Solutions to Problem Set 5

I. Problems to be graded on completion.

14. $2^2 1^2 + 4(2)(1) = 12(1)$, so $(2,1)$ lies on the curve. Now

$$\begin{aligned} 2xy^2 + 2x^2yy' + 4y + 4xy' &= 12y' \\ 4 + 8y' + 4 + 8y' &= 12y' \\ y' &= -2 \end{aligned}$$

so the tangent line is given by $\dfrac{y-1}{x-2} = -2$ or $\mathbf{y = -2x + 5}$.

16. $0 + \cos(1 \cdot 0) + 3 \cdot 1^2 = 4$, so $(1,0)$ lies on the curve. Now

$$\begin{aligned} y' - \sin(xy^2)(y^2 + 2xyy') + 6x &= 0 \\ y' - 0 + 6 &= 0 \\ y' &= -6 \end{aligned}$$

so the tangent line is given by $\dfrac{y-0}{x-1} = -6$ or $\mathbf{y = -6x + 6}$.

18. $\sqrt{1} + 4 \cdot 1^2 = 5$, so $(4,1)$ lies on the curve. Now

$$\begin{aligned} \tfrac{1}{2}y^{-1/2}y' + y^2 + 2xyy' &= 0 \\ \tfrac{1}{2}y' + 1 + 8y' &= 0 \\ y' &= -\tfrac{2}{17} \end{aligned}$$

so the tangent line is given by $\dfrac{y-1}{x-4} = -\tfrac{2}{17}$ or $\mathbf{y = -\tfrac{2}{17}x + \tfrac{25}{17}}$.

37. a. $y' = \dfrac{-y}{x+3y^2}$.

b.

$$\begin{aligned} xy'' + 2y' + 3y^2y'' + 6y(y')^2 &= 0 \\ (x+3y^2)y'' - \frac{2y}{x+3y^2} + \frac{6y^3}{(x+3y^2)^2} &= 0 \\ (x+3y^2)^3y'' - (x+3y^2)2y + 6y^3 &= 0 \\ (x+3y^2)^3y'' - 2xy &= 0 \\ y'' &= \frac{2xy}{(x+3y^2)^3} \end{aligned}$$

38.

$$\begin{aligned}
3x^2 - 8yy' &= 0\\
y' &= \frac{3x^2}{8y}\\
6x - 8(y')^2 - 8yy'' &= 0\\
6x - \frac{9x^4}{8y^2} - 8yy'' &= 0\\
48xy^2 - 9x^4 - 64y^3y'' &= 0\\
y'' &= \frac{48xy^2 - 9x^4}{64y^3}
\end{aligned}$$

22. Let α be the angle between the minute hand and 12:00, β the angle between the hour hand and 12:00, $\theta = \beta - \alpha$ the angle between the minute hand and the hour hand, and x the distance between the tips of the hands. Always, $\frac{d\alpha}{dt} = \frac{2\pi \text{ rad}}{\text{hour}} = \frac{\pi}{30}\frac{\text{rad}}{\text{min}}$ and $\frac{d\alpha}{dt} = \frac{2\pi \text{ rad}}{12 \text{ hours}} = \frac{\pi}{360}\frac{\text{rad}}{\text{min}}$, so $\frac{d\theta}{dt} = \frac{d\beta}{dt} - \frac{d\alpha}{dt} = \frac{11}{360}$. Moreover,

$$\begin{aligned}
x^2 &= 4^2 + 5^2 - 2 \cdot 4 \cdot 5 \cos\theta\\
2x\frac{dx}{dt} &= 40 \sin\theta \frac{d\theta}{dt}.
\end{aligned}$$

At $3:00$, $\theta = \frac{\pi}{4}$ and $x = \sqrt{4^2 + 5^2 - 0} = \sqrt{41}$, so

$$\frac{dx}{dt} = \frac{11\pi}{18\sqrt{41}} \approx 0.3\frac{\text{in}}{\text{min}}.$$

1. Let θ be the angle between the beam and the line from the beacon to A and x be the distance from A to the point the beam is lighting up. Always, $\frac{d\theta}{dt} = \frac{10 \text{ rev}}{\text{min}} = \frac{10 \text{ rev}}{\text{min}}\frac{2\pi \text{ rad}}{\text{rev}} = 20\pi$, and

$$\begin{aligned}
\tan\theta &= \frac{x}{2}\\
\theta &= \arctan\frac{x}{2}\\
\frac{d\theta}{dt} &= \frac{1}{1 + (\frac{x}{2})^2}\frac{1}{2}\frac{dx}{dt}
\end{aligned}$$

so when $x = 2$,

$$\frac{dx}{dt} = 80\pi \approx 251\frac{\text{mi}}{\text{min}} \approx 4.2\frac{\text{mi}}{\text{sec}}.$$

2. Let θ be the acute angle in the diagram and x be the length of the shadow. Always, $\frac{d\theta}{dt} = \frac{2\pi \text{ rad}}{\text{day}} = \frac{2\pi \text{ rad}}{1440 \text{ min}} = \frac{\pi}{720}$, and

$$\begin{aligned}
\tan\theta &= \frac{25}{x}\\
\theta &= \arctan(25x^{-1})\\
\frac{d\theta}{dt} &= \frac{1}{1 + (25x^{-1})^2}(-25x^{-2})\frac{dx}{dt} = \frac{-25}{x^2 + 25^2}\frac{dx}{dt}
\end{aligned}$$

so when $x = 50$,

$$\frac{dx}{dt} = \frac{-125\pi}{720} = \frac{-24\pi}{144} \approx -0.55\ \frac{\text{m}}{\text{min}} \approx -0.9\ \frac{\text{cm}}{\text{sec}}.$$

But the minus sign is implausible, and we see that $\frac{d\theta}{dt}$ should have been negative since the sun is setting, so our answer is 0.9 cm/s.

