Worksheet 14

March 26 and 31, 2008

We define the Gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$$

for $x > 0$.

1. (a) Compute $\Gamma(1)$.
   (b) Compute $\Gamma(2)$. Observe that you can recycle your answer to part (a).
   (c) Compute $\Gamma(3)$. Observe that you can recycle your answer to part (b).
   (d) You see where this is going: show that $\Gamma(x+1) = x\Gamma(x)$ in general. This is called the "functional equation" for the Gamma function.
   (e) Compute $\Gamma(4)$, $\Gamma(5)$, and $\Gamma(6)$ using the functional equation.
   (f) What is $\Gamma(n)$ when n is a positive integer?

2. (a) Make the change of variables $t = u^2$ to get another formula for $\Gamma(x)$.
   (b) Compute $\Gamma(\frac{1}{2})$ using the trick of Gauss: observe that $\Gamma(\frac{1}{2}) = \int_{-\infty}^{\infty} e^{-x^2}\, dx$, so

   $$\Gamma(\tfrac{1}{2})^2 = \int_{-\infty}^{\infty} e^{-x^2}\, dx \int_{-\infty}^{\infty} e^{-y^2}\, dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2} e^{-y^2}\, dx\, dy$$

   and change to polar coordinates. Notice that $\Gamma(1)$ shows up.
   (c) Graph $y = e^{-x^2}$, with inflection points and everything. Where have you seen this graph before? What did you call it then?
   (d) Sketch the graph of $z = e^{-x^2-y^2}$.
   (e) Compute $\Gamma(\frac{3}{2})$, $\Gamma(\frac{5}{2})$, and $\Gamma(\frac{7}{2})$ using the functional equation.
   (f) What should $2\frac{1}{2}!$ be? How does it compare to $2!$ and $3!$?

(Continued overleaf.)

3. (a) What is the circumference of a circle of radius r? The area of a sphere?

   (b) Let A_n denote the surface "area" of the unit sphere in n-dimensional space. (I put "area" in quotes because if $n = 4$ we should call it the surface volume, and if $n > 4$ we don't have a word for it. In any event, we're talking about the boundary, not the interior.) What is A_3? A_2? Argue that the area of the sphere of radius r will be $A_n r^{n-1}$.

   (c) Suppose that we are in n-dimensional space with coordinates $x_1, x_2, \ldots, x_n$. Argue that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-x_1^2-x_2^2-\cdots-x_n^2}\,dx_1\,dx_2\cdots dx_n = (\sqrt{\pi})^n.$$

   (d) Suppose a function of n variables depends only on the radius $r = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$:

$$f(x_1, x_2, \ldots, x_n) = g(r).$$

   Argue that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(x_1, x_2, \ldots, x_n)\,dx_1\,dx_2\cdots dx_n = \int_0^{\infty} g(r)\cdot A_n r^{n-1}\,dr.$$

   Start by thinking about $n = 2$ and $n = 3$, which you can visualize.

   (e) From parts (c) and (d) and from 2(a), get a formula for A_n involving $\pi^{n/2}$ and the Gamma function. Write it neatly and put a box around it.

4. (a) Does your formula give the right value of A_2? A_3?

   (b) What is the surface area of a sphere of radius r in 4-dimensional space? 5? Everything up to 8? Notice that you always end up with whole powers of π.

   (c) What is the area of a disc of radius r? The volume of a ball? How are those related to the circumference of a circle and area of a sphere? What is the volume of the ball in 4-dimensional space? 5?

   (d) What does the sphere in 1-dimensional space look like? What does your formula say that its surface area is? The volume of the corresponding ball? Does this make sense?

   (e) What is $\lim_{n\to\infty} A_n$? Hint: Start by considering only even n.