Convergence Tests

1 Building Blocks

Geometric Series: $\sum r^n$ converges if $|r| < 1$ and diverges if $|r| \geq 1$.

p-Series: $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

$\mathbf{n^{th}}$ Term Test: If $a_n \not\to 0$ then $\sum a_n$ diverges. If $a_n \to 0$ then $\sum a_n$ *might* converge.

2 Tests for Series with Positive Terms Only

Integral Test: If $a_n = f(n)$ and $a_n \searrow 0$ then $\sum_{n=0}^{\infty} a_n$ acts like $\int_0^{\infty} f(x)\, dx$.

Basic Comparison Test: Suppose that $a_n \leq b_n$. If the larger series $\sum b_n$ converges then so does the smaller series $\sum a_n$. If the smaller series $\sum a_n$ diverges then so does the larger series $\sum b_n$.

Limit Comparison Test: Suppose that $\frac{a_n}{b_n} \to L$. If $0 < L < \infty$ then $\sum a_n$ acts like $\sum b_n$. If $L = 0$ or ∞, choose a different b_n for comparison.

Ratio Test: Suppose that $\frac{a_{n+1}}{a_n} \to \rho$. Then $\sum a_n$ converges if $\rho < 1$ and diverges if $\rho > 1$. If $\rho = 1$, use another test.

3 Tests for Series with Positive and Negative Terms

Absolute Convergence Test: If $\sum |a_n|$ converges then so does $\sum a_n$.

Alternating Series Test: If $a_n \searrow 0$ then $a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + \dots$ converges. Consider using this test when you see $(-1)^n$.

4 Tests that We Won't Cover in this Course

Root Test: Suppose that $a_n \geq 0$ and $\sqrt[n]{a_n} \to \rho$. Then $\sum a_n$ converges if $\rho < 1$ and diverges if $\rho > 1$.

Raabe's Test: Suppose that $a_n > 0$ and $n\left(\frac{a_n}{a_{n+1}} - 1\right) \to \rho$. Then $\sum a_n$ converges if $\rho > 1$ and diverges if $\rho < 1$.

Gauss's Test: Suppose that $a_n > 0$, b_n is bounded, $r > 1$, and $\frac{a_n}{a_{n+1}} = 1 + \frac{h}{n} + \frac{b_n}{n^r}$. Then a_n converges if $h > 1$ and diverges if $h \leq 1$.

Bertrand's Test: Suppose that $\frac{a_n}{a_{n+1}} = 1 + \frac{1}{n} + \frac{\rho_n}{n \ln n}$ and $\rho_n \to \rho$. Then $\sum a_n$ diverges if $\rho > 1$ and converges if $\rho < 1$.

Abel's Test: Suppose that $\sum a_n$ converges and $b_n \searrow 0$. Then $\sum a_n b_n$ converges.

Cauchy's Test: If $a_n \searrow 0$ then $\sum a_n$ acts like $\sum 2^n a_{2^n} = a_1 + 2a_2 + 4a_4 + 8a_8 + \ldots$.

Ermakoff's Test: Suppose that $a_n = f(n)$, $a_n \searrow 0$, and $\frac{e^x f(e^x)}{f(x)} \to \rho$. Then $\sum a_n$ diverges if $\rho > 1$ and converges if $\rho < 1$.

Kummer's Test: Suppose that $a_n > 0$, $b_n > 0$, and $b_n \frac{a_n}{a_{n+1}} - b_{n+1} \to \rho$. If $\rho > 0$ then $\sum a_n$ converges. If $\rho < 0$ and $\sum \frac{1}{b_n}$ diverges then $\sum a_n$ diverges.

References

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[2] Bromwich, T. J. I'a., *An Introduction to the Theory of Infinite Series*, 2nd ed., London: Macmillan, 1931.

[3] Rudin, W., *Principles of Mathematical Analysis*, 3rd ed., New York: McGraw-Hill, 1976.