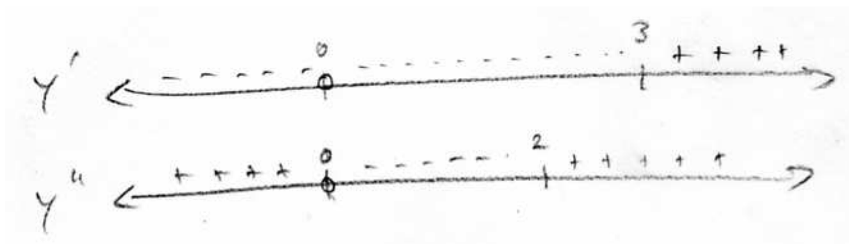
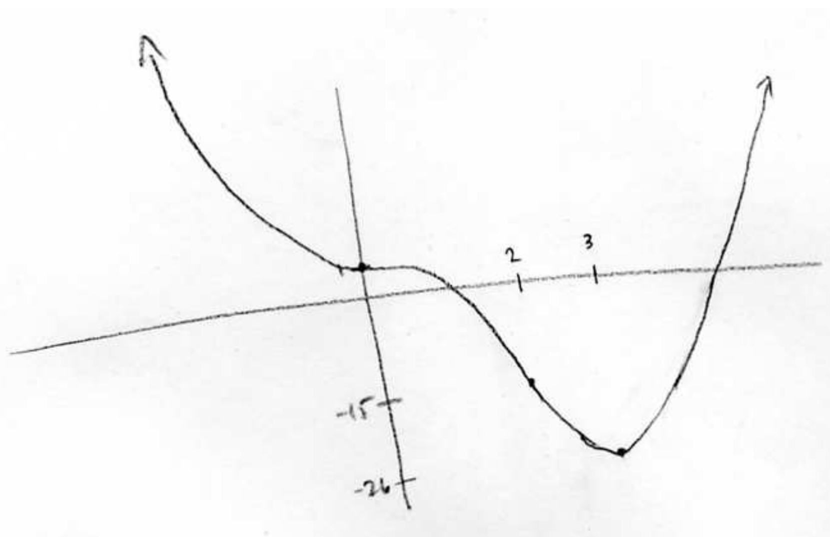
Solutions to Problem Set 7

I. Problems to be graded on completion.

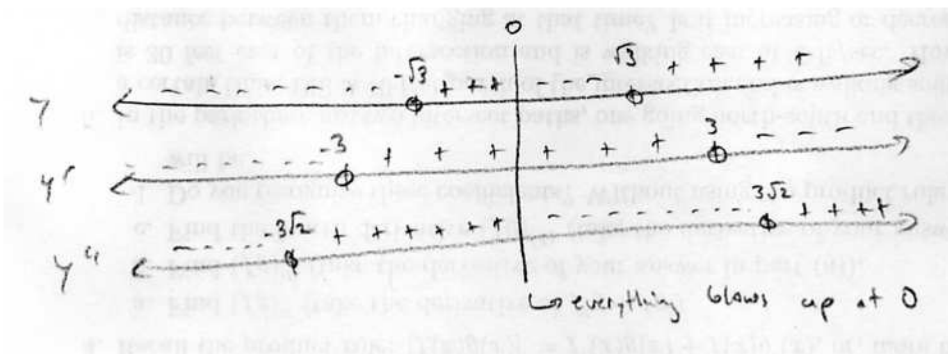
1. $y = x^4 - 4x^3 + 1$, so $y' = 4x^3 - 12x^2 = 4x^2(x-3)$, so $y'' = 12x^2 - 24x = 12x(x-2)$. We cannot solve the equation $y = 0$. When $y' = 0$, $x = 0$ or $x = 3$. When $y'' = 0$, $x = 0$ or $x = 2$. The signs of y' and y'' are as follows:



