

Solutions to Homework 1

§8.1 #46. Substitute $u = e^{2x} + e^{-2x}$, so $du = 2(e^{2x} - e^{-2x})dx$. Then there are two ways to evaluate the integral, depending on how we want to deal with the limits of integration. If we use dashes for the limits when u is in play, then

$$\begin{aligned} \int_0^1 \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} dx &= \frac{1}{2} \int_-^+ \frac{du}{u} \\ &= \frac{1}{2} \ln |u| \Big|_-^+ \\ &= \frac{1}{2} \ln |e^{2x} + e^{-2x}| \Big|_0^1 \\ &= \frac{1}{2} \ln |e^2 + e^{-2}| - \frac{1}{2} \ln |e^0 + e^0| \\ &= \frac{1}{2} \ln |e^2 + e^{-2}| - \frac{1}{2} \ln 2. \end{aligned}$$

If we change the limits when we make the substitution, observing that when $x = 1$, $u = e^2 + e^{-2}$ and when $x = 0$, $u = e^0 + e^0 = 2$, then

$$\begin{aligned} \int_{x=0}^{x=1} \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} dx &= \frac{1}{2} \int_{u=2}^{u=e^2+e^{-2}} \frac{du}{u} \\ &= \frac{1}{2} \ln |u| \Big|_2^{e^2+e^{-2}} \\ &= \frac{1}{2} \ln |e^2 + e^{-2}| - \frac{1}{2} \ln 2. \end{aligned}$$

§8.2 #12 (a). This is just a very long computation.

$$\begin{aligned} \int \cos^6 \theta \sin^2 \theta \, d\theta &= \int \left(\frac{1 + \cos 2\theta}{2} \right)^3 \left(\frac{1 - \cos 2\theta}{2} \right) d\theta \\ &= \frac{1}{16} \int (1 + 2 \cos 2\theta - 2 \cos^3 2\theta - \cos^4 2\theta) d\theta \\ &= \frac{1}{16} \int d\theta + \frac{1}{8} \int \cos 2\theta \, d\theta - \frac{1}{8} \int \cos^3 2\theta \, d\theta - \frac{1}{16} \int \cos^4 2\theta \, d\theta \\ &= \frac{1}{16} \theta + \frac{1}{16} \sin 2\theta - \frac{1}{8} \int (1 - \sin^2 2\theta) \cos 2\theta \, d\theta - \frac{1}{16} \int \left(\frac{1 + \cos 4\theta}{2} \right)^2 d\theta \\ &= \frac{1}{16} \theta + \frac{1}{16} \sin 2\theta - \frac{1}{8} \int \cos 2\theta \, d\theta + \frac{1}{8} \int \sin^2 2\theta \cos 2\theta \, d\theta - \frac{1}{64} \int (1 + 2 \cos 4\theta + \cos^2 4\theta) d\theta \\ &\quad \text{in the second integral here, substitute } u = \sin 2\theta, \text{ so } du = 2 \cos 2\theta \, d\theta \\ &= \frac{1}{16} \theta + \frac{1}{16} \sin 2\theta - \frac{1}{16} \sin 2\theta + \frac{1}{16} \int u^2 du - \frac{1}{64} \int d\theta - \frac{1}{32} \int \cos 4\theta \, d\theta - \frac{1}{64} \int \cos^2 4\theta \, d\theta \\ &= \frac{1}{16} \theta + \frac{1}{48} u^3 - \frac{1}{64} \theta - \frac{1}{128} \sin 4\theta - \frac{1}{64} \int \left(\frac{1 + \cos 8\theta}{2} \right) d\theta \\ &= \frac{3}{64} \theta + \frac{1}{48} \sin^3 2\theta - \frac{1}{128} \sin 4\theta - \frac{1}{128} \int d\theta - \frac{1}{128} \int \cos 8\theta \, d\theta \\ &= \frac{3}{64} \theta + \frac{1}{48} \sin^3 2\theta - \frac{1}{128} \sin 4\theta - \frac{1}{128} \theta - \frac{1}{1024} \sin 8\theta + C \\ &= \frac{5}{128} \theta + \frac{1}{48} \sin^3 2\theta - \frac{1}{128} \sin 4\theta - \frac{1}{1024} \sin 8\theta + C \end{aligned}$$

(b). The same problem, only we start by using the half-angle formula $\sin \theta \cos \theta = \frac{\sin 2\theta}{2}$.

$$\begin{aligned}
 \int \cos^6 \theta \sin^2 \theta \, d\theta &= \int \cos^4 \theta \left(\frac{\sin 2\theta}{2} \right)^2 d\theta \\
 &= \int \left(\frac{1 + \cos 2\theta}{2} \right)^2 \left(\frac{\sin 2\theta}{2} \right)^2 d\theta \\
 &= \frac{1}{16} \int (1 + 2 \cos 2\theta + \cos^2 2\theta) \sin^2 2\theta \, d\theta \\
 &= \frac{1}{16} \int \sin^2 2\theta \, d\theta + \frac{1}{8} \int \cos 2\theta \sin^2 2\theta \, d\theta + \frac{1}{16} \int \cos^2 2\theta \sin^2 2\theta \, d\theta \\
 &\quad \text{in the second integral here, substitute } u = \sin 2\theta, \text{ so } du = 2 \cos 2\theta \, d\theta \\
 &= \frac{1}{16} \int \left(\frac{1 - \cos 4\theta}{2} \right) d\theta + \frac{1}{16} \int u^2 du + \frac{1}{16} \int \left(\frac{\sin 4\theta}{2} \right)^2 d\theta \\
 &= \frac{1}{32} \int d\theta - \frac{1}{32} \int \cos 4\theta \, d\theta + \frac{1}{48} u^3 + \frac{1}{64} \int \sin^2 4\theta \, d\theta \\
 &= \frac{1}{32} \theta - \frac{1}{128} \sin 4\theta + \frac{1}{48} \sin^3 2\theta + \frac{1}{64} \int \left(\frac{1 - \cos 8\theta}{2} \right) d\theta \\
 &= \frac{1}{32} \theta - \frac{1}{128} \sin 4\theta + \frac{1}{48} \sin^3 2\theta + \frac{1}{128} \int d\theta - \frac{1}{128} \int \cos 8\theta \, d\theta \\
 &= \frac{1}{32} \theta - \frac{1}{128} \sin 4\theta + \frac{1}{48} \sin^3 2\theta + \frac{1}{128} \theta - \frac{1}{1024} \sin 8\theta + C \\
 &= \frac{5}{128} \theta - \frac{1}{128} \sin 4\theta + \frac{1}{48} \sin^3 2\theta - \frac{1}{1024} \sin 8\theta + C
 \end{aligned}$$

