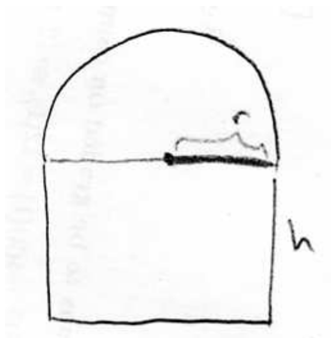
Solutions to Practice Exam 2

1. Let h be the height of the rectangle and r be the radius of the circle, which is half the width of the rectangle, as below:



The perimiter is $\pi r + 2h + 2r = (\pi + 2)r + 2h$. Since this is constrained to be 24, $h = 12 - (\frac{\pi}{2} + 1)r$. The area is

$$\begin{aligned} A &= \tfrac{1}{2}\pi r^2 + 2rh \\ &= \tfrac{\pi}{2}r^2 + 2r[12 - (\tfrac{\pi}{2} + 1)r] \\ &= \tfrac{\pi}{2}r^2 + 24r - (\pi + 2)r^2 \\ &= 24r - (\tfrac{\pi}{2}\pi + 2)r^2 \end{aligned}$$

We wish to choose r to maximize A, so we set $0 = \dfrac{dA}{dr} = 24 - (\pi + 4)r$, so $r = \dfrac{24}{\pi + 4} \approx 3.36$, so after a little arithmetic, we find that $h = \dfrac{24}{\pi + 4}$ as well. That is, the rectangle should be a twice as wide as it is tall.

2. Let t be the time in hours and $x(t)$ be the driver's position. By the mean value theorem there must be a time c between 0 and 2 for which

$$v(c) = x'(c) = \frac{x(2:00) - x(12:00)}{2:00 - 12:00} = \frac{150 \text{ mi}}{2 \text{ hr}} = 75 \ \frac{\text{mi}}{\text{hr}}$$

so he must have been speeding at some point.

3. Consider $\log(e^x + 3x)^{1/x} = \dfrac{1}{x}\log(e^x + 3x) = \dfrac{\log(e^x + 3x)}{x}$. As $x \to 0$, this is of the form $\dfrac{0}{0}$, so we can use L'Hôpital's rule:

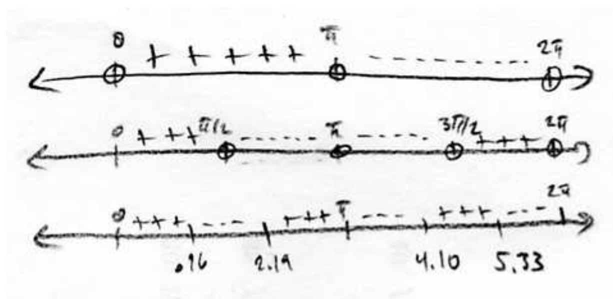
$$\lim_{x \to 0} \frac{\log(e^x + 3x)}{x} = \lim_{x \to 0} \frac{\frac{1}{e^x + 3x}(e^x + 3)}{1} = 4.$$

Thus $\lim_{x \to 0}(e^x + 3x)^{1/x} = \lim_{x \to 0} e^{\log(e^x + 3x)^{1/x}} = e^{\lim_{x \to 0} \log(e^x + 3x)^{1/x}} = e^4 \approx 54.6.$

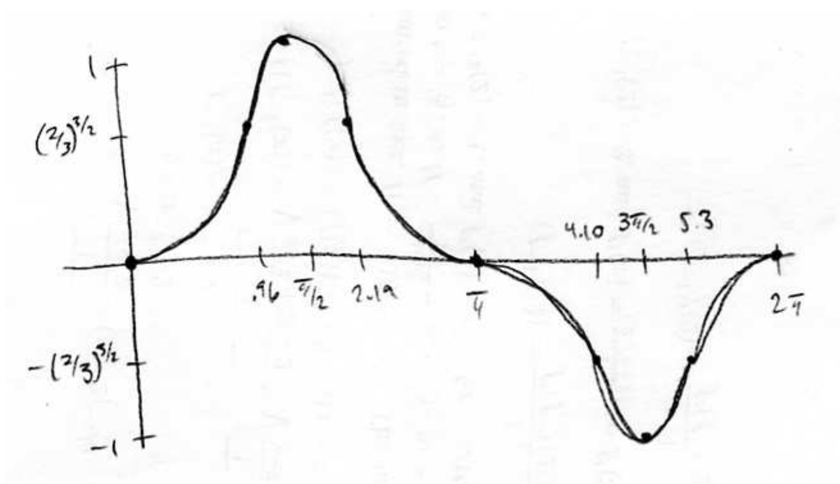
4. $y = (\sin x)^3$, so $y' = 3(\sin x)^2 \cos x$, so

