

## Solutions to Problem Set 9

### I. Problems to be graded on completion.

2.

$$\int_{-1}^2 (x^3 - x + 2) dx = \left[ \frac{x^4}{4} - \frac{x^2}{2} + 2x \right]_{-1}^2 = 8 + \frac{1}{4}.$$

4.

$$\int_{-3}^1 [0 - (x^2 + 2x - 3)] dx = \int_{-3}^1 (-x^2 - 2x + 3) dx = \left[ -\frac{x^3}{3} - x^2 + 3x \right]_{-3}^1 = 10 + \frac{2}{3}.$$

6. If  $x + 4 = x^2 - 2$  then  $0 = x^2 - x - 6 = (x - 3)(x + 2)$ , so  $x = -2$  or  $3$ .

$$\int_{-2}^3 [x + 4 - (x^2 - 2)] dx = \int_{-2}^3 [-x^2 + x + 6] dx = \left[ -\frac{x^3}{3} + \frac{x^2}{2} + 6x \right]_{-2}^3 = 20 + \frac{5}{6}.$$

8. If  $x^2 = -x + 2$  then  $x^2 + x - 2 = 0$ , so  $(x + 2)(x - 1) = 0$ , so  $x = -2$  or  $1$ .

$$\int_{-2}^1 (-x + 2 - x^2) dx = \left[ -\frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-2}^1 = 4 + \frac{1}{2}.$$

10. If  $\sqrt{x} = -x + 6$  then  $x = (-x + 6)^2 = x^2 - 12x + 36$ , so  $0 = x^2 - 13x + 36 = (x - 4)(x - 9)$ , so  $x = 4$  or  $9$ , but the second solution doesn't actually work since  $\sqrt{9} = 3$  while  $-9 + 6 = -3$ . If  $-x + 6 = 0$  then  $x = 6$ .

$$\int_0^4 \sqrt{x} dx + \int_4^6 (-x + 6) dx = \frac{x^{3/2}}{3/2} \Big|_0^4 + \left[ -\frac{x^2}{2} + 6x \right]_4^6 = \frac{16}{3} + 2 = 7 + \frac{1}{3}.$$

6.

$$\int_0^2 2f(x) dx = 2 \left( \int_0^1 f(x) dx + \int_1^2 f(x) dx \right) = 2(2 + 3) = 10.$$

8.

$$\int_0^1 [2f(s) + g(s)] ds = 2 \int_0^1 f(s) ds + \int_0^1 g(s) ds = 2(2) + (-1) = 3.$$

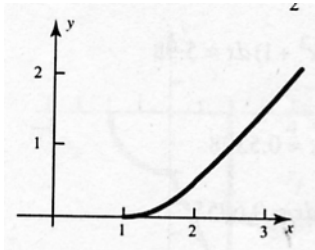
10.

$$\int_1^1 [3f(x) + 2g(x)] dx = 0.$$

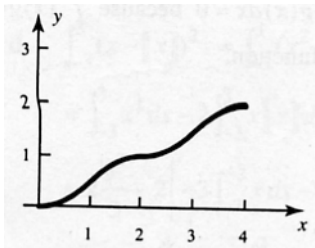
12.

$$\begin{aligned} \int 0^2 [\sqrt{3}f(t) + \sqrt{2}g(t) + \pi] dt &= \sqrt{3} \left( \int_0^1 f(t) dt + \int_1^2 f(t) dt \right) + \sqrt{2} \int_0^2 g(x) dx + \pi x \Big|_0^2 \\ &= \sqrt{3}(2 + 3) + \sqrt{2}(4) + \pi(2) - \pi(0) = 5\sqrt{3} + 4\sqrt{2} + 2\pi. \end{aligned}$$

2. When  $1 \leq x \leq 2$ ,  $A(x) = \frac{1}{2}x^2$ . When  $x \geq 2$ ,  $A(x) = x - \frac{1}{2}$ .



4. First observe that each of the little triangles has area  $\frac{1}{2}$ , so  $A(1) = \frac{1}{2}$ ,  $A(2) = 1$ ,  $A(3) = \frac{3}{2}$ , and  $A(4) = 2$ . When  $0 \leq x \leq 1$ ,  $A(x) = \frac{1}{2}x^2$ . When  $0 \leq x \leq 2$ , we take that curve and spin it  $180^\circ$ .



13.  $G'(x) = 2x$ .

