

## Solutions to Exam 2

1. (a) This limit is of the form  $\infty^0$ , so we should take the logarithm and use l'Hôpital's rule. Let  $L = \lim_{n \rightarrow \infty} (1 + 3n)^{1/n}$ . Then

$$\begin{aligned} \ln L &= \ln \left( \lim_{n \rightarrow \infty} (1 + 3n)^{1/n} \right) \\ &= \lim_{n \rightarrow \infty} \ln \left( (1 + 3n)^{1/n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \ln(1 + 3n) \\ &= \lim_{n \rightarrow \infty} \frac{\ln(1 + 3n)}{n}. \end{aligned}$$

Now this is of the form  $\infty/\infty$ , so by l'Hôpital's rule,

$$\ln L = \lim_{n \rightarrow \infty} \frac{\frac{3}{1+3n}}{1} = 0$$

so  $L = e^0 = 1$ .

- (b) This limit is of the form  $\infty - \infty$ , so we should get things over a common denominator and if necessary use l'Hôpital's rule. The trick is to multiply and divide by  $\sqrt{n+2} + \sqrt{n}$ :

$$\begin{aligned} a_n &= \sqrt{n+2} - \sqrt{n} \\ &= (\sqrt{n+2} - \sqrt{n}) \frac{\sqrt{n+2} + \sqrt{n}}{\sqrt{n+2} + \sqrt{n}} \\ &= \frac{(\sqrt{n+2} - \sqrt{n})(\sqrt{n+2} + \sqrt{n})}{\sqrt{n+2} + \sqrt{n}} \\ &= \frac{(\sqrt{n+2})^2 - (\sqrt{n})^2}{\sqrt{n+2} + \sqrt{n}} \\ &= \frac{(n+2) - n}{\sqrt{n+2} + \sqrt{n}} \\ &= \frac{2}{\sqrt{n+2} + \sqrt{n}} \end{aligned}$$

which goes to 0 as  $n \rightarrow \infty$ .

2. (a) This series is positive and decreasing, so it behaves the same as the corresponding integral:

$$\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \int_{\ln 2}^{\infty} \frac{du}{u^2} = -\frac{1}{u} \Big|_{\ln 2}^{\infty} = -0 - \left( -\frac{1}{\ln 2} \right)$$

where we substituted  $u = \ln x$ , so  $du = \frac{dx}{x}$ . The integral converges, so the series converges.

- (b) We know that  $|\sin n| \leq 1$  for all  $n$ , so  $\sin^2 n = |\sin n|^2 \leq 1$ , so

$$\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^{3/2}} \leq \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}.$$

The right-hand side is a  $p$ -series with  $p = 3/2 > 1$ , so it converges, so by the comparison test, the left-hand side converges as well.

3. (a) The terms go to zero:

$$\frac{\sqrt{n}}{n+4} = \frac{\sqrt{n}}{n+4} \frac{1}{\frac{1}{n}} = \frac{\frac{1}{\sqrt{n}}}{1 + \frac{4}{n}} \rightarrow \frac{0}{1+0}$$

so by the alternating series test, the series converges. For absolute convergence, we compare  $\sum \frac{\sqrt{n}}{n+4}$  to  $\sum \frac{\sqrt{n}}{n} = \sum \frac{1}{\sqrt{n}}$  using the limit comparison test:

$$\frac{\frac{1}{\sqrt{n}}}{\frac{\sqrt{n}}{n+4}} = \frac{n+4}{n} = 1 + \frac{4}{n} \rightarrow 1.$$

Since 1 is between 0 and  $\infty$ ,  $\sum \frac{\sqrt{n}}{n+4}$  behaves like  $\sum \frac{1}{\sqrt{n}}$ , which is a  $p$ -series with  $p = 1/2 \leq 1$ , so it diverges. Thus  $\sum \frac{\sqrt{n}}{n+4}$  converges, but not absolutely, so it converges conditionally.

- (b) As  $n \rightarrow \infty$ ,  $\pi/n \rightarrow 0$ , so  $\cos(\pi/n) \rightarrow \cos 0 = 1$ , so the terms do not go to zero, so the series diverges.

4. We look at the derivative:

$$\begin{aligned} \frac{d}{dx} \ln(1+x^2) &= \frac{2x}{1+x^2} \\ &= 2x \frac{1}{1-(-x^2)} \\ &= 2x(1-x^2+x^4-x^6+x^8-\dots) \\ &= 2x-2x^3+2x^5-2x^7+2x^9-\dots \end{aligned}$$

where the third line is valid when  $|-x^2| < 1$ , that is, when  $|x| < \sqrt{1} = 1$ . Now we take the anti-derivative:

$$\begin{aligned} \ln(1+x^2) &= \int (2x-2x^3+2x^5-2x^7+2x^9-\dots) dx \\ &= x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 - \frac{1}{4}x^8 + \frac{1}{5}x^{10} - \dots + C. \end{aligned}$$

Plugging in  $x = 0$ , we have  $C = \ln 1 = 0$ , so our final answer is

$$\ln(1+x^2) = x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 - \frac{1}{4}x^8 + \frac{1}{5}x^{10} - \dots$$

5. The auxiliary equation is  $r^2 + 2r + 1 = 0$ , so  $(r+1)^2 = 0$ , so  $r = -1$ , so the basic solutions to the homogeneous equation are  $e^{-x}$  and  $xe^{-x}$ . For a particular solution to the inhomogeneous equation, we guess  $y_p = A \sin x + B \cos x$ , so  $y'_p = A \cos x - B \sin x$ , so  $y''_p = -A \sin x - B \cos x$ . Now

$$\begin{aligned} \cos x &= y''_p + 2y'_p + y_p \\ &= (-A \sin x - B \cos x) + 2(A \cos x - B \sin x) + (A \sin x + B \cos x) \\ &= 2A \cos x - 2B \sin x \end{aligned}$$

so  $2A = 1$  and  $2B = 0$ , so  $A = 1/2$  and  $B = 0$ , so  $y_p = \frac{1}{2} \sin x$ . Thus the general solution to the inhomogeneous equation is

$$y = C_1 e^{-x} + C_2 x e^{-x} + \frac{1}{2} \sin x.$$

When  $x = 0$ ,  $y = 0$ , so  $0 = C_1 + 0 + 0$ , so  $C_1 = 0$ , so  $y = C_2 x e^{-x} + \frac{1}{2} \sin x$ . We differentiate this:  $y' = C_2 e^{-x} - C_2 x e^{-x} + \frac{1}{2} \cos x$ . When  $x = 0$ ,  $y' = 0$ , so  $0 = C_2 - 0 + \frac{1}{2}$ , so  $C_2 = -1/2$ , so our final answer is

$$y = -\frac{1}{2} x e^{-x} + \frac{1}{2} \sin x.$$