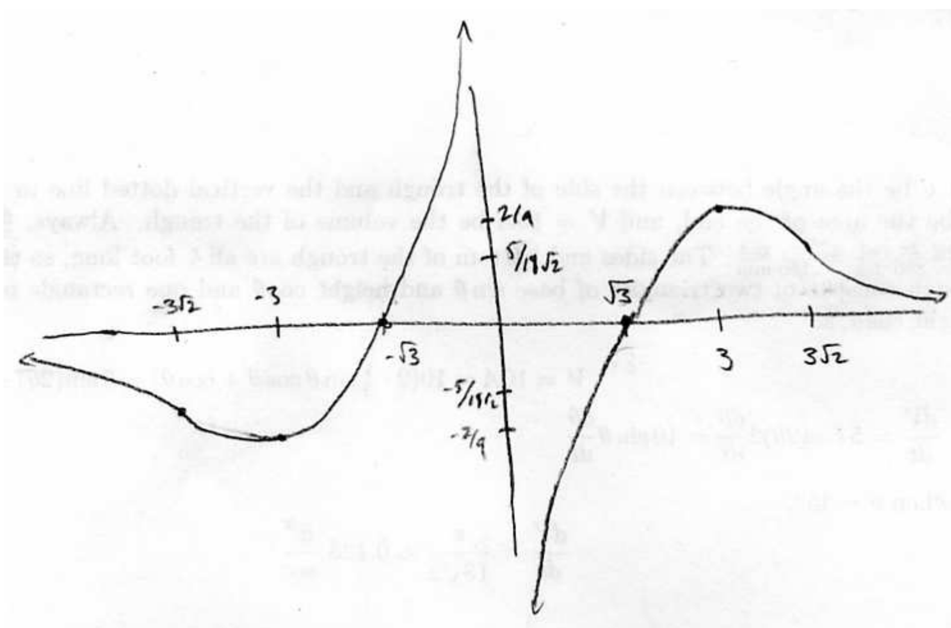
When $x = 0$, $y = 1$. When $x = 2$, $y = -15$. When $x = 3$, $y = -26$. As $x \to \infty$, $y \to \infty$. As $x \to -\infty$, $y \to \infty$.



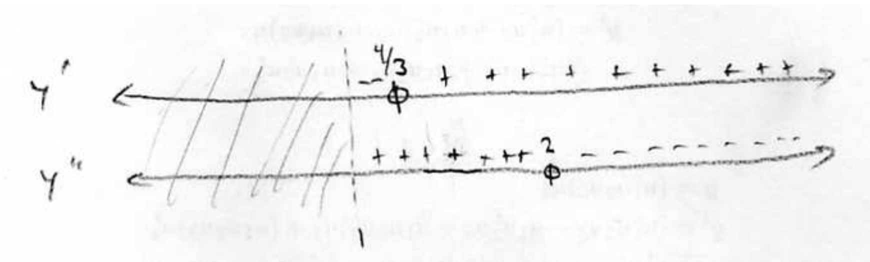
2. $y = x^{-1} - 3x^{-3} = \dfrac{x^2-3}{x^3}$, so $y' = -x^{-2} + 9x^{-4} = \dfrac{9-x^2}{x^4}$, so $y'' = 2x^{-3} - 36x^{-5} = \dfrac{2(x^2-18)}{x^5}$. When $y = 0$, $x = \pm\sqrt{3}$. When $y' = 0$, $x = \pm 3$. When $y'' = 0$, $x = \pm 3\sqrt{2}$. When $x = 0$, y, y', and y'' blow up. The signs of y, y', and y'' are as follows:



