Final Exam Solutions

May 13, 2008

1. Evaulate

$$\iint_V \operatorname{curl} \mathbf{F} \cdot \mathbf{n} \, d\sigma$$

where

$$\mathbf{F} = e^x \mathbf{i} + y^2 \mathbf{j} + z \tan\left(\frac{xyz}{4}\right) \mathbf{k}$$

and V is the part of the sphere $x^2 + y^2 + z^2 = 5$ above the plane $z = 2$ and $\mathbf{n}$ is the upward unit normal.

Solution: We use Stokes' theorem

$$\iint_V \operatorname{curl} \mathbf{F} \cdot \mathbf{n} \, d\sigma = \oint_{\partial V} \mathbf{F} \cdot \mathbf{T} \, ds.$$

The boundary is the circle of radius $\sqrt{5 - 2^2} = 1$ in the plane $z = 2$, which we parametrize:

$$x = \cos t \qquad y = \sin t \qquad z = 2,$$

or in other words $\mathbf{r} = \cos t \, \mathbf{i} + \sin t \, \mathbf{j} + 2\mathbf{k}$, so $d\mathbf{r} = (-\sin t \, \mathbf{i} + \cos t \, \mathbf{j}) \, dt$. Thus

$$\mathbf{F} \cdot d\mathbf{r} = (-e^x \sin t + y^2 \cos t) \, dt = (-e^{\cos t} \sin t + \sin^2 t \cos t) \, dt.$$

We integrate this from 0 to 2π:

$$\int_0^{2\pi} (-e^{\cos t} \sin t + \sin^2 t \cos t) \, dt = -\int_0^{2\pi} e^{\cos t} \sin t \, dt + \int_0^{2\pi} \sin^2 t \cos t \, dt.$$

In the first integral we let $u = \cos t$, so $du = -\sin t \, dt$, and in the second we let $v = \sin t$, so $dv = \cos t \, dt$:

$$\int_1^1 e^u \, du + \int_0^0 v^2 \, dv = 0.$$

Alternatively, Stokes' theorem says that the answer depends only on the boundary of the surface, so we can choose another surface with the same boundary and find the flux of $\operatorname{curl} \mathbf{F}$ through that. Choose the unit disc in the plane $z = 2$, which does have $\mathbf{n} = \mathbf{k}$, and $\operatorname{curl} \mathbf{F} \cdot \mathbf{k} = 0$, so again the answer is zero.

2. Calculate

$$\iint_{\partial V} \mathbf{F} \cdot \mathbf{n} \, d\sigma$$

on the solid V given by $x^2 + y^2 + z^2 \leq 1$, where

$$\mathbf{F} = (2x + yz)\mathbf{i} + 3y\mathbf{j} + z\mathbf{k}$$

and ∂V denotes the boundary of V.

Solution: We use the divergence theorem

$$\iint_{\partial V} \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_V \operatorname{div} \mathbf{F} \, dx \, dy \, dz.$$

Since $\operatorname{div} \mathbf{F} = 2 + 3 + 1 = 6$, we are integrating 6 over the unit sphere. The volume of the unit sphere is $\frac{4}{3}\pi$, so we get $6 \cdot \frac{4}{3}\pi = 8\pi$.

3. Find the tangent plane to the surface

$$4x^2 + y^2 + 9z^2 = 14$$

at $x = 1$, $y = 1$, $z = 1$.

Solution: Let $f(x, y, z) = 4x^2 + y^2 + 9z^2$, so we are dealing with a level surface of f. Then ∇f is normal to this surface. $\nabla f(x, y, z) = 8x\mathbf{i} + 2y\mathbf{j} + 18z\mathbf{k}$, so $\nabla f(1, 1, 1) = 8\mathbf{i} + 2\mathbf{j} + 18\mathbf{k}$, so the plane is $8x + 2y + 18z = 28$. (We got the 28 by plugging $(1, 1, 1)$ into $8x + 2y + 18z$.) We could divide by 2 to get $4x + y + 9z = 14$.

4. Find a unit vector in the direction which makes $f(x, y) = x^2 - y^2 \cos \pi x$ increase most rapidly at the point $(2, -1)$.

Solution: $\nabla f(x, y) = (2x + \pi y^2 \sin \pi x)\mathbf{i} - 2y \cos \pi x \mathbf{j}$, so $\nabla f(2, -1) = 4\mathbf{i} + 2\mathbf{j}$. We make this a unit vector:

$$\frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j}.$$

5. (a) Show that the line integral

$$\int_C (3x^2 + 2y^2)dx + (4xy - 5)dy$$

is path-independent by finding a potential function f.

Solution: Since $f_y = 4xy - 5$, we have $f = 2xy^2 - 5y + g(x)$. Thus $f_x = 2y^2 + g'(x)$, so $g'(x) = 3x^2$, so $g(x) = x^3 + C$, so

$$f = 2xy^2 - 5y + x^3 + C.$$

We only need one solution, so we could take $C = 0$.

(b) Use the potential function in (a) to evaluate the integral where C is any curve joining $(0, 0)$ to $(1, 1)$.

Solution: $f(1, 1) - f(0, 0) = (2 - 5 + 1) - (0 - 0 + 0) = -2$.

6. Find the largest volume of a Coke whose area cannot exceed 6π square inches. You may assume that the Coke can is a perfect cylinder. Hint: Do not forget the top and bottom of the Coke can.

Solution: Let r be the radius of the can and h the height. We must maximize the volume $f(r, h) = \pi r^2 h$ subject to the constraint $g(r, h) = 2\pi r^2 + 2\pi rh = 6\pi$. Thus we solve

$$\begin{aligned} \nabla f &= \lambda \nabla g \\ 2\pi rh\mathbf{i} + \pi r^2\mathbf{j} &= \lambda[(4\pi r + 2\pi h)\mathbf{i} + 2\pi r\mathbf{j}] \\ 2rh\mathbf{i} + r^2\mathbf{j} &= \lambda[(4r + 2h)\mathbf{i} + 2r\mathbf{j}] \end{aligned}$$

$$\begin{aligned} 2rh &= \lambda(4r + 2h) \\ r^2 &= 2\lambda r. \end{aligned}$$

From the last of these, $r = 2\lambda$ (the possibility $r = 0$ is absurd). Substituting this into the second-to-last, we have

$$\begin{aligned} 4\lambda h &= \lambda(4r + 2h) \\ 2h &= 2r + h \\ h &= 2r. \end{aligned}$$

Thus the can is as wide as it is tall. The area is now $2\pi r^2 + 2\pi rh = 6\pi r^2$. Setting this equal to 6π, we find $r = 1$, so $h = 2$, so the volume is $\pi r^2 h = 2\pi$.