Worksheet 9

February 20, 2008

1. (a) Differentiate x^x and $(\sin x)^{\cos x}$ the old-fashioned way.
   (b) Let $F(u, v) = u^v$. Find F_u and F_v.
   (c) Find the derivatives in part (a) by observing that $x^x = F(x, x)$ and $(\sin x)^{\cos x} = F(\sin x, \cos x)$ and using the chain rule.

2. Give a new proof the product rule by letting $F(u, v) = uv$ and using the chain rule to differentiate $F(f(x), g(x))$.

3. Let $f(x, y, z)$ be a real-valued function of three variables and $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ be a vector-valued function.

   (a) Observe that $f(\mathbf{r}(t))$ makes sense.
   (b) Give a real-world example of a function of three variables $f(x, y, z)$ and a vector-valued function $\mathbf{r}(t)$. What does $f(\mathbf{r}(t))$ mean in your example?
   (c) Express $\frac{\partial}{\partial t} f(\mathbf{r}(t))$ in terms of the gradient ∇f and the velocity vector $\mathbf{r}'(t)$.

4. Let $\mathbf{v} = \langle 3, 4\rangle$ and $\mathbf{w} = \langle 5, 12\rangle$, and let $f(x, y)$ be a differentiable function with $D_{\mathbf{v}} f(0, 0) = 3$ and $D_{\mathbf{w}} f(0, 0) = 1$. Find $f_x(0, 0)$ and $f_y(0, 0)$.

5. Sketch some level curves of $z = y^2 - x^2$. Pick some points on these curves and compute ∇f at those points. Add those vectors to your sketch.

6. According to Newton's law of gravitation, the force exerted on a particle of mass m located at the point (x, y, z) by a particle of mass M located at the origin is given by

$$\mathbf{F}(x, y, z) = \frac{-GMm}{|\mathbf{r}|^3}\,\mathbf{r}$$

   where $\mathbf{r} = \langle x, y, z\rangle$ and $G = 6.673 \times 10^{-11}$. Show that $\mathbf{F}$ is the gradient of

$$f(x, y, z) = \frac{GMm}{\sqrt{x^2 + y^2 + z^2}} = GMm|\mathbf{r}|^{-1}.$$