Math 541
Problem Set 3

1.3.2. $\sigma = (1\ 13\ 5\ 10)(3\ 15\ 8)(4\ 14\ 11\ 7\ 12\ 9)$.
$\tau = (1\ 14)(2\ 9\ 15\ 13\ 4)(3\ 10)(5\ 12\ 7)(8\ 11)$.

1.3.15. First we show that an $m$-cycle has order $m$. If $\sigma = (a_1\ a_2\ \ldots\ a_m)$ is a $m$-cycle, then $\sigma(a_1) = a_2$, $\sigma^2(a_1) = a_3$, $\sigma^3(a_1) = a_4$, and similarly $\sigma^k(a_1)$ is different from $a_1$ for all $k < m$, but $\sigma^m$ is the identity.

Now let $\sigma_1, \ldots, \sigma_k$ be disjoint cycles of lengths $m_1, \ldots, m_k$, let $\sigma = \sigma_1 \cdots \sigma_m$, and let $m$ be the least common multiple of $m_1, \ldots, m_k$. Since disjoint cycles commute, $\sigma^m = {\sigma_1}^m \cdots {\sigma_k}^m = 1$, so $|\sigma| \leq m$. On the other hand, let $p = |\sigma|$. Then $1 = \sigma^p = {\sigma_1}^p \cdots {\sigma_k}^p$, and ${\sigma_1}^p, \ldots, {\sigma_k}^p$ are still disjoint cycles, so ${\sigma_i}^p = 1$ for all $i = 1, \ldots, k$. Thus $p$ is a multiple of each $m_i$, so $m \leq p = |\sigma|$.

1.4.8. Every field contains a 0 and a 1 which are not equal, so let

$$A = \begin{pmatrix} 1 & 1 & & & \\ 0 & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & & & \\ 1 & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

Then

$$AB = \begin{pmatrix} 2 & 1 & & & \\ 1 & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \qquad BA = \begin{pmatrix} 1 & 1 & & & \\ 1 & 2 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

where $2 = 1 + 1 \neq 1$. Thus $AB \neq BA$, so $GL_n(F)$ is not abelian.

1.6.1. (a) $\varphi(x^n) = \varphi(\overbrace{x \cdot x \cdots x}^{n \text{ times}}) = \overbrace{\varphi(x)\varphi(x)\cdots\varphi(x)}^{n \text{ times}} = \varphi(x)^n$.

(b) First, $\varphi(1) = \varphi(1 \cdot 1) = \varphi(1)\varphi(1)$, so $1 = \varphi(1)$, so the claim is true for $n = 0$. Next, $\varphi(x)\varphi(x^{-1}) = \varphi(xx^{-1}) = \varphi(1) = 1$, so $\varphi(x^{-1}) = \varphi(x)^{-1}$. If $n > 0$ then

$$\varphi(x^{-n}) = \varphi(\overbrace{x^{-1}x^{-1}\cdots x^{-1}}^{n \text{ times}}) = \overbrace{\varphi(x^{-1})\varphi(x^{-1})\cdots\varphi(x^{-1})}^{n \text{ times}} = \overbrace{\varphi(x)^{-1}\varphi(x)^{-1}\cdots\varphi(x)^{-1}}^{n \text{ times}} = \varphi(x)^{-n}.$$

1.6.9. An element of $S_4$ is one of the following: the identity, a transposition, a product of disjoint transpositions, a 3-cycle, or a 4-cycle. Thus an element of $S_4$ has order at most 4. But $D_{24}$ has an element of order 24, namely $r$.