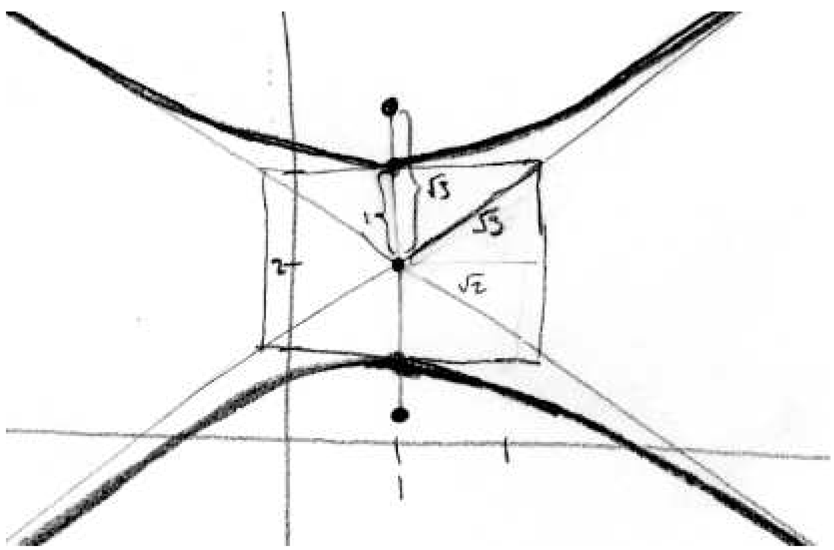
Solutions to Exam 3

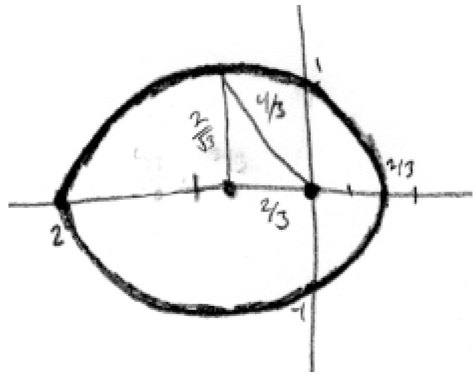
1. We complete the square:

$$
\begin{aligned}
2(y^2 - 4y) - (x^2 - 2x) &= -5 \\
2(y^2 - 4y + 4) - (x^2 - 2x + 1) &= -5 + 2 \cdot 4 - 1 \\
2(y-2)^2 - (x-1)^2 &= 2 \\
(y-2)^2 - \frac{(x-1)^2}{2} &= 1.
\end{aligned}
$$

This is a vertical hyperbola with center $(1, 2)$. The distance from the center to the vertex is 1, so the vertices are at $(1, 2 \pm 1)$, that is, at $(1, 1)$ and $(1, 3)$. The distance from the center to the focus is $\sqrt{1+2} = \sqrt{3}$. The eccentricity is the distance from the center to the focus divided by the distance from the center to the vertex, which is $\sqrt{3}$.

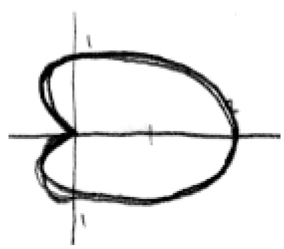


2. (a) The denominator $2 + \cos\theta$ is never zero, so the conic is an ellipse.
   (b) When $\theta = 0$, $r = 2/3$. When $\theta = \pi/2$, $r = 1$. When $\theta = \pi$, $r = 2$. When $\theta = 3\pi/2$, $r = 1$.