28.

$$\begin{aligned}
\log y &= \log(x^2+3x) + \log(x-2) + \log(x^2+1) \\
\frac{y'}{y} &= \frac{2x+3}{x^2+3x} + \frac{1}{x-2} + \frac{2x}{x^2+1} \\
y' &= (x^2+3x)(x-2)(x^2+1)\left(\frac{2x+3}{x^2+3x} + \frac{1}{x-2} + \frac{2x}{x^2+1}\right) \\
&= (2x+3)(x-2)(x^2+1) + (x^2+3x)(x^2+1) + (x^2+3x)(x-2)(2x)
\end{aligned}$$

30.

$$\begin{aligned}
\log y &= \tfrac{2}{3}\log(x^2+3) + 2\log(3x+2) - \tfrac{1}{2}\log(x+1) \\
\frac{y'}{y} &= \frac{4x}{3(x^2+3)} + \frac{6}{3x+2} - \frac{1}{2(x+1)} \\
y' &= \frac{(x^2+3)^{2/3}(3x+2)^2}{\sqrt{x+1}}\left(\frac{4x}{3(x^2+3)} + \frac{6}{3x+2} - \frac{1}{2(x+1)}\right) \\
&= \frac{4x(3x+2)^2}{3(x^2+3)^{1/3}\sqrt{x+1}} + \frac{6(x^2+3)^{2/3}(3x+2)}{\sqrt{x+1}} - \frac{(x^2+3)^{2/3}(3x+2)^2}{2(x+1)^{3/2}}
\end{aligned}$$

16.

$$\begin{aligned}
x &= -\frac{y}{3} + 1 \\
y &= -3(x-1) \\
f^{-1}(x) &= -3x+3 \\
f(f^{-1}(x)) &= -\frac{f^{-1}(x)}{3} + 1 = -\frac{-3x+3}{3} + 1 = \frac{3x-3}{3} + 1 = x - 1 + 1 = x \\
f^{-1}(f(x)) &= -3f(x) + 3 = -3\left(-\frac{x}{3} + 1\right) + 3 = x - 3 + 3 = x
\end{aligned}$$

20.

$$\begin{aligned}
x &= \sqrt{\frac{1}{y-2}} = (y-2)^{-1/2} \\
y &= x^{-2} + 2 \\
f^{-1}(x) &= x^{-2} + 2 \\
f(f^{-1}(x)) &= \sqrt{\frac{1}{x^{-2}+2-2}} = \sqrt{\frac{1}{x^{-2}}} = \sqrt{x^2} = x \\
f^{-1}(f(x)) &= (f(x))^{-2} + 2 = ((x-2)^{-1/2})^{-2} + 2 = x - 2 + 2 = x
\end{aligned}$$

30. If H is measured in feet, $H = -16t^2 + v_0 t$. The maximum height is acheived when $0 = H' = -32t + v_0$, so $t = \frac{v_0}{32}$, so $H = \frac{-16v_0^2}{32^2} + \frac{v_0^2}{32} = \frac{v_0^2}{64}$. $v_0 = \sqrt{64H} = 8\sqrt{H}$.

38. $f(1) = 2$, so $f^{-1}(2) = 1$, and $f'(x) = x^5 + 5x - 4$, so $f'(1) = 10$, so

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(1)} = \frac{1}{10}.$$

40. $f(3) = 2$, so $f^{-1}(2) = 3$, and $f'(x) = \frac{1}{2\sqrt{x+1}}$, so $f'(3) = \frac{1}{4}$, so

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(3)} = 4.$$

II. Problems to be graded on correctness.

1. a.

$$(x-y)^2 - y = 0$$
$$x^2 - 2xy + y^2 - y = 0$$
$$y^2 - (2x+1)y + x^2 = 0$$
$$y = \frac{(2x+1) \pm \sqrt{(2x+1)^2 - 4x^2}}{2} = x + \frac{1}{2} \pm \sqrt{\frac{4x+1}{4}} = x + \frac{1}{2} \pm \sqrt{x + \frac{1}{4}}$$

b.

$$\frac{dy}{dx} = 1 + 0 \pm \frac{1}{2}(x + \frac{1}{4})^{-1/2} \cdot 1 = 1 \pm \frac{1}{2\sqrt{x+\frac{1}{4}}} = 1 \pm \frac{1}{\sqrt{4x+1}}$$

c.

