Solutions to Practice Exam 1

1.
$$\int \tan x \sec^{-2} x \; dx = \int \frac{\sin x}{\cos x} \cos^2 x \; dx = \int \sin x \cos x \; dx = \int u \; du = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2 x + C$$
where we substituted $u = \sin x$, so $du = \cos x \; dx$. Alternatively, we could write
$$\int \tan x \sec^{-2} x \; dx = \int \sec^{-3} x \sec x \tan x \; dx$$
and substitute $u = \sec x$, so $du = \sec x \tan x \; dx$.

2.
$$\int \cos 2x \cos x \; dx = \frac{1}{2}\int (\cos 3x + \cos x) \; dx = \frac{1}{6}\sin 3x + \frac{1}{2}\sin x + C$$
where we have evaluated $\int \cos 3x \; dx$ by substituting $u = 3x$. Alternatively,
$$\int \cos 2x \cos x \; dx = \int (1 - 2\sin^2 x)\cos x \; dx = \int (1 - 2u^2) \; du = u - \frac{2}{3}u^3 + C = \sin x - \frac{2}{3}\sin^3 x + C.$$
This is agrees with the previous answer, since
$$\begin{aligned} \sin 3x = \sin(2x + x) = \sin x \cos 2x + \sin 2x \cos x &= \sin x(1 - 2\sin^2 x) + 2(\sin x \cos x)\cos x \\ &= \sin x - 2\sin^3 x + 2\sin x(1 - \sin^2 x) = 3\sin x - 4\sin^3 x. \end{aligned}$$

3. Complete the square:
$$-x^2 - 2x = -(x^2 + 2x) = -(x^2 + 2x + 1 - 1) = -[(x + 1)^2 - 1] = 1 - (x + 1)^2.$$
Thus we should substitute $x + 1 = \sin t$, so $dx = \cos t \; dt$. If we do this,
$$\sqrt{-x^2 - 2x} = \sqrt{1 - (x + 1)^2} = \sqrt{1 - \sin^2 t} = \sqrt{\cos^2 t} = \cos t,$$
so
$$\begin{aligned} \int \frac{2x\sqrt{-x^2 - 2x}}{x + 1} \; dx &= \int \frac{2(\sin t - 1)\cos t}{\sin t} \cos t \; dt \\ &= \int \frac{2\sin t \cos^2 t - 2\cos^2 t}{\sin t} \; dt \\ &= \int 2\cos^2 t \; dt - 2\int \frac{1 - \sin^2 t}{\sin t} \; dt \\ &= \int (1 + \cos 2t) \; dt - 2\int (\csc t - \sin t) \; dt \\ &= t + \frac{1}{2}\sin 2t - 2\ln|\csc t - \cot t| - 2\cos t + C \\ &= t + \sin t \cos t - 2\ln\left|\frac{1}{\sin t} - \frac{\cos t}{\sin t}\right| - 2\cos t + C \\ &= t + (\sin t - 2)\cos t - 2\ln\left|\frac{1 - \cos t}{\sin t}\right| + C. \end{aligned}$$

Since $\sin t = x+1$, $t = \sin^{-1}(x+1)$, and from our earlier calculation, we know that $\cos t = \sqrt{-x^2-2x}$, so we end up with

$$\sin^{-1}(x+1) + (x-1)\sqrt{-x^2-2x} - 2\ln\left|\frac{1-\sqrt{-x^2-2x}}{x+1}\right| + C.$$

4. For the rationalizing substitution, we could let $u = 3x-2$, so $x = \frac{1}{3}(u+2)$, so $dx = \frac{1}{3}\,du$. Instead, we let $u = (3x-2)^{1/5}$, so $u^5 = 3x-2$, so $x = \frac{1}{3}(u^5+2)$, so $dx = \frac{5}{3}u^4\,du$, so

$$\begin{aligned}
\int x^2(3x-2)^{4/5}\,dx &= \int \left(\frac{1}{3}(u^5+2)\right)^2 u^4 \frac{5}{3}u^4\,du \\
&= \frac{5}{27}\int (u^{10}+4u^5+4)u^8\,du \\
&= \frac{5}{27}\int (u^{18}+4u^{13}+4u^8)\,du \\
&= \frac{5}{27}\left(\frac{1}{19}u^{19}+\frac{4}{14}u^{14}+\frac{4}{9}u^9\right)+C \\
&= \frac{5}{513}(3x-2)^{19/5}+\frac{20}{378}(3x-2)^{13/5}+\frac{20}{243}(3x-2)^{9/5}+C.
\end{aligned}$$

