**Solutions to Exam 1**

**I. (A)** Let $u = \sqrt{x}$, so $du = \dfrac{dx}{2\sqrt{x}}$.

$$\begin{aligned}\int \frac{\cos(\sqrt{x})}{\sqrt{x}}\ dx &= 2\int \cos u\ du \\ &= 2\sin u + C \\ &= 2\sin(\sqrt{x}) + C\end{aligned}$$

**I. (B)** Let $u = 3x+2$, so $x = \dfrac{u-2}{3}$ and $dx = \dfrac{du}{3}$.

$$\begin{aligned}\int x\sqrt{3x+2}\ dx &= \frac{1}{9}\int (u^{3/2} - 2u^{1/2})\ du \\ &= \frac{2}{45}u^{5/2} - \frac{4}{27}u^{3/2} + C \\ &= \frac{2}{45}(3x+2)^{5/2} - \frac{4}{27}(3x+2)^{3/2} + C\end{aligned}$$

It was also possible to let $u = \sqrt{3x+2}$ or to integrate by parts.

**II. (A)** Let $u = x$ and $dv = e^x\ dx$, so $du = dx$ and $v = e^x$.

$$\begin{aligned}\int xe^x\ dx &= xe^x - \int e^x dx \\ &= xe^x - e^x + C\end{aligned}$$

**II. (B)** Let $u = \sin x$, so $du = \cos x\ dx$, and apply the result of part (A).

$$\begin{aligned}\int \sin x \cos x e^{\sin x}\ dx &= \int ue^u\ du \\ &= ue^u - e^u + C \\ &= \sin x e^{\sin x} - e^{\sin x} + C\end{aligned}$$

**III. (A)** Factor $x^2 - 5x + 4$ as $(x-4)(x-1)$. We wish to find $A$ and $B$ such that

$$\frac{1}{(x-4)(x-1)} = \frac{A}{x-4} + \frac{B}{x-1}$$

so

$$1 = A(x-1) + B(x-4).$$

Letting $x = 4$, we get $A = 1/3$, and letting $x = 1$, we get $B = -1/3$.

$$\begin{aligned}\int \frac{dx}{x^2-5x+4} &= \int \frac{dx}{(x-4)(x-1)} \\ &= \frac{1}{3}\int \frac{dx}{x-4} - \frac{1}{3}\int \frac{dx}{x-1} \\ &= \frac{1}{3}\ln|x-4| - \frac{1}{3}\ln|x-1| + C\end{aligned}$$

**III. (B)** Let $u = \cos x$, so $du = -\sin x\ dx$, and apply the result of part (A).

$$\begin{aligned}\int \frac{-\sin x\ dx}{\cos^2 x - 5\cos x + 4} &= \int \frac{du}{u^2 - 5u + 4} \\ &= \frac{1}{3}\ln|u-4| - \frac{1}{3}\ln|u-1| + C \\ &= \frac{1}{3}\ln|\cos x - 4| - \frac{1}{3}\ln|\cos x - 1| + C\end{aligned}$$

**IV. (A)** Let $u = x^2 - 1$, so $du = 2x\ dx$. Then

$$\begin{aligned}\int_2^3 \frac{x\ dx}{\sqrt{x^2-1}} &= \frac{1}{2}\int_-^- \frac{du}{\sqrt{u}} \\ &= \sqrt{u}\ \Big|_-^- \\ &= \sqrt{x^2-1}\ \Big|_2^3 \\ &= 2\sqrt{2} - \sqrt{3}\end{aligned}$$

**IV. (B)** Let $u = 1 - x^2$, so $du = -2x\ dx$. Then

$$\begin{aligned}\int_{\frac{1}{3}}^{\frac{1}{2}} \frac{x\ dx}{\sqrt{1-x^2}} &= -\frac{1}{2}\int_-^- \frac{du}{\sqrt{u}} \\ &= -\sqrt{u}\ \Big|_-^- \\ &= -\sqrt{1-x^2}\ \Big|_{\frac{1}{3}}^{\frac{1}{2}} \\ &= -\frac{\sqrt{3}}{2} + \frac{2\sqrt{2}}{3}\end{aligned}$$

**V. (A)** We wish to find $A$, $B$, and $C$ such that

$$\frac{3x^2 + x + 1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx + C}{x^2+1}$$

so

$$3x^2 + x + 1 = A(x^2+1) + (Bx + C)x.$$

Letting $x = 0$, we get $A = 1$, so

$$\begin{aligned}3x^2 + x + 1 &= x^2 + 1 + Bx^2 + Cx \\ 2x^2 + x &= Bx^2 + Cx\end{aligned}$$

so $B = 2$ and $C = 1$.

$$\begin{aligned}\int \frac{3x^2 + x + 1}{x(x^2+1)} &= \int \frac{dx}{x} + \int \frac{(2x+1)\ dx}{x^2+1} \\ &= \int \frac{dx}{x} + \int \frac{2x\ dx}{x^2+1} + \int \frac{dx}{x^2+1}\end{aligned}$$

In the second-to-last of these integrals, let $u = x^2 + 1$, so $du = 2x\ dx$, so

$$\begin{aligned}
\int \frac{3x^2 + x + 1}{x(x^2+1)} &= \int \frac{dx}{x} + \int \frac{du}{u} + \int \frac{dx}{x^2+1} \\
&= \ln|x| + \ln|u| + \arctan x + C \\
&= \ln|x| + \ln|x^2+1| + \arctan x + C
\end{aligned}$$

**V. (B)** Let $x = \sec t$, so $dx = \sec t \tan t\ dt$ and $t = \operatorname{arcsec} x$. Then

$$\begin{aligned}
\int_2^3 \frac{dx}{x\sqrt{x^2-1}} &= \int_{-}^{-} \frac{\sec t \tan t\ dt}{\sec t \sqrt{\sec^2 t - 1}} \\
&= \int_{-}^{-} dt \\
&= t\ \Big|_{-}^{-} \\
&= \operatorname{arcsec} x\ \Big|_2^3 \\
&= \operatorname{arcsec} 3 - \operatorname{arcsec} 2
\end{aligned}$$

**VI. (A)**

$$\begin{aligned}
\int_0^\beta \sin(x)\cos(ax)\ dx &= \frac{1}{2}\int_0^\beta (\sin(1+a)x + \sin(1-a)x)\ dx \\
&= \left[\frac{-\cos(1+a)x}{2(1+a)} + \frac{-\cos(1-a)x}{2(1-a)}\right]_0^\beta \\
&= \frac{1-\cos(1+a)\beta}{2(1+a)} + \frac{1-\cos(1-a)\beta}{2(1-a)}
\end{aligned}$$

where for the last step we used the fact that $\cos 0 = 1$.

**VI. (A)** Let $u = \sin x$, so $du = \cos x\ dx$. Then

$$\begin{aligned}
\int_0^\beta \sin x \cos x\ dx &= \int_{-}^{-} u\ du \\
&= \frac{1}{2}u^2\ \Big|_{-}^{-} \\
&= \frac{1}{2}\sin^2 x\ \Big|_0^\beta \\
&= \frac{1}{2}\sin^2 \beta
\end{aligned}$$

where for the last step we used the fact that $\sin 0 = 0$.