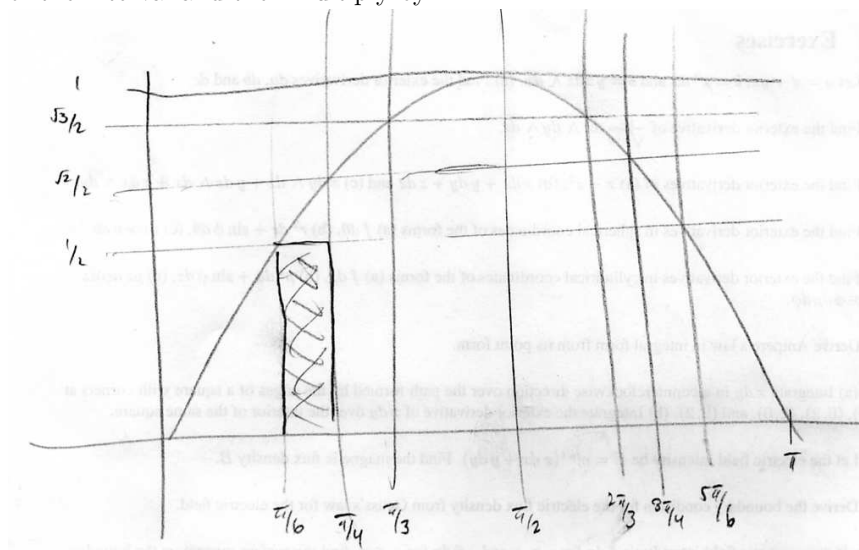
14.  $G(x) = -\int_1^x 2t \, dt$ , so  $G'(x) = -2x$ .

20. Let  $F(z)$  be an anti-derivative of  $\sqrt{2z + \sin z}$ . Then  $G(x) = F(x^2 + x) - F(1)$ , so  $G'(x) = F'(x^2 + x)(2x + 1) - 0 = \sqrt{2(x^2 + x) + \sin(x^2 + x)}(2x + 1)$ .

22. Let  $F(t)$  be an anti-derivative of  $t^5$ . Then  $G(x) = F(\sin x) - F(\cos x)$ , so  $G'(x) = F'(\sin x)(\cos x) - F'(\cos x)(-\sin x) = (\sin x)^5 \cos x + (\cos x)^5 \sin x$ .

## II. Problems to be graded on correctness.

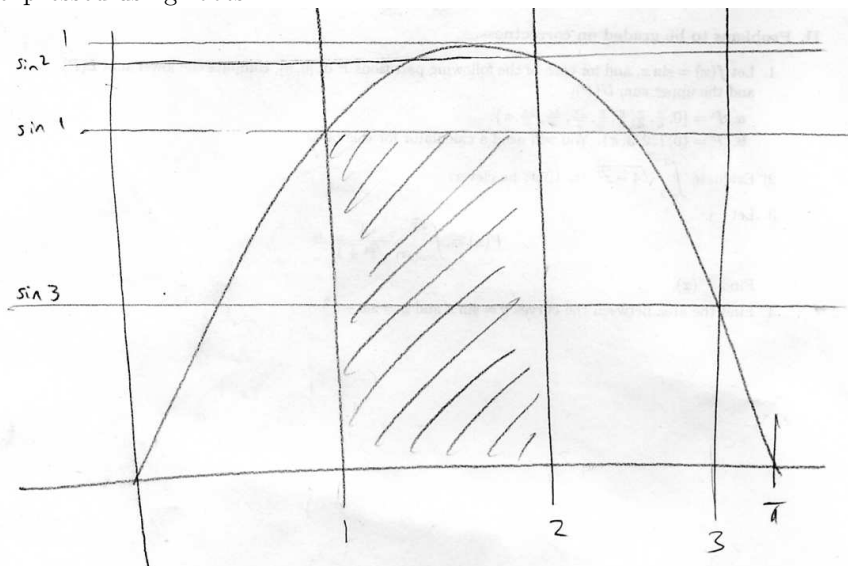
1. a. If we draw a picture, we see that we can just find the lower and upper sums for the first half of the interval and then multiply by 2.



