Final Exam Solutions

1. First we evaluate the definite integral. We complete the square and let $x-2 = 2\sin t$, so $dx = 2\cos t\ dt$ and $t = \sin^{-1}(x/2-1)$:

$$\begin{aligned}\int \frac{dx}{\sqrt{4x-x^2}} &= \int \frac{dx}{\sqrt{4-4+4x-x^2}} = \int \frac{dx}{\sqrt{4-(x^2-4x+4)}} = \int \frac{dx}{\sqrt{4-(x-2)^2}} \\ &= \int \frac{2\cos t\ dt}{\sqrt{4-4\sin^2 t}} = \int \frac{2\cos t\ dt}{\sqrt{4\cos^2 t}} = \int \frac{2\cos t\ dt}{2\cos t} = \int dt = t + C = \sin^{-1}(x/2-1) + C.\end{aligned}$$

Now
$$\int_0^1 \frac{dx}{4x-x^2} = \sin^{-1}(x/2-1)\Big|_0^1 = \sin^{-1}(-1/2) - \sin^{-1}(-1) = -\frac{\pi}{6} - \left(-\frac{\pi}{2}\right) = \frac{\pi}{3}.$$

2. Let $u = \tan^{-1} x$ and $dv = x\ dx$, so $du = dx/(x^2+1)$ and $v = x^2/2$:

$$\begin{aligned}\int_0^1 x\tan^{-1} x\ dx &= \frac{1}{2}x^2\tan^{-1} x\Big|_0^1 - \frac{1}{2}\int_0^1 \frac{x^2}{x^2+1}\ dx = \frac{\pi}{8} - 0 - \frac{1}{2}\int_0^1 \frac{x^2+1-1}{x^2+1}\ dx \\ &= \frac{\pi}{8} - \frac{1}{2}\int_0^1 \left(1 - \frac{1}{x^2+1}\right)\ dx = \frac{\pi}{8} - \frac{1}{2}\Big[x - \tan^{-1} x\Big]_0^1 = \frac{\pi}{8} - \frac{1}{2}\left(1 - \frac{\pi}{4} - 0\right) = \frac{\pi}{4} - \frac{1}{2}.\end{aligned}$$

3. We wish to find A, B, and C such that

$$\frac{1}{x(x^2-2x-3)} = \frac{1}{x(x-3)(x+1)} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x+1}$$

so
$$1 = A(x-3)(x+1) + Bx(x+1) + Cx(x-3).$$

Letting $x=0$, we get $A = -1/3$. Letting $x=3$, we get $B = 1/12$. Letting $x=-1$, we get $C = 1/4$. Thus

$$\begin{aligned}\int \frac{dx}{x(x^2-2x-3)} &= -\frac{1}{3}\int \frac{dx}{x} + \frac{1}{12}\int \frac{dx}{x-3} + \frac{1}{4}\int \frac{dx}{x+1} \\ &= -\frac{1}{3}\ln|x| + \frac{1}{12}\ln|x-3| + \frac{1}{4}\ln|x+1| + C.\end{aligned}$$

4. First we make the substitution $u = x-1$, so $du = dx$; when $x=0$, $u=-1$, and when $x=2$, $u=1$:

$$\int_0^2 \frac{x\ dx}{(x-1)^3} = \int_{-1}^1 \frac{(u+1)\ du}{u^3} = \int_{-1}^1 \left(\frac{1}{u^2} + \frac{1}{u^3}\right) du.$$

But
$$\int \left(\frac{1}{u^2} - \frac{1}{u^3}\right) du = -\frac{1}{u} - \frac{1}{2u^2} + C$$

which blows up at $u = 0$, so the integral diverges. If we want to be very careful,

$$\int_{-1}^{0}\left(\frac{1}{u^2}+\frac{1}{u^3}\right)du = \lim_{b\to 0^-}\int_{-1}^{b}\left(\frac{1}{u^2}+\frac{1}{u^3}\right)du = \lim_{b\to 0^-}\left[-\frac{1}{u}-\frac{1}{2u^2}\right]_{-1}^{b}$$

$$= \lim_{b\to 0^-}\left[-\frac{1}{b}-\frac{1}{2b^2}-\left(1-\frac{1}{2}\right)\right] = \lim_{b\to 0^-}\left(-\frac{2b+1}{2b^2}-\frac{1}{2}\right) = -\infty$$

and if the integral from -1 to 0 diverges then the integral from -1 to 1 does as well.

5. (a) Observe that the n^{th} partial sum is

$$\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right) = 1-\frac{1}{n+1}$$

and that as $n \to \infty$, this approaches 1, so the series converges (in fact the sum is 1). Alternatively,

$$\sum\left(\frac{1}{k}-\frac{1}{k+1}\right) = \sum\frac{1}{k(k+1)} = \sum\frac{1}{k^2+k}$$

and we can compare this to $\sum \frac{1}{k^2}$ using the limit comparison test:

$$\frac{\frac{1}{k^2+k}}{\frac{1}{k^2}} = \frac{k^2}{k^2+k} = \frac{1}{1+\frac{1}{k}} \to 1$$

and 1 is between 0 and ∞, so the series behave the same. But $\sum \frac{1}{k^2}$ is a p-series with $p = 2 > 1$, so it converges.

