1. The region S is the triangle bounded by the lines $x = 0$, $y = 1$, and $y = x$. The indefinite integral $\int \sin(y^2)\ dy$ cannot be done, so we make y the outside integral and x the inside integral:

$$\begin{aligned} \int_0^1 \int_0^y \sin(y^2)\ dx\ dy &= \int_0^1 y \sin(y^2)\ dy \\ &= \int_0^1 \tfrac{1}{2} \sin(u)\ du \\ &= -\tfrac{1}{2}\cos(u)\Big|_0^1 \\ &= \tfrac{1}{2}(1 - \cos 1) \end{aligned}$$

where in the second line we substituted $u = y^2$.

2. This is §15.9 #11. Let the corner of the box in the first octant be at (x, y, z); it is enough to maximize the volume $f(x, y, z) = xyz$ in the first octant and then multiply by 8. The constraint is $g(x, y, z) = x^2 + \frac{1}{4}y^2 + \frac{1}{9}z^2 = 3$. Compute the gradients:

$$\nabla f(x, y, z) = (yz, xz, xy) \qquad \nabla g(x, y, z) = (2x, \tfrac{2}{4}y, \tfrac{2}{9}z).$$

We want to solve $\nabla f = \lambda \nabla g$. Together with the constraint, this gives four equations:

$$yz = 2\lambda x \tag{1a}$$
$$xz = \tfrac{2}{4}\lambda y \tag{1b}$$
$$xy = \tfrac{2}{9}\lambda z \tag{1c}$$
$$x^2 + \tfrac{1}{4}y^2 + \tfrac{1}{9}z^2 = 3. \tag{1d}$$

In equation (1a), multiply through by x, in (1b) by y, and in (1c) by z:

$$xyz = 2\lambda x^2 \tag{2a}$$
$$xyz = \tfrac{2}{4}\lambda y^2 \tag{2b}$$
$$xyz = \tfrac{2}{9}\lambda z^2. \tag{2c}$$

Since x, y, and z are the dimensions of a box, none of them equals zero, so from (2a) we see that $\lambda \neq 0$, so we can divide by it. In (2a), (2b), and (2c), divide through by 2λ and substitute into (1d):

$$\frac{xyz}{2\lambda} + \frac{xyz}{2\lambda} + \frac{xyz}{2\lambda} = 3$$

so $xyz = 2\lambda$. Substitute this into (2a), (2b), and (2c):

$$\begin{aligned} 2\lambda &= 2\lambda x^2 \\ 2\lambda &= \tfrac{2}{4}\lambda y^2 \\ 2\lambda &= \tfrac{2}{9}\lambda z^2. \end{aligned}$$

Divide through by 2λ and solve: $x = 1$, $y = 2$, $z = 3$ (choose the positive square root since we assumed that (x, y, z) was in the first octant). Thus the volume of the whole box is

$$8xyz = 48.$$

3. This is §16.6 Example 3. Begin by writing the surface as $z = \sqrt{R^2 - x^2 - y^2}$. Find the partial derivatives:

$$\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{R^2 - x^2 - y^2}}$$
$$\frac{\partial z}{\partial y} = \frac{-y}{\sqrt{R^2 - x^2 - y^2}}$$

The integrand will be

$$\begin{aligned}\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} &= \sqrt{\frac{x^2}{R^2 - x^2 - y^2} + \frac{y^2}{R^2 - x^2 - y^2} + \frac{R^2 - x^2 - y^2}{R^2 - x^2 - y^2}} \\ &= \sqrt{\frac{R^2}{R^2 - x^2 - y^2}} \\ &= \frac{R}{\sqrt{R^2 - x^2 - y^2}}\end{aligned}$$

Following the hint, we work in polar coordinates, so the integrand becomes $\frac{R}{\sqrt{R^2 - r^2}}$. To find the region over which we will integrate, we project the surface down onto the xy-plane to get an annulus (donut) $\sqrt{R^2 - A^2} \leq r \leq \sqrt{R^2 - B^2}$. The surface goes all the way around, so θ runs from 0 to 2π.

$$\begin{aligned}&\int_0^{2\pi} \int_{\sqrt{R^2 - A^2}}^{\sqrt{R^2 - B^2}} \frac{R}{\sqrt{R^2 - r^2}} r \, dr \, d\theta \\ &= 2\pi R \int_{\sqrt{R^2 - A^2}}^{\sqrt{R^2 - B^2}} (R^2 - r^2)^{-1/2} r \, dr \\ &= -\pi R \int_{A^2}^{B^2} u^{-1/2} \, du \\ &= -2\pi R u^{1/2} \Big|_{A^2}^{B^2} \\ &= 2\pi R(A - B)\end{aligned}$$

where in the third line we substituted $u = R^2 - r^2$, so $du = -2r \, dr$.