# Solutions to Exam 1

1.

$$\begin{aligned}
\int \tan^5 x \; dx &= \int \tan^3 x(\sec^2 x - 1) \; dx \\
&= \int \tan^3 x \sec^2 x \; dx - \int \tan^3 x \; dx \\
&= \int \tan^3 x \sec^2 x \; dx - \int \tan x(\sec^2 x - 1) \; dx \\
&= \int \tan^3 x \sec^2 x \; dx - \int \tan x \sec^2 x \; dx + \int \tan x \; dx \\
&= \int u^3 \; du - \int u \; du + \int \tan x \; dx \\
&= \frac{1}{4}u^4 - \frac{1}{2}u^2 + \ln|\sec x| + C \\
&= \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + \ln|\sec x| + C
\end{aligned}$$

where we let $u = \tan x$, so $du = \sec^2 x \; dx$.

2. Let $x = 2\sec t$, so $dx = 2\sec t \tan t \; dt$. Then

$$\begin{aligned}
\int \frac{dx}{\sqrt{x^2-4}} &= \int \frac{2\sec t \tan t \; dt}{\sqrt{4\sec^2 t - 4}} \\
&= \int \frac{2\sec t \tan t \; dt}{\sqrt{4(\sec^2 t - 1)}} \\
&= \int \frac{2\sec t \tan t \; dt}{\sqrt{4\tan^2 t}} \\
&= \int \frac{2\sec t \tan t \; dt}{2\tan t} \\
&= \int \sec t \; dt \\
&= \ln|\sec t + \tan t| + C \\
&= \ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| + C \\
&= \ln\left|x + \sqrt{x^2-4}\right| - \ln 2 + C
\end{aligned}$$

where in the second-to-last line we drew a triangle: $\sec t = \frac{x}{2}$, so the hypotenuse is $x$ and the adjacent side is 2, so the opposite side is $\sqrt{x^2-4}$, so $\tan x = \frac{\sqrt{x^2-4}}{2}$.

3. We do partial fractions:
$$\frac{x^2}{(x-3)(x+2)^2} = \frac{A}{x-3} + \frac{B}{x+2} + \frac{C}{(x+2)^2}.$$
Clearing denominators,
$$x^2 = A(x+2)^2 + B(x-3)(x+2) + C(x-3).$$
Plugging in $x=-2$, we get $C=-\frac{4}{5}$. Plugging in $x=3$, we get $A=\frac{9}{25}$. Plugging in $x=0$, we get $0=\frac{9}{25}\cdot 4 - 6B + \frac{4}{5}\cdot 3$, so $B=\frac{16}{25}$. Thus
$$\begin{aligned}\int \frac{x^2}{(x-3)(x+2)^2}\,dx &= \frac{9}{25}\int \frac{1}{x-3}\,dx + \frac{16}{25}\int \frac{1}{x+2}\,dx - \frac{4}{5}\int \frac{1}{(x+2)^2}\,dx \\ &= \frac{9}{25}\ln|x-3| + \frac{16}{25}\ln|x+2| + \frac{4}{5}\cdot\frac{1}{x+2} + C\end{aligned}$$
where for the first term we substituted $u=x-3$ and for the second and third we substituted $u=x+2$.

4. You could let $u=\tan^{-1}x$ and $dv = x\,dx$, but it is slicker to let $x=\tan t$, so $dx=\sec^2 t\,dt$. Observe that $\tan^{-1}1=\pi/4$ and $\tan^{-1}0=0$.
$$\int_0^1 x\tan^{-1}x\,dx = \int_0^{\pi/4}\tan t\cdot t\cdot\sec^2 t\,dt.$$
Now let $u=t$ and $dv=\tan t\sec^2 t\,dt$, so $du=dt$ and $v=\frac{1}{2}\tan^2 t$:
$$\begin{aligned}&\left[\frac{1}{2}t\tan^2 t\right]_0^{\pi/4} - \frac{1}{2}\int_0^{\pi/4}\tan^2 t\,dt \\ &= \frac{1}{2}\cdot\frac{\pi}{4}\cdot 1^2 - 0 - \frac{1}{2}\int_0^{\pi/4}(\sec^2 t - 1)\,dt \\ &= \frac{\pi}{8} - \frac{1}{2}\Big[\tan t - t\Big]_0^{\pi/4} \\ &= \frac{\pi}{8} - \frac{1}{2}\left(1-\frac{\pi}{4}-0\right) \\ &= \frac{\pi}{4} - \frac{1}{2}.\end{aligned}$$

5. First we evaluate the indefinite integral, letting $u=\ln x$ and $dv = x^{-2}\,dx$, so $du = x^{-1}\,dx$ and $v=-x^{-1}$:
$$\int \frac{\ln x\,dx}{x^2} = -x^{-1}\ln x + \int x^{-2}\,dx = -\frac{\ln x}{x} - x^{-1} + C = \frac{-\ln x - 1}{x} + C.$$
Now we observe that
$$\lim_{x\to\infty}\frac{-\ln x - 1}{x} = \lim_{x\to\infty}\frac{-1/x}{1} = 0,$$
so
$$\int_e^\infty \frac{\ln x\,dx}{x^2} = \lim_{a\to\infty}\int_e^a \frac{\ln x\,dx}{x^2} = \lim_{a\to\infty}\left[\frac{-\ln x - 1}{x}\right]_e^a = \lim_{a\to\infty}\frac{-\ln a - 1}{a} - \left(\frac{-\ln e - 1}{e}\right) = 0 + \frac{2}{e}.$$
In particular, the integral converges.