(b) The sequence $\frac{1}{k(\ln k)^2}$ is positive and decreasing, so by the integral test, the series $\sum \frac{1}{k(\ln k)^2}$ behaves like the integral

$$\int_{2}^{\infty}\frac{dx}{x(\ln x)^2} = \int_{\ln 2}^{\infty}\frac{du}{u^2} = -\frac{1}{u}\bigg|_{\ln 2}^{\infty} = 0-\left(-\frac{1}{\ln 2}\right)$$

where we let $u = \ln x$, so $du = \frac{dx}{x}$. The integral converges, so the series converges, although the sum is probably not $1/\ln 2$.

6. (a) The terms do not go to zero:

$$\lim_{n\to\infty}\frac{2n}{3n-4} = \lim_{n\to\infty}\frac{2}{3-\frac{4}{n}} = \frac{2}{3}.$$

Thus the series diverges.

(b) We use the absolute ratio test:

$$\frac{\frac{2^{n+1}}{(n+1)3^{n+2}}}{\frac{2^n}{n3^{n+1}}} = \frac{2^{n+1}}{2^n}\cdot\frac{n}{n+1}\cdot\frac{3^{n+1}}{3^{n+2}} = 2\cdot\frac{1}{1+\frac{1}{n}}\cdot\frac{1}{3} \to \frac{2}{3}$$

and $2/3 < 1$, so the series converges absolutely.

7.

$$\frac{1}{16+x^2} = \frac{1}{16}\cdot\frac{1}{1+(x^2/16)} = \frac{1}{16}\cdot\frac{1}{1-(-x^2/16)}$$

$$= \frac{1}{16}\left[1+\left(-\frac{x^2}{16}\right)+\left(-\frac{x^2}{16}\right)^2+\left(-\frac{x^2}{16}\right)^3+\cdots\right]$$

$$= \frac{1}{16}-\frac{x^2}{16^2}+\frac{x^4}{16^3}-\frac{x^6}{16^4}+\frac{x^8}{16^5}-\cdots.$$

This converges when $|-x^2/16| < 1$, so $x^2/16 < 1$, so $x^2 < 16$, so $|x| < 4$.

8. The auxiliary equation is $r^2 - 1 = 0$, so $r = \pm 1$, so the basic solutions to the homogeneous equation are $y = e^x$ and $y = e^{-x}$. For a particular solution to the inhomogeneous equation, we guess $y = A\sin x + B\cos x + Cx + D$, so $y'' = -A\sin x - B\cos x$, so $y'' - y = -2A\sin x - 2B\cos x - Cx - D$, so $A = -1/2$, $B = 0$, $C = -1$, and $D = 0$. Thus our general solution is

$$y = C_1 e^x + C_2 e^{-x} - \frac{1}{2}\sin x - x.$$

Observe that $y' = C_1 e^x - C_2 e^{-x} - \frac{1}{2}\cos x - 1$. When $x = 0$, $y = 0$, so $0 = C_1 + C_2 - 0 - 0$, so $C_2 = -C_1$; and $y' = 0$, so $0 = C_1 - C_2 - 3/2$, so $2C_1 = 3/2$, so $C_1 = 3/4$ and $C_2 = -3/4$. Thus our solution to the initial value problem is

$$y = \frac{3}{4}e^x - \frac{3}{4}e^{-x} - \frac{1}{2}\sin x - x.$$

9. This is a cardioid with the dimple on the left. When $\theta = 0$, $r = 4$. When $\theta = \pi/2$, $r = 2$. When $\theta = \pi$, $r = 0$. When $\theta = 3\pi/2$, $r = 2$. The area is

$$\frac{1}{2}\int_0^{2\pi} (2 + 2\cos\theta)^2 \; d\theta$$

but we can find the area of half and double it:

$$\begin{aligned}\int_0^{\pi} (2 + 2\cos\theta)^2 \; d\theta = \int_0^{\pi} (4 + 8\cos\theta + 4\cos^2\theta)d\theta = \int_0^{\pi} [4 + 8\cos\theta + 2(1 + \cos 2\theta)]d\theta \\ = \int_0^{\pi} [(+8\cos\theta + 2\cos 2\theta)d\theta = \Big[6\theta + 8\sin\theta + \sin 2\theta\Big]_0^{\pi} = 6\pi\end{aligned}$$

10. We take two derivatives:

$$\begin{aligned}\mathbf{r} &= \langle \cosh t, \sinh t, t\rangle \\ \mathbf{r}' &= \langle \sinh t, \cosh t, 1\rangle \\ \mathbf{r}'' &= \langle \cosh t, \sinh t, 0\rangle.\end{aligned}$$

When $t = 0$,

$$\begin{aligned}\mathbf{r}' &= \langle 0, 1, 1\rangle \\ |\mathbf{r}'| &= \sqrt{2} \\ \mathbf{r}'' &= \langle 1, 0, 0\rangle \\ \mathbf{r}' \times \mathbf{r}'' &= \langle 0, 1, -1\rangle \\ |\mathbf{r}' \times \mathbf{r}''| &= \sqrt{2}\end{aligned}$$

so

$$\begin{aligned}\mathbf{T} &= \frac{\mathbf{r}'}{|\mathbf{r}'|} = \left\langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle \\ \mathbf{B} &= \frac{\mathbf{r}' \times \mathbf{r}''}{|\mathbf{r}' \times \mathbf{r}''|} = \left\langle 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\rangle \\ \mathbf{N} &= \mathbf{B} \times \mathbf{T} = \langle 1, 0, 0\rangle \\ \kappa &= \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3} = \frac{1}{2}.\end{aligned}$$