$$(x-y)^2 - y = 0$$
$$2(x-y)(1-y') - y' = 0$$
$$2(x-y) - 2(x-y)y' - y' = 0$$
$$2(x-y) = [2(x-y)+1]y'$$
$$y' = \frac{2(x-y)}{2(x-y)+1}$$

d.

$$2(x-y) = 2\left(x - \left(x + \frac{1}{2} \pm \sqrt{x+\frac{1}{4}}\right)\right) = 2\left(-\frac{1}{2} \mp \sqrt{x+\frac{1}{4}}\right) = -1 \mp 2\sqrt{x+\frac{1}{4}} = -1 \mp \sqrt{4x+1}$$

$$y' = \frac{2(x-y)}{2(x-y)+1} = \frac{-1 \mp \sqrt{4x+1}}{-1 \mp \sqrt{4x+1} + 1} = \frac{-1 \mp \sqrt{4x+1}}{\mp\sqrt{4x+1}}$$
$$= \frac{-1}{\mp\sqrt{4x+1}} + \frac{\mp\sqrt{4x+1}}{\mp\sqrt{4x+1}} = \frac{\pm 1}{\sqrt{4x+1}} + 1 = 1 \pm \frac{1}{\sqrt{4x+1}}$$

2. a.

$$y = \tfrac{1}{4}(x^2+1)^2(3x+2)^{-1/3}$$
$$y' = \tfrac{1}{4}2(x^2+1)2x(3x+2)^{-1/3} + \tfrac{1}{4}(x^2+1)^2(-\tfrac{1}{3})(3x+2)^{-4/3}3$$
$$= x(x^2+1)(3x+2)^{-1/3} - \tfrac{1}{4}(x^2+1)^2(3x+2)^{-4/3}$$

b.

$$\log y = 2\log(x^2+1) - \log 4 - \frac{1}{3}\log(3x+2)$$
$$\frac{y'}{y} = 2\frac{2x}{x^2+1} - 0 - \frac{1}{3}\frac{3}{3x+2}$$
$$= \frac{4x}{x^2+1} - \frac{1}{3x+2}$$
$$y' = \tfrac{1}{4}(x^2+1)^2(3x+2)^{-1/3}\left(\frac{4x}{x^2+1} - \frac{1}{3x+2}\right)$$
$$= x(x^2+1)(3x+2)^{-1/3} - \tfrac{1}{4}(x^2+1)^2(3x+2)^{-4/3}$$

3. Let θ be the angle between the side of the trough and the vertical dotted line in the diagram, A be the area of the end, and $V = 10A$ be the volume of the trough. Always, $\frac{d\theta}{dt} = \frac{1\text{ deg}}{\text{min}} = \frac{1\text{ deg}}{\text{min}}\frac{2\pi\text{ rad}}{360\text{ deg}} = \frac{\pi}{180}\frac{\text{rad}}{\text{min}}$. The sides and bottom of the trough are all 1 foot long, so the end of the trough consists of two triangles of base $\sin\theta$ and height $\cos\theta$ and one rectangle of base 1 and height $\cos\theta$, so

$$V = 10A = 10(2 \cdot \tfrac{1}{2}\sin\theta\cos\theta + \cos\theta) = 5\sin(2\theta) + 10\cos\theta$$

$$\frac{dV}{dt} = 5\cos(2\theta)2\frac{d\theta}{dt} = 10\sin\theta\frac{d\theta}{dt}$$

so when $\theta = 45°$,

$$\frac{dV}{dt} = \frac{\pi}{18\sqrt{2}} \approx 0.123\ \frac{\text{ft}^3}{\text{sec}}.$$

4. a.

$$\begin{aligned} y &= (u_1u_2)u_3 \\ y' &= (u_1'u_2 + u_1u_2')u_3 + (u_1u_2)u_3 \\ &= u_1'u_2u_3 + u_1u_2'u_3 + u_1u_2u_3' \end{aligned}$$

b.

$$\begin{aligned} y &= (u_1u_2u_3)u_4 \\ y' &= (u_1'u_2u_3 + u_1u_2'u_3 + u_1u_2u_3')u_4 + (u_1u_2u_3)u_4' \\ &= u_1'u_2u_3u_4 + u_1u_2'u_3u_4 + u_1u_2u_3'u_4 + u_1u_2u_3u_4' \end{aligned}$$

c.

$$\begin{aligned} \log y &= \log u_1 + \log u_2 + \cdots + \log u_n \\ \frac{y'}{y} &= \frac{u_1'}{u_1} + \frac{u_2'}{u_2} + \cdots + \frac{u_n'}{u_n} \\ y' &= (u_1u_2\cdots u_n)\left(\frac{u_1'}{u_1} + \frac{u_2'}{u_2} + \cdots + \frac{u_n'}{u_n}\right) \\ &= u_1u_2\cdots u_n\frac{u_1'}{u_1} + u_1u_2\cdots u_n\frac{u_2'}{u_2} + \cdots + u_1u_2\cdots u_n\frac{u_n'}{u_n} \\ &= u_1'u_2\cdots u_n + u_1u_2'\cdots u_n + \cdots + u_1u_2\cdots u_n' \end{aligned}$$