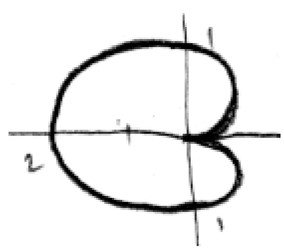


   (c) We see that the vertices on the major axis are $(2/3, 0)$ and $(-2, 0)$ in rectangular coordinates. The center is halfway between these, at $(-2/3, 0)$. One focus is at the origin, so the distance from the center to the focus is $2/3$, and the distance from the center to the vertex is $4/3$, so the eccentricity is $\frac{2/3}{4/3} = 1/2$. The distance from the center to the vertex on the major axis is the same as the distance from the focus to the vertex on the minor axis, so the distance from the center to the vertex on the minor axis is $\sqrt{(4/3)^2 - (2/3)^2} = 2/\sqrt{3} \approx 1.155$. Thus the vertices on the minor axis are at $(-2/3, \pm 2/\sqrt{3})$.

3. We begin with a sketch. For $r = 1 + \cos\theta$, when $\theta = 0$, $r = 2$; when $\theta = \pi/2$, $r = 1$; when $\theta = \pi$, $r = 0$; and when $\theta = 3\pi/2$, $r = 1$. Thus the picture is



For $r = 1 - \cos\theta$, when $\theta = 0$, $r = 0$; when $\theta = \pi/2$, $r = 1$; when $\theta = \pi$, $r = 2$; and when $\theta = 3\pi/2$, $r = 1$. Thus the picture is



so we expect three intersections. We solve:

$$\begin{aligned} 1 + \cos\theta &= 1 - \cos\theta \\ 2\cos\theta &= 0 \\ \cos\theta &= 0 \\ \theta &= \pi/2 \text{ or } 3\pi/2. \end{aligned}$$

Thus we have found two intersections, one at $\theta = \pi/2$, $r = 1$ and the other at $\theta = 3\pi/2$, $r = 1$. In rectangular coordinates, these are $(0, \pm 1)$. But there is also an intersection at the origin $(0, 0)$.

4. If we write $\mathbf{r}(t) = \langle e^t, e^{2t} \rangle$ then $\mathbf{r}'(t) = \langle e^t, 2e^{2t} \rangle$, so $\mathbf{r}'(0) = \langle 1, 2 \rangle$, so $|\mathbf{r}'(0)| = \sqrt{1^2 + 2^2} = \sqrt{5}$, so

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle.$$

If we write $x(t) = e^t$ and $y(t) = e^{2t}$ then $x'(t) = e^t$ and $y'(t) = 2e^{2t}$, so $x''(t) = e^t$ and $y''(t) = 4e^{2t}$, so $x'(0) = 1$, $y'(0) = 2$, $x''(0) = 1$, and $y''(0) = 4$. Thus

$$\kappa = \frac{|x''y' - y''x'|}{((x')^2 + (y')^2)^{3/2}} = \frac{2}{5^{3/2}} = \frac{2}{5\sqrt{5}} \approx 0.179.$$

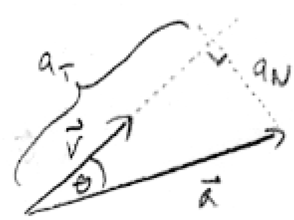
5. We begin with $\mathbf{r}(t) = \langle 2t^3, 3t^2 \rangle$.

   (a) $\mathbf{r}'(t) = \langle 6t^2, 6t \rangle$, so $\mathbf{r}'(1) = \langle 6, 6 \rangle$.

   (b) $|\langle 6, 6 \rangle| = 6\sqrt{2}$.

   (c) $\mathbf{r}''(t) = \langle 12t, 6 \rangle$, so $\mathbf{r}''(1) = \langle 12, 6 \rangle$.

   (d) We draw a triangle:



   Thus

$$a_T = |\mathbf{a}| \cos\theta = |\mathbf{a}| \frac{\mathbf{a} \cdot \mathbf{v}}{|\mathbf{a}||\mathbf{v}|} = \frac{\mathbf{a} \cdot \mathbf{v}}{|\mathbf{v}|} = \frac{12 \cdot 6 + 6 \cdot 6}{6\sqrt{2}} = 9\sqrt{2}.$$

   (e) ${a_T}^2 + {a_N}^2 = |\mathbf{a}|^2$, so $(9\sqrt{2})^2 + {a_N}^2 = 12^2 + 6^2$, so $a_N = \sqrt{18} = 3\sqrt{2}$.