**I.** (25 points)
**(A)**(10 points) Solve the following differential equation.

$$y' = \frac{\cos x}{y}$$

**Solution**: We compute

$$yy' = \cos x,$$

and note that the left-hand side is a derivative. So we have

$$\begin{aligned} \frac{d}{dx}\left(\frac{1}{2}y^2\right) &= \cos x \\ \frac{1}{2}y^2 &= \sin x + C \\ y^2 &= 2\sin x + 2C, \end{aligned}$$

or

$$y = \pm\sqrt{2\sin x + 2C}.$$

**(B)**(15 points) Solve the initial value differential equation

$$y' = y + e^x, \qquad \text{with } y(0) = 5$$

**Solution**: We have

$$\begin{aligned} y' - y &= e^x \\ e^{-x}y' - e^{-x}y &= 1 \\ \frac{d}{dx}(e^{-x}y) &= 1 \\ e^{-x}y &= x + C \end{aligned}$$

Now $y = 5$ when $x = 0$, so $e^{-0} \cdot 5 = 0 + C$; thus $C = 5$.

$$y = (x + 5)e^x.$$

**II.** (25 points) Solve each differential equation.
**(A)**(10 points)

$$y'' + 3y' - 10y = 0$$

**Solution**: We get the auxiliary equation $r^2 + 3r - 10 = 0$, so $r = -5$ or $r = 2$. Thus we have two distinct real roots to the auxiliary equation, and get for a general solution

$$y = C_1 e^{-5t} + C_2 e^{2t}.$$

**(B)**(15 points)

$$y'' + 3y' - 10y = e^{2t}$$

**Solution**: We calculated the homogeneous solution in part **(A)**. Since $e^{2t}$ is one of the homogeneous solutions, we guess that a particular solution is $y_p = Ate^{2t}$. Then

$$y_p'' + 3y_p' - 10y_p = 7Ae^{2t} = e^{2t}.$$

Therefore we decide that $A = 1/7$. So the general solution is

$$y = C_1 e^{-5t} + C_2 e^{2t} + \frac{1}{7} te^{2t}.$$

**III.** (25 points) Solve the following differential equation.

$$y^{(4)} + y'' = 0.$$

**Solution**: The auxiliary equation is

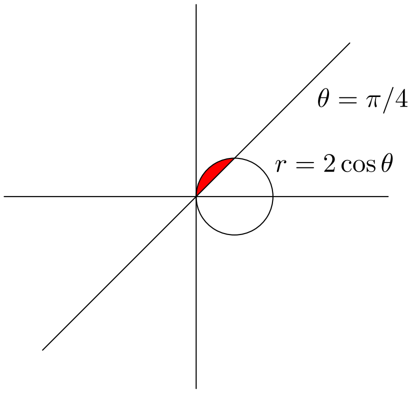
$$r^4 + r^2 = 0,$$

so we have roots $r = 0$ repeated twice, and $r = \pm i$. Thus the general solution is

$$y = C_1 + C_2 x + C_3 \cos x + C_4 \sin x.$$

**IV.** (25 points) In each part find the shaded area. Set up the integral but do not evaluate

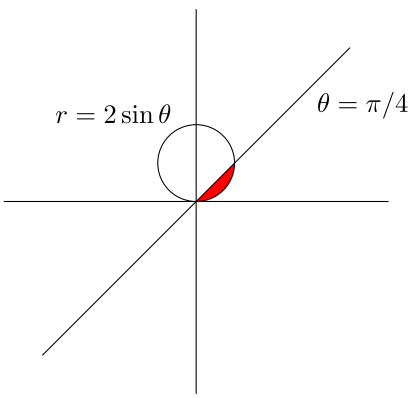
**(A)**(7 points)




**Solution:**

$$A = \frac{1}{2}\int_{\pi/4}^{\pi/2} (2\cos\theta)^2\, d\theta.$$

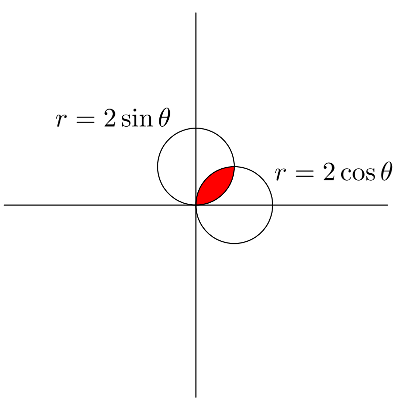
**(B)**(8 points)




**Solution:**

$$A = \frac{1}{2}\int_{0}^{\pi/4} (2\sin\theta)^2\, d\theta.$$

**(C)**(10 points)




**Solution:** We note that the area shaded is simply the sum of the two previous areas, so

$$A = \frac{1}{2}\int_{0}^{\pi/4} (2\sin\theta)^2\, d\theta + \frac{1}{2}\int_{\pi/4}^{\pi/2} (2\cos\theta)^2\, d\theta.$$

One could also note that by symmetry the areas are equal, so that the answer could also be twice that of **(A)** or **(B)**.

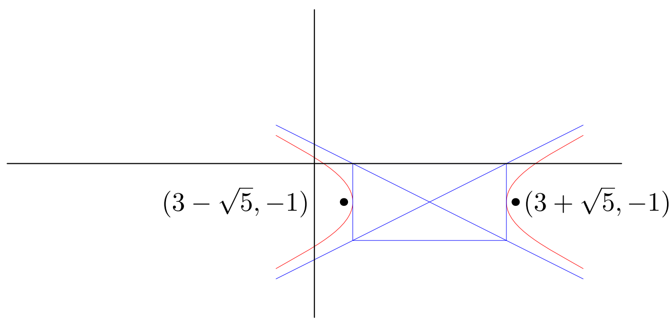
**V.** (25 points) Let $x^2 - 4y^2 - 6x - 8y + 1 = 0$ be the equation of a conic.
**(A)**Put the equation in the standard form. Identify the conic.

**Solution**: We have

$$\begin{aligned}(x^2 - 6x + 9) - 4(y^2 + 2y + 1) &= -1 + 9 - 4 \\ (x-3)^2 - 4(y+1)^2 &= 4 \\ \frac{(x-3)^2}{4} - (y+1)^2 &= 1,\end{aligned}$$

which is the equation of a hyperbola.
**(B)**Give the coordinates of the foci and compute the eccentricity.

**Solution**: We have that $c$, the distance from the center of the hyperbola to a focus is $c = \sqrt{4+1} = \sqrt{5}$. Thus the eccentricity is $e = c/a = \sqrt{5}/2$, and the foci are at $(3 \pm \sqrt{5}, -1)$. The graph is shown below, with the foci indicated:



**VI.** (25 points) Let $xy = 1$ be given.
**(A)**(15 points) Eliminate the cross product term by a suitable rotation of the axes.
**(B)**(10 points) Graph the equation showing the rotated axes.

**Solution**: The angle $\theta$ that eliminates the cross product term satisfies

$$\cot 2\theta = \frac{A - C}{B} = \frac{0 - 0}{1} = 0,$$

so $\theta = \pi/4$. Thus we get

$$x = \frac{u - v}{\sqrt{2}} \qquad \text{and} \qquad y = \frac{u + v}{\sqrt{2}}.$$

Subsituting for $x$ and $y$, we get

$$\frac{u^2}{2} - \frac{v^2}{2} = 1.$$

The graph is shown below: