

I. (30 points) Evaluate, if possible, the following improper integrals.

(A) $\int_1^3 \frac{1}{(x-1)^3} dx$

Solution: The denominator of the integrand is 0 at $x = 1$. So we have to take a limit and see what the result is. By definition,

$$\begin{aligned}\int_1^3 \frac{dx}{(x-1)^3} &= \lim_{b \rightarrow 1^+} \int_b^3 \frac{dx}{(x-1)^3} \\ &= \lim_{b \rightarrow 1^+} \left. -\frac{1}{2}(x-1)^{-2} \right|_b^3 \\ &= \lim_{b \rightarrow 1^+} \frac{1}{2}(b-1)^{-2} - \frac{1}{8},\end{aligned}$$

which diverges.

(B) $\int_0^\infty \frac{1}{1+x^2} dx$

Solution: This time the problem is that we have an unbounded domain, rather than, as in the previous problem, an unbounded range. Thus we have, by definition,

$$\begin{aligned}\int_0^\infty \frac{dx}{1+x^2} &= \lim_{a \rightarrow \infty} \int_0^a \frac{dx}{1+x^2} \\ &= \lim_{a \rightarrow \infty} \tan^{-1} x \Big|_0^a \\ &= \lim_{a \rightarrow \infty} (\tan^{-1} a - \tan^{-1} 0) \\ &= \lim_{a \rightarrow \infty} \tan^{-1} a \\ &= \frac{\pi}{2}.\end{aligned}$$

II. (30 points) Indicate if the following series converge or diverge. Mention the test you used and justify your answer

(A)

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2 + 1}{n^4 + 3}$$

Solution: The easiest way to see that this converges is the Alternating Series Test. We let $a_n = (n^2 + 1)/(n^4 + 3)$ and see that a_n is a decreasing positive sequence with limit 0, so the series converges by the Alternating Series Test.

(B)

$$\sum_{n=1}^{\infty} \frac{n^2}{n^3 + 1}$$

Solution: This diverges by the Integral Test, or by a Limit Comparison Test with $1/n$. We compare it to the integral

$$\int_1^{\infty} \frac{x^2}{x^3 + 1} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{x^2}{x^3 + 1} dx = \lim_{b \rightarrow \infty} \frac{1}{3} \ln \left(\frac{b^3 + 1}{2} \right),$$

which diverges, so the series diverges. The alternative is to note that

$$\lim_{n \rightarrow \infty} \frac{n^2/(n^3 + 1)}{1/n} = \lim_{n \rightarrow \infty} \frac{n^3}{n^3 + 1} = 1,$$

and because the harmonic series $\sum k^{-1}$ diverges, the original series diverges as well.

III. (30 points)

(A) Write $2.1212\overline{12}$ as a fraction of two integers.

Solution: The number $2.1212\overline{12}$ means that the “12” is repeated *ad infinitum*. Then we see that the number represents the series

$$\begin{aligned} 2.1212\overline{12} &= 2 + \frac{12}{100} + \frac{12}{100^2} + \frac{12}{100^3} + \cdots \\ &= 2 + \frac{12}{100} \left(1 + \frac{1}{100} + \frac{1}{100^2} + \frac{1}{100^3} + \cdots \right) \\ &= 2 + \frac{12}{100} \cdot \frac{1}{1 - \frac{1}{100}} \\ &= 2 + \frac{12}{100} \cdot \frac{100}{99} \\ &= 2 + \frac{4}{33} \\ &= \frac{70}{33} \end{aligned}$$

(B) Find the radius of convergence for

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{2^n} x^n.$$

Solution: Use the Absolute Ratio Test to see that

$$\lim_{n \rightarrow \infty} \left| \frac{\frac{(n+1)^2}{2^{n+1}} (-1)^{n+1} x^{n+1}}{\frac{n^2}{2^n} (-1)^n x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x}{2} \right| \cdot \frac{(n+1)^2}{n^2} = \frac{|x|}{2}.$$

The series will converge when the ratio is less than 1, so we have $|x| < 2$, which gives us a radius of convergence of 2.

IV. (30 points)

(A) Find the Maclaurin series for $\frac{1}{(1+2x)^2}$.

Solution: We take derivatives

$$\begin{aligned} f(x) &= (1+2x)^{-2} & f(0) &= 1 \\ f'(x) &= -4(1+2x)^{-3} & f'(0) &= -4 \\ f''(x) &= 24(1+2x)^{-4} & f''(0) &= 24 \\ f'''(x) &= -192(1+2x)^{-5} & f'''(0) &= -192 \\ f^{(4)}(x) &= 1920(1+2x)^{-6} & f^{(4)}(0) &= 1920 \end{aligned}$$

Thus, we get for the Maclaurin series

$$\begin{aligned} f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots &= 1 - 4x + 12x^2 - 32x^3 + 80x^4 - \dots \\ &= \sum_{k=0}^{\infty} (k+1)(-2)^k x^k. \end{aligned}$$

Alternatively, one could note that

$$y = (1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \dots$$

so

$$\frac{dy}{dx} = (1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$$

Now make the assignment $x \leftarrow -2x$, so the series becomes

$$(1 - (-2x))^{-2} = 1 + 2(-2x) + 3(-2x)^2 + 4(-2x)^3 + \dots,$$

or

$$(1+2x)^{-2} = \frac{1}{(1+2x)^2} = 1 - 4x + 12x^2 - 32x^3 + \dots$$

(B) Suppose $1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3$ are the first four terms in Maclaurin series of $f(x) = (1+x)^{\frac{1}{3}}$. Find $f(0)$, $f'(0)$, $f''(0)$ and $f^{(3)}(0)$.

Solution: We note that the coefficient of x^n is $f^{(n)}(0)/n!$. Call the coefficient a_n , and then $f^{(n)}(0) = n! \cdot a_n$. So we get

$$\begin{aligned} f(0) &= 0! \cdot 1 = 0! = 1, \\ f'(0) &= 1! \cdot \frac{1}{3} = 1 \cdot \frac{1}{3} = \frac{1}{3}, \\ f''(0) &= 2! \cdot \left(-\frac{1}{9}\right) = 2 \cdot \left(-\frac{1}{9}\right) = -\frac{2}{9}, \end{aligned}$$

and

$$f^{(3)}(0) = 3! \cdot \frac{5}{81} = 6 \cdot \frac{5}{81} = \frac{30}{81} = \frac{10}{27}.$$

V. (30 points)

(A) Find the 4th Taylor polynomial, P_4 , based at $a = 0$ for $\ln(1 + x)$. Do NOT manipulate a well known series to get your answer.

Solution: We take derivatives. Let $f(x) = \ln(1 + x)$, so $f(0) = 0$. Then

$$\begin{aligned} f'(x) &= (1+x)^{-1} & f'(0) &= 1 \\ f''(x) &= -(1+x)^{-2} & f''(0) &= -1 \\ f^{(3)}(x) &= 2(1+x)^{-3} & f^{(3)}(0) &= 2 \\ f^{(4)}(x) &= -6(1+x)^{-4} & f^{(4)}(0) &= -6. \end{aligned}$$

Then we compute write out the first few terms of the series

$$\begin{aligned} P_4(x) &= \frac{f(0)x^0}{0!} + \frac{f'(0)x^1}{1!} + \frac{f''(0)x^2}{2!} + \frac{f^{(3)}(0)x^3}{3!} + \frac{f^{(4)}(0)x^4}{4!} \\ &= \frac{0x^0}{0!} + \frac{1x^1}{1!} + \frac{(-1)x^2}{2!} + \frac{2x^3}{3!} + \frac{(-6)x^4}{4!} \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}. \end{aligned}$$

(B) Use your answer in part (A) to estimate $\ln(0.8)$.

Solution: In this case, we have $x = -0.2$ in the expression above. Then we evaluate

$$(-0.2) - \frac{(-0.2)^2}{2} + \frac{(-0.2)^3}{3} - \frac{(-0.2)^4}{4}$$

to get roughly -0.2230667 . When we ask a calculator to produce the answer, we get approximately -0.223144 .