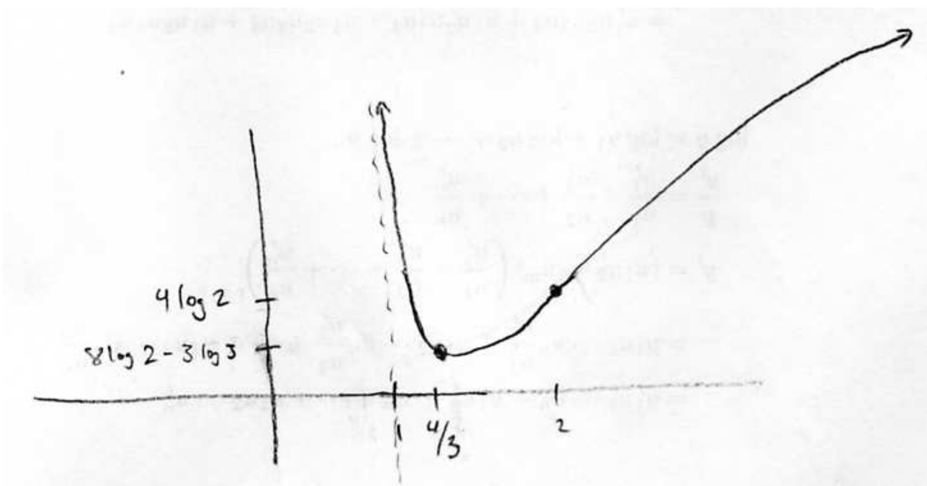
When $x = \pm\sqrt{3}$, $y = \pm 2/9 \approx .222$. When $x = \pm 3\sqrt{2}$, $y = \pm 5/17\sqrt{2} \approx .196$. As $x \to 0^+$, $y \to -\infty$. As $x \to 0^-$, $y \to \infty$. As $x \to \infty$, $y \to 0$. As $x \to -\infty$, $y \to 0$.



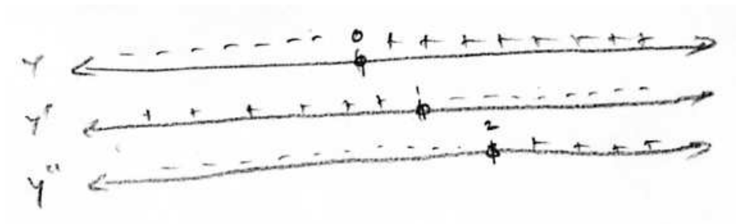
3. We cannot only take the logarithm of a positive number, so y is only defined when $\dfrac{x^4}{x-1} > 0$. Since $x^4 \geq 0$ always, this implies that $x - 1 > 0$, so $x > 1$. Now $y = 4\log x - \log(x-1)$, so $y' = 4x^{-1} - (x-1)^{-1} = \dfrac{3x-4}{x(x-1)}$, so $y'' = -4x^{-2} + (x-1)^{-2} = \dfrac{-3x^2+8x-4}{x^2(x-1)^2} = \dfrac{-(3x-2)(x-2)}{x^2(x-1)^2}$. When $y = 0$, $\dfrac{x^4}{x-1} = 1$, so $x^4 = x - 1$, so $x^4 - x + 1 = 0$, which we cannot solve. When $y' = 0$, $x = \frac{4}{3}$. When $y'' = 0$, $x = 2/3$ or $x = 2$, but we are only interested in $x > 1$, so we discard $x = 2/3$. The signs of y' and y'' are as follows:



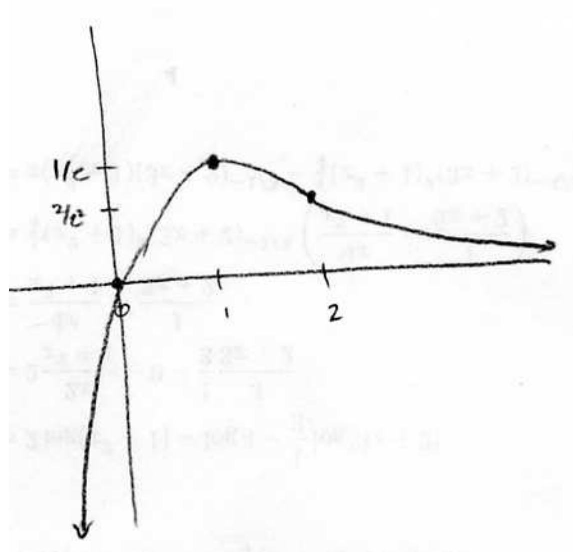
When $x = 4/3$, $y = \log(256/27) = 8\log 2 - 3\log 3 \approx 2.25$. When $x = 2$. $y = \log 16 = 4\log 2 \approx 2.77$. As $x \to 1^+$, $y \to -\infty$. As $x \to \infty$, $y \to \infty$.