(c). The two answers are clearly equal.

§8.3 #8. Substitute $u = (1 - x)^{1/3}$, so $x = 1 - u^3$, so $dx = -3u^2 du$. Then

$$\begin{aligned}
 \int x(1-x)^{2/3} dx &= \int (1-u^3)u^2(-3u^2)du \\
 &= -3 \int (u^4 - u^7) du \\
 &= -3 \int u^4 du + 3 \int u^7 du \\
 &= -\frac{3}{5}u^5 + \frac{3}{8}u^8 + C \\
 &= -\frac{3}{5}(1-x)^{5/3} + \frac{3}{8}(1-x)^{8/3} + C.
 \end{aligned}$$

§8.3 #18.

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2 + 4x + 5}} &= \int \frac{dx}{\sqrt{(x+2)^2 + 1}} \\ &\text{substitute } x+2 = \tan t, \text{ so } dx = \sec^2 t \, dt \\ &= \int \frac{\sec^2 t \, dt}{\sqrt{\tan^2 t + 1}} \\ &= \int \frac{\sec^2 t}{\sec t} dt \\ &= \int \sec t \, dt \\ &= \ln |\sec t + \tan t| + C \\ &= \ln |\sqrt{\tan^2 t + 1} + \tan t| + C \\ &= \ln |\sqrt{(x+2)^2 + 1} + x + 2| + C \\ &= \ln |\sqrt{x^2 + 4x + 5} + x + 2| + C.\end{aligned}$$

§8.3 #29 (a). Substitute $u = x^2 + 9$, so $du = 2x \, dx$. Then

$$\begin{aligned}\int \frac{x \, dx}{x^2 + 9} &= \frac{1}{2} \int \frac{du}{u} \\ &= \frac{1}{2} \ln |u| + C \\ &= \frac{1}{2} \ln(x^2 + 9) + C\end{aligned}$$

(b). Substitute $x = 3 \tan t$, so $dx = 3 \sec^2 t \, dt$. Then

$$\begin{aligned}\int \frac{x \, dx}{x^2 + 9} &= \int \frac{9 \tan t \sec^2 t \, dt}{9 \tan^2 t + 9} \\ &= \int \frac{\tan t \sec^2 t \, dt}{\sec^2 t} \\ &= \int \tan t \, dt \\ &= \ln |\sec t| + C \\ &= \ln \sqrt{\tan^2 t + 1} + C \\ &= \ln \sqrt{\left(\frac{x}{3}\right)^2 + 1} + C\end{aligned}$$

Now we show that this is the same as the previous answer:

$$\begin{aligned}\ln \sqrt{\left(\frac{x}{3}\right)^2 + 1} + C &= \ln \frac{\sqrt{x^2 + 9}}{3} + C \\ &= \frac{1}{2} \ln(x^2 + 9) - \ln 3 + C.\end{aligned}$$

The extra $\ln 3$ is just a constant, so it can be swept under the $+ C$, so we have shown that the two answers are equal.

§8.1 #68 (extra credit). First we establish the given identity:

$$\begin{aligned}\sec x &= \frac{1}{\cos x} \\ &= \frac{\sin x + 1}{\cos x(1 + \sin x)} \\ &= \frac{\sin x + \sin^2 x + \cos^2 x}{\cos x(1 + \sin x)} \\ &= \frac{\sin x + \sin^2 x + \cos^2 x}{\cos x(1 + \sin x)} \\ &= \frac{\sin x(1 + \sin x) + \cos^2 x}{\cos x(1 + \sin x)} \\ &= \frac{\sin x}{\cos x} + \frac{\cos x}{1 + \sin x}\end{aligned}$$

Now the integral becomes

$$\int \sec x \, dx = \int \frac{\sin x}{\cos x} dx + \int \frac{\cos x}{1 + \sin x} dx.$$

In the first integral, we substitute $u = \cos x$, so $du = -\sin x \, dx$, and in the second, we substitute $v = 1 + \sin x$, so $dv = \cos x \, dx$. Now

$$\begin{aligned}\int \sec x \, dx &= -\int \frac{du}{u} + \int \frac{dv}{v} \\ &= -\ln|u| + \ln|v| + C \\ &= \ln\left|\frac{v}{u}\right| + C \\ &= \ln\left|\frac{1 + \sin x}{\cos x}\right| + C \\ &= \ln\left|\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right| + C \\ &= \ln|\sec x + \tan x| + C\end{aligned}$$