Practice Exam 2

1. First we take four derivatives:

$$\begin{aligned}
f(x) &= \ln(\cos x)\\
f'(x) &= \frac{-\sin x}{\cos x} = -\tan x\\
f''(x) &= -\sec^2 x\\
f'''(x) &= -2\sec x\cdot\sec x\tan x = -2\sec^2 x\tan x\\
f^{(4)}(x) &= -4\sec x\cdot\sec x\tan x\cdot\tan x + -2\sec^2 x\cdot\sec^2 x\\
&= -6\sec^4 x + 4\sec^2 x
\end{aligned}$$

   (a) We plug in 0: $f(0) = \ln 1 = 0$, $f'(0) = 0$, $f''(0) = -1$, $f'''(0) = 0$, and $f^{(4)}(0) = -2$, so

$$\begin{aligned}
f(x) &= \frac{-1}{2!}x^2 + \frac{-2}{4!}x^4 + \cdots\\
&= -\frac{x^2}{2} - \frac{x^4}{12} + \cdots .
\end{aligned}$$

   (b) We plug in $\frac{\pi}{4}$. Observe that $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$, so $\tan\frac{\pi}{4} = 1$ and $\sec\frac{\pi}{4} = \sqrt{2}$. Thus $f(\frac{\pi}{4}) = \ln\frac{1}{\sqrt{2}} = -\frac{1}{2}\ln 2$, $f'(\frac{\pi}{4}) = -1$, $f''(\frac{\pi}{4}) = -2$, $f'''(\frac{\pi}{4}) = -4$, and $f^{(4)}(\frac{\pi}{4}) = -16$, so

$$\begin{aligned}
f(x) &= -\frac{1}{2}\ln 2 + \frac{-1}{1!}\left(x-\frac{\pi}{4}\right) + \frac{-2}{2!}\left(x-\frac{\pi}{4}\right)^2 + \frac{-4}{3!}\left(x-\frac{\pi}{4}\right)^3 + \frac{-16}{4!}\left(x-\frac{\pi}{4}\right)^4 + \cdots\\
&= -\frac{1}{2}\ln 2 - \left(x-\frac{\pi}{4}\right) - \left(x-\frac{\pi}{4}\right)^2 - \frac{2}{3}\left(x-\frac{\pi}{4}\right)^3 - \frac{2}{3}\left(x-\frac{\pi}{4}\right)^4 + \cdots
\end{aligned}$$

2. Observe that this can be written as $\displaystyle\sum_{n=1}^{\infty}\frac{3^n x^{2n}}{(n+1)^2}$. Now

$$\left|\frac{\frac{3^{n+1}x^{2n+2}}{(n+2)^2}}{\frac{3^n x^{2n}}{(n+1)^2}}\right| = \left|\frac{3^{n+1}}{3^n}\frac{x^{2n+2}}{x^{2n}}\frac{(n+1)^2}{(n+2)^2}\right| = \left|3x^2\left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right)^2\right| \to 3|x|^2$$

   so the series converges if $3|x|^2 < 1$, that is, if $|x| < \frac{1}{\sqrt{3}}$, and diverges if $|x| > \frac{1}{\sqrt{3}}$.

   If $x = \frac{1}{\sqrt{3}}$, the series is

$$\sum_{n=0}^{\infty}\frac{1}{(n+1)^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

   which converges since $2 > 1$. If $x = -\frac{1}{\sqrt{3}}$, the series is $\sum\frac{(-1)^n}{(n+1)^2}$ which converges absolutely. Thus the convergence set is the closed interval $[-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}]$.

3.

$$\begin{aligned}\frac{d}{dx}\tan^{-1}x &= \frac{1}{1+x^2}\\ &= \frac{1}{1-(-x^2)}\\ &= 1-x^2+x^4-x^6+\cdots\\ \tan^{-1}x &= x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots+C.\end{aligned}$$

Putting $x=0$, we see that $C=0$. Thus

$$x^2\tan^{-1}x = x^3-\frac{x^5}{3}+\frac{x^7}{5}-\frac{x^9}{7}+\cdots.$$

4. First we solve the homogeneous equation. The auxiliary equation is $r^2+2r-15=0$, so $r=3$ or $r=-5$, so the basic solutions are $y=e^{3x}$ and $y=e^{-5x}$.

   (a) We would guess $y_p=Ae^{3x}$, but this is already a homogeneous solution, so we guess $y_p=Axe^{3x}$. Then $y_p'=Ae^{3x}+3Axe^{3x}=Ae^{3x}(1+3x)$ and $y_p''=3Ae^{3x}(1+3x)+3Ae^{3x}=Ae^{3x}(6+9x)$.

   $$\begin{aligned}e^{3x} &= y_p''+2y_p'-15y_p\\ &= Ae^{3x}(6+9x)+2Ae^{3x}(1+3x)-15Axe^{3x}\\ &= 8Ae^{3x}\end{aligned}$$

   so $A=\frac{1}{8}$, so the general solution is

   $$y=C_1e^{3x}+C_2e^{-5x}+\frac{1}{8}xe^{3x}.$$

   (b) The basic solutions to the homogeneous equation are $u_1=e^{3x}$ and $u_2=e^{-5x}$, so $u_1'=3e^{3x}$ and $u_2'=-5e^{-5x}$. We wish to find v_1 and v_2 such that

   $$\begin{aligned}e^{3x}v_1'+e^{-5x}v_2'&=0\\ 3e^{3x}v_1'-5e^{5x}v_2'&=e^{3x}.\end{aligned}$$

   Multiplying the first equation by 5 and adding it to the second, we get $8e^{3x}v_1'=e^{3x}$, so $v_1'=\frac{1}{8}$. Substituting this into the first equation, we get $\frac{1}{8}e^{3x}+e^{-5x}v_2'=0$, so $e^{-5x}v_2'=-\frac{1}{8}e^{3x}$, so $v_2'=-\frac{1}{8}e^{8x}$. Thus $v_1=\frac{1}{8}x$ and $v_2=\frac{1}{64}e^{8x}$, so

   $$y_p=u_1v_1+u_2v_2=\frac{1}{8}xe^{3x}+\frac{1}{64}e^{8x}e^{-5x}=\frac{1}{8}xe^{3x}+\frac{1}{64}e^{3x}.$$

   The general solution is

   $$y=C_1e^{3x}+C_2e^{-5x}+\frac{1}{8}xe^{3x}+\frac{1}{64}e^{3x}=\left(C_1+\frac{1}{64}\right)e^{3x}+C_2e^{-5x}+\frac{1}{8}xe^{3x}$$

   which is just our answer to part (a) with a different choice of C_1.