Math 541
Problem Set 1

0.1.5. (a) No. $1/2 = 2/4$, but $f(1/2) = 1$ while $f(2/4) = 2$.

(b) Yes. If $a/b = a'/b'$ then there is a c such that $a' = ca$ and $b' = cb$, so

$$\frac{a'^2}{b'^2} = \frac{a^2c^2}{b^2c^2} = \frac{a^2}{b^2}.$$

0.1.7. The relation $\sim$ is reflexive: for all $a \in A$, $f(a) = f(a)$, so $a \sim a$. It is symmetric: if $a \sim b$ then $f(a) = f(b)$, so $f(b) = f(a)$, so $b \sim a$. It is transitive: if $a \sim b$ and $b \sim c$ then $f(a) = f(b)$ and $f(b) = f(c)$, so $f(a) = f(c)$, so $a \sim c$. Thus $\sim$ is an equivalence relation.

Every equivalence class of $\sim$ is a fiber of f, as follows. Let $a \in A$. Then the equivalence class of a equals the fiber over $f(a)$, for $a' \sim a$ iff $f(a') = f(a)$ iff $a' \in f^{-1}(\{f(a)\})$.

Conversely, every fiber of f is an equivalence class of $\sim$. Let $b \in B$. Since f is surjective, there is an $a \in A$ such that $f(a) = b$. The fiber over b is the equivalence class of a as we showed above.

0.2.1. (a) The greatest common factor is $1 = 2 \cdot 20 - 3 \cdot 13$. The least common multiple is 260.

(b) The Euclidean Algorithm gives

$$\begin{aligned} 372 &= 5 \cdot 69 + 27 \\ 69 &= 2 \cdot 27 + 15 \\ 27 &= 1 \cdot 15 + 12 \\ 15 &= 1 \cdot 12 + 3 \\ 12 &= 4 \cdot 3 \end{aligned}$$

so the greatest common factor is 3, which we can also find by writing $372 = 2^2 \cdot 3 \cdot 31$ and $69 = 3 \cdot 23$. The least common multiple is $372 \cdot 69/3 = 8556$. Rewriting our divisions above,

$$\begin{aligned} 27 &= 372 - 5 \cdot 69 \\ 15 &= 69 - 2 \cdot 27 = 69 - 2 \cdot (372 - 5 \cdot 69) = -2 \cdot 372 + 11 \cdot 69 \\ 12 &= 27 - 15 = (372 - 5 \cdot 69) - (-2 \cdot 372 + 11 \cdot 69) = 3 \cdot 372 - 16 \cdot 69 \\ 3 &= 15 - 12 = (-2 \cdot 372 + 11 \cdot 69) - (3 \cdot 372 - 16 \cdot 69) = \mathbf{-5 \cdot 372 + 27 \cdot 69}. \end{aligned}$$

0.2.3. Since n is composite, let a be a positive divisor of n different from 1 and n, and let $b = n/a$. Then $a < n$, so $n \nmid a$, and $a > 1$, so $b = n/a < n$, so $n \nmid b$, but $ab = n$, so in particular $n \mid ab$.

0.2.4.

$$ax + by = a\left(x_0 + \frac{b}{d}t\right) + b\left(y_0 - \frac{a}{d}t\right) = ax_0 + \frac{ab}{d}t + by_0 - \frac{ab}{d}t = ax_0 + by_0 = N.$$