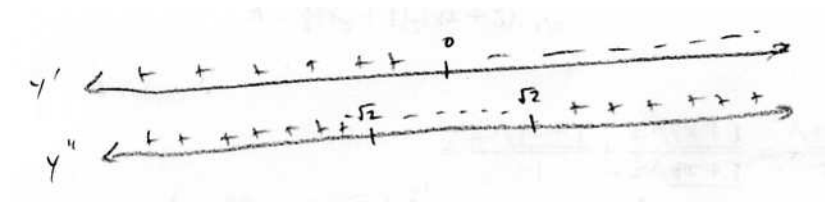
4. $y = xe^{-x}$, so $y' = e^{-x} - xe^{-x} = (1-x)e^{-x}$, so $y'' = -e^{-x} - (1-x)e^{-x} = (x-2)e^{-x}$. e^{-x} is always positive, so we can disregard it when thinking about zeros and signs. When $y = 0$, $x = 0$. When $y' = 0$, $x = 1$. When $y'' = 0$, $x = 2$. The signs of y, y', and y'' are as follows:



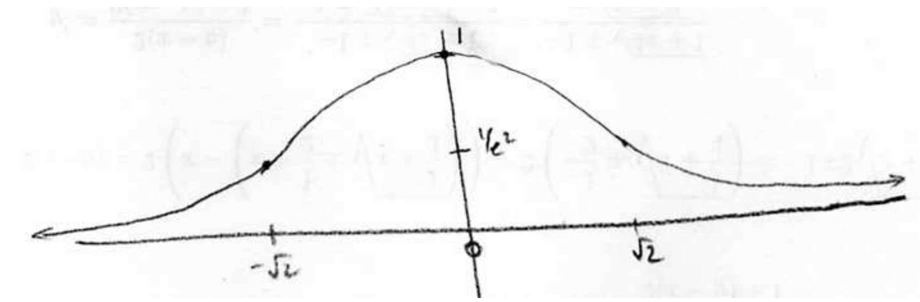
When $x = 1$, $y = 1/e \approx .368$. When $x = 2$, $y = 2/e^2 \approx .271$. As $x \to \infty$, $\lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0$ by L'Hôpital's rule. As $x \to -\infty$, $y \to -\infty$.



5. $y = e^{-x^2}$, so $y' = -2xe^{-x^2}$, so $y'' = -2e^{-x^2} + 4x^2e^{-x^2} = 2(x^2-2)e^{-x^2}$. y is always positive. When $y' = 0$, $x = 0$. When $y'' = 0$, $x = \pm\sqrt{2}$. The signs of y' and y'' are as follows:

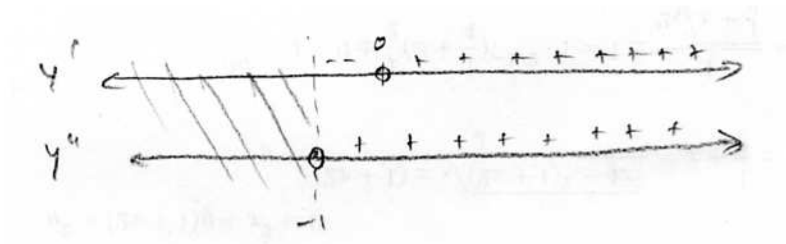


When $x = 0$, $y = 1$. When $x = \pm\sqrt{2}$, $y = 1/e^2 \approx .135$. As $x \to \pm\infty$, $y \to 0$.



II. Problems to be graded on correctness.

1. $y = (1+x)^r - (1+rx)$, so $y = r(1+x)^{r-1} - r$, so $y'' = r(r-1)(1+x)^{r-2}$. We cannot solve the equation $y = 0$ in general. When $y' = 0$, $r(1+x)^{r-1} = r$, so $(1+x)^{r-1} = 1$, so $1+x = 1^{1/(r-1)} = 1$, so $x = 0$. When $y'' = 0$, $(1+x)^r - 2 = 0$, so $1 + x = 0^{1/(r-2)} = 0$, so $x = -1$. The signs of y' and y'' are as follows:



When $x = -1$, $y = r - 1$. When $x = 0$, $y = 0$. As $x \to \infty$, we have $\infty - 1 - \infty$, but $(1 + x)^r$ goes to infinity faster than rx since $r > 2$, so $y \to \infty$.