By parts, first let $u = x^2$ and $dv = (3x-2)^{4/5}\,dx$, so $du = 2x\,dx$ and $v = \frac{1}{3}\frac{5}{9}(3x-2)^{9/5}$:

$$\int x^2(3x-2)^{4/5}\,dx = \frac{5}{27}x^2(3x-2)^{9/5} - \frac{10}{27}\int x(3x-2)^{9/5}\,dx.$$

Next, let $u = x$ and $dv = (3x-2)^{9/5}\,dx$, so $du = dx$ and $v = \frac{1}{3}\frac{5}{14}(3x-2)^{14/5}$:

$$\begin{aligned}
&= \frac{5}{27}x^2(3x-2)^{9/5} - \frac{10}{27}\left(\frac{5}{42}x(3x-2)^{14/5} - \frac{5}{42}\int (3x-2)^{14/5}\,dx\right) \\
&= \frac{5}{27}x^2(3x-2)^{9/5} - \frac{25}{576}x(3x-2)^{14/5} + \frac{25}{576}\cdot\frac{1}{3}\cdot\frac{5}{19}(3x-2)^{19/5}+C
\end{aligned}$$

5. First, we do partial fractions:

$$\begin{aligned}
\frac{1}{x^2(x-1)} &= \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1} \\
1 &= Ax(x-1)+B(x-1)+Cx^2
\end{aligned}$$

Letting $x=0$, we see that $B=-1$. Letting $x=1$, we see that $C=1$. Letting $x=2$, we see that $1 = 2A+B+4C = 2A-1+4$, so $A=-1$. Thus

$$\begin{aligned}
\int_1^\infty \frac{dx}{x^3-x^2} &= \int_1^\infty \left(\frac{-1}{x}+\frac{-1}{x^2}+\frac{1}{x-1}\right)dx \\
&= \left[-\ln|x|+\frac{1}{x}+\ln|x-1|\right]_1^\infty \\
&= \left[\frac{1}{x}+\ln\left|\frac{x-1}{x}\right|\right]_1^\infty \\
&= \left[\frac{1}{x}+\ln\left|1-\frac{1}{x}\right|\right]_1^\infty
\end{aligned}$$

If we "plug in" ∞, we get $0+\ln(1+0)=0$, so there is no trouble there, but if we plug in 1, we get $1-\ln 0 = 1-(-\infty)=\infty$, so the integral diverges.

6.
$$\int \frac{x^3+1}{x^3-x^2}\ dx = \int \left(\frac{x^3}{x^3-x^2} + \frac{1}{x^3-x^2}\right)\ dx$$

For the first term,
$$\int \frac{x^3}{x^3-x^2}\ dx = \int \frac{x}{x-1}\ dx = \int \frac{x-1+1}{x-1}\ dx = \int \left(1+\frac{1}{x-1}\right) dx = x + \ln|x-1| + C$$

and we evaluated $\int \frac{1}{x^3-x^2}\ dx$ in the previous problem, so the answer is
$$x + \frac{1}{x} + 2\ln|x-1| - \ln|x| + C.$$

7. In problem 3 we saw that
$$\int \frac{2x\sqrt{-x^2-2x}}{x+1}\ dx = \sin^{-1}(x+1) + (x-1)\sqrt{-x^2-2x} - 2\ln\left|\frac{1-\sqrt{-x^2-2x}}{x+1}\right| + C$$

so if the right-hand side is continuous on the interval $[-2, 0]$ we can write
$$\int_{-2}^{0} \frac{2x\sqrt{-x^2-2x}}{x+1}\ dx = \left[\sin^{-1}(x+1) + (x-1)\sqrt{-x^2-2x} - 2\ln\left|\frac{1-\sqrt{-x^2-2x}}{x+1}\right|\right]_{-2}^{0} = \pi.$$

But the logarithmic term may not be finite at $x = -1$, so we must evaluate
$$\lim_{x\to-1} \ln\left|\frac{1-\sqrt{-x^2-2x}}{x+1}\right| = \ln\left|\lim_{x\to-1} \frac{1-\sqrt{-x^2-2x}}{x+1}\right| = \ln\left|\lim_{x\to-1} \frac{\frac{x+1}{\sqrt{-x^2-2x}}}{1}\right| = \ln 0 = -\infty.$$

Thus the integral diverges.