$$\begin{aligned} y'' &= 6 \sin x (\cos x)^2 - 3(\sin x)^3 \\ &= 3 \sin x[2(\cos x)^2 - (\sin x)^2] \\ &= 3 \sin x[2(1 - (\sin x)^2) - (\sin x)^2] \\ &= 3 \sin x[2 - 3(\sin x)^2]. \end{aligned}$$

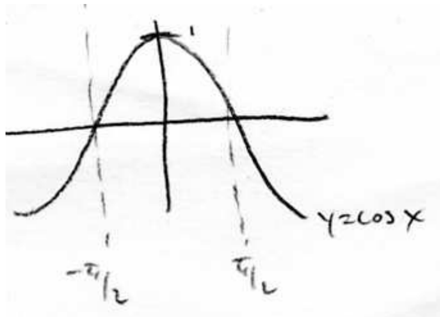
When $y = 0$, $\sin x = 0$, so $x = 0$, π, or 2π. When $y' = 0$, either $\sin x = 0$, so $x = 0$, π, or 2π; or $\cos x = 0$, so $x = \pi/2$ or $3\pi/2$. When $y'' = 0$, either $\sin x = 0$, so $x = 0$, π, or 2π; or $2 = 3(\sin x)^2$, so $\sin x = \pm\sqrt{2/3}$, so $x = .96$, 2.19, 4.10, or 5.33. The signs of y, y', and y'' are as follows:



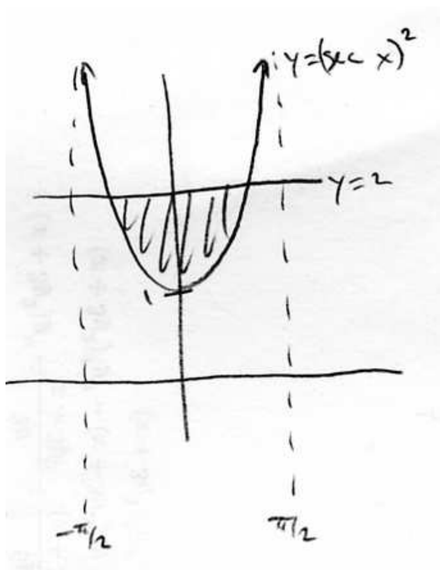
When $x = \pi/2$, $y = 1$. When $x = 3\pi/2$, $y = -1$. When $x = .96$ or 2.19, $y = (2/3)^{3/2} \approx .54$. When $x = 4.10$ or 5.33, $y = -(2/3)^{3/2} \approx -.54$.



5. $y = \cos x$ looks like this,



so $y = (\sec x)^2 = \dfrac{1}{(\cos x)^2}$ will be 1 when $\cos x$ is 1 and go to infinity when $\cos x$ goes to 0:



First we find the intersection. If $(\sec x)^2 = 2$ then $\sec x = \sqrt{2}$ (since we're working on between $-\pi/2$ and $\pi/2$ we can ignore the negative square root), so $\cos x = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$, so $x = \pi/4$ or $-\pi/4$. Now

$$\int_{-\pi/4}^{\pi/4} [2 - (\sec x)^2]\, dx = \Big[2x - \tan x\Big]_{-\pi/4}^{\pi/4} = \left(\frac{\pi}{2} - 1\right) - \left(-\frac{\pi}{2} + 1\right) = \pi - 2 \approx 1.14.$$