$$U(P) = 2 \left( \frac{\pi}{6} \cdot \frac{1}{2} + \frac{\pi}{12} \cdot \frac{\sqrt{2}}{2} + \frac{\pi}{12} \cdot \frac{\sqrt{3}}{2} + \frac{\pi}{6} \cdot 1 \right) = \frac{\pi}{12} (6 + \sqrt{2} + \sqrt{3}) \approx 2.394.$$

$$L(P) = 2 \left( \frac{\pi}{6} \cdot 0 + \frac{\pi}{12} \cdot \frac{1}{2} + \frac{\pi}{12} \cdot \frac{\sqrt{2}}{2} + \frac{\pi}{6} \cdot \frac{\sqrt{3}}{2} \right) = \frac{\pi}{12} (1 + \sqrt{2} + 2\sqrt{3}) \approx 1.539.$$

- b. Here the trick is to note that in the second slot, the maximum occurs in the middle. Note that  $\sin 1 \approx .841$ ,  $\sin 1 \approx .909$ , and  $\sin 3 \approx .141$  are not algebraic numbers—they cannot be expressed using roots.



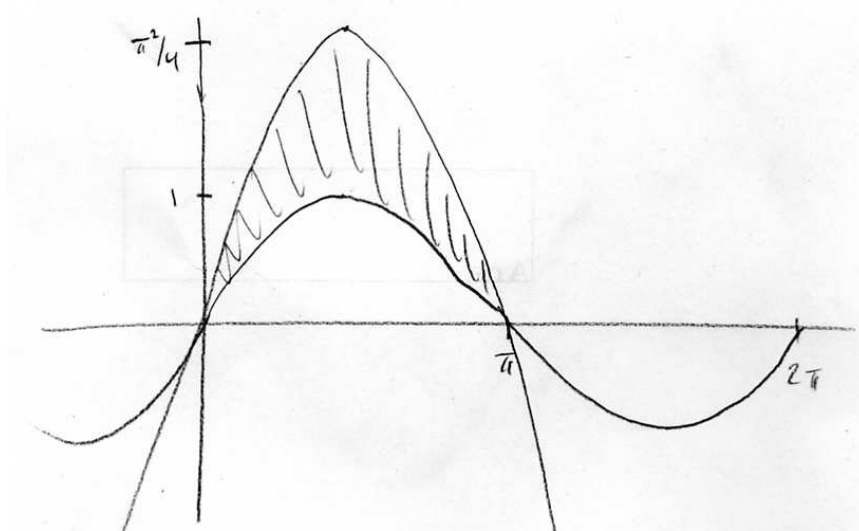
$$U(P) = \sin 1 + 1 + \sin 2 + (\sin 3)(\pi - 3) \approx 2.771.$$

$$L(P) = 0 + \sin 1 + \sin 3 + (0)(\pi - 3) \approx .983.$$

2. If  $y = \sqrt{4 - x^2}$  then  $y^2 = 4 - x^2$ , so  $x^2 + y^2 = 4$ ; thus  $y = \sqrt{4 - x^2}$  is the equation of a semi-circle of radius 2. The area of this is  $2\pi$ .
3. Let  $G(t)$  be an anti-derivative of  $\frac{1}{\sqrt{t^4 + 1}}$ . Then  $F(x) = G(x^{1/2}) - G(\cos(x^2))$ , so

$$\begin{aligned} F'(x) &= G'(x^{1/2})\left(\frac{1}{2}x^{-1/2}\right) - G'(\cos(x^2))(-\sin(x^2)2x) \\ &= \frac{1}{\sqrt{x^2 + 1}} \cdot \frac{1}{2\sqrt{x}} - \frac{1}{\sqrt{(\cos(x^2))^4 + 1}}(-\sin(x^2)2x) \\ &= \frac{1}{2\sqrt{x^3 + x}} + \frac{2x \sin(x^2)}{\sqrt{(\cos(x^2))^4 + 1}}. \end{aligned}$$

4. We can't solve the equation  $\sin x = \pi x - x^2$  by turn-the-crank methods, but if we just go ahead and graph the two functions we see that we don't need to.  $y = \pi x - x^2$  is a parabola with roots at 0 and  $\pi$  and a maximum at  $\left(\frac{\pi}{2}, \frac{\pi^2}{4}\right)$ . Observe that  $\frac{\pi^2}{4} > 2$ .



$$\int_0^\pi (\pi x - x^2 - \sin x) dx = \left[ \pi \frac{x^2}{2} - \frac{x^3}{3} + \cos x \right]_0^\pi = \frac{\pi^3}{6} - 2 \approx 3.168.$$