \sin x$
7. a. $f'(x) = g(x) + xg'(x)$, so $f'(0) = 3$.
   b. $f'(x) = 6xg(x) + 3x^2g'(x) - 5$, so $f'(0) = -5$.
   c. $f(x) = g'(x) + (g(x))^{-2}g'(x)$, so $f'(0) = 3 + \frac{3}{4}$.

II. Problems to be graded on correctness.

1. First we rewrite the function:

$$\tan(e^{2-3x^2}\cos(x^{-1})).$$

Now we take the derivative using the chain rule (several times) and the product rule.

$$\begin{aligned}
&\left[\tan(e^{2-3x^2}\cos(x^{-1}))\right]' \\
=\ &\left[\sec(e^{2-3x^2}\cos(x^{-1}))\right]^2\left[e^{2-3x^2}\cos(x^{-1})\right]' \\
=\ &\left[\sec(e^{2-3x^2}\cos(x^{-1}))\right]^2\left[\left(e^{2-3x^2}\right)'\cos(x^{-1})+e^{2-3x^2}\left(\cos(x^{-1})\right)'\right] \\
=\ &\left[\sec(e^{2-3x^2}\cos(x^{-1}))\right]^2\left[e^{2-3x^2}(2-3x^2)'\cos(x^{-1})+e^{2-3x^2}\left(-\sin(x^{-1})\right)(x^{-1})'\right] \\
=\ &\left[\sec(e^{2-3x^2}\cos(x^{-1}))\right]^2\left[e^{2-3x^2}(0-6x)\cos(x^{-1})+e^{2-3x^2}\left(-\sin(x^{-1})\right)(-x^{-2})\right] \\
=\ &\left[\sec(e^{2-3x^2}\cos(x^{-1}))\right]^2 e^{2-3x^2}\left[-6x\cos(\tfrac{1}{x})+\tfrac{1}{x^2}\sin(\tfrac{1}{x})\right]
\end{aligned}$$

where in the last step we have just simplified.

2. a. $\cos(x+8)$
   b. $\cos x\cos 8-\sin x\sin 8$
   c. $\sin(x+8)=\sin x\cos 8+\cos x\sin 8$. The derivative of this is $\cos x\cos 8-\sin x\sin 8$.
3. a. $e^{x^{-1}+\log x}(-x^{-2}+\frac{1}{x})$
   b. $e^{x^{-1}+\log x}=e^{x^{-1}}e^{\log x}=e^{x^{-1}}\cdot x=xe^{x^{-1}}$. The derivative of this is $e^{x^{-1}}+xe^{x^{-1}}(-x^{-2})$
   c. Simplifying part (a):

$$e^{x^{-1}+\log x}(-x^{-2}+\tfrac{1}{x})=xe^{x^{-1}}(x^{-1}-x^{-2})=e^{x^{-1}}(1-x^{-1})$$

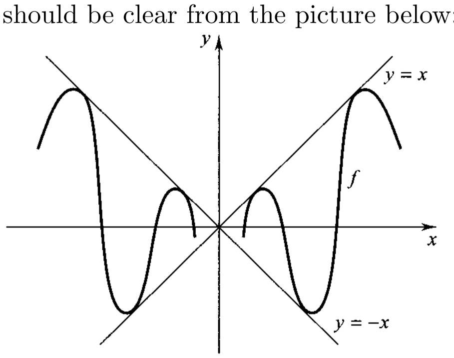
   Simplifying part (b):

$$e^{x^{-1}}+xe^{x^{-1}}(-x^{-2})=e^{x^{-1}}-e^{x^{-1}}x^{-1}=e^{x^{-1}}(1-x^{-1})$$

4. a. Since $-1\le\sin(1/h)\le 1$ for all $h\neq 0$,

$$-|h|\le h\sin(1/h)\le|h|.$$

   The absolute values are necessary because h can be on either side of 0. The inequality above should be clear from the picture below:



   Now

$$\lim_{h\to 0}-|h|=\lim_{h\to 0}|h|=0,$$

   so by the squeeze theorem,

$$\lim_{h\to 0}h\sin(1/h)=0.$$

b.

$$\lim_{h\to 0} h^2 \sin(1/h) = \lim_{h\to 0}\Big(h \cdot h\sin(1/h)\Big) = \Big(\lim_{h\to 0} h\Big)\Big(\lim_{h\to 0} h\sin(1/h)\Big) = 0 \cdot 0 = 0$$

c.

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{h\sin(1/h) - 0}{h} \\ &= \lim_{h\to 0} \sin(1/h) \end{aligned}$$

We know that this last limit does not exist.

d.

$$\begin{aligned} g'(0) &= \lim_{h\to 0} \frac{g(0+h) - g(0)}{h} \\ &= \lim_{h\to 0} \frac{h^2\sin(1/h) - 0}{h} \\ &= \lim_{h\to 0} h\sin(1/h) \\ &= 0 \end{aligned}$$

The fact that this limit exists is what it means for g to be differentiable at 0.

e. When $x \neq 0$, $g'(x) = 2x\sin(x^{-1}) + x^2\cos(x^{-1})(-x^{-2}) = 2x\sin(1/x) - \cos(1/x)$. Saying that g' is continuous at 0 is the same as saying that

$$\lim_{x\to 0} g'(x) = g'(0),$$

that is, that

$$\lim_{x\to 0}\left(2x\sin\frac{1}{x} - \cos\frac{1}{x}\right) = 0.$$

We have shown that $\lim_{x\to 0} 2x\sin(1/x) = 0$, but even the one-sided limit

$$\lim_{x\to 0^+} \cos\frac{1}{x}$$

does not exist. To see this, substitute $u = 1/x$, so as $x \to 0^+$, $u \to \infty$:

$$\lim_{x\to 0^+} \cos\frac{1}{x} = \lim_{u\to\infty} \cos u.$$

As $u \to \infty$, $\cos u$ oscillates back and forth between -1 and 1, never approaching a limit. Thus g' is not continuous at 0.