Math 541
Problem Set 4

1.7.8. (a) Let $\sigma, \tau \in S_A$ and let $a_1, \ldots, a_k \in A$ be distinct. Then σ is a bijection, so $\sigma(a_1), \cdots, \sigma(a_k)$ are distinct, so we really do have a map $G \times B \to B$ and not to some different range. Now $\sigma \cdot (\tau \cdot \{a_1, \ldots, a_k\}) = \sigma \cdot \{\tau(a_1), \ldots, \tau(a_k)\} = \{\sigma(\tau(a_1)), \ldots, \sigma(\tau(a_k))\} = (\sigma \circ \tau)\{a_1, \ldots, a_k\}$, and $1 \cdot \{a_1, \ldots, a_k\} = \{1(a_1), \ldots, 1(a_k)\} = \{a_1, \ldots, a_k\}$, so we have a group action.

(b) (1 2) acts as follows:

$$\{1,2\} \mapsto \{1,2\} \qquad \{1,3\} \mapsto \{2,3\} \qquad \{1,4\} \mapsto \{2,4\}$$
$$\{2,3\} \mapsto \{1,3\} \qquad \{2,4\} \mapsto \{1,4\} \qquad \{3,4\} \mapsto \{3,4\}.$$

(1 2 3) acts as follows:

$$\{1,2\} \mapsto \{2,3\} \qquad \{1,3\} \mapsto \{1,2\} \qquad \{1,4\} \mapsto \{2,4\}$$
$$\{2,3\} \mapsto \{1,3\} \qquad \{2,4\} \mapsto \{3,4\} \qquad \{3,4\} \mapsto \{1,4\}.$$

2.1.2. Throughout, let H denote the subset in question. In each case, H is not closed under products.

(a) $(1\ 2) \circ (1\ 3) = (1\ 3\ 2)$, which is not a 2-cycle. Even more problematic, $1 \notin H$.

(b) Again, $1 \notin H$, but also $s \cdot sr = r$, which does not have order 2, hence is not a reflection.

(c) Let $n = ab$, where $1 < a < n$ and $1 < b < n$. If $x \in G$ has order n then x^a has order b, so $x^a \notin H$.

(d) $1 + 1 = 2$, which is not odd.

(e) $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \notin H$ since $\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}}\right)^2 = \frac{5}{6} + \sqrt{\frac{2}{3}}$.

2.1.8. If $K \subseteq H$ then $H \cup K = H$, which is a subgroup. Similarly, if $H \subseteq K$ then $H \cup K = K$. Inversely, suppose there exist $h \in H - K$ and $k \in K - H$. If $hk \in H$ then $k = h^{-1} \cdot hk \in H$ since H is a subgroup, but $k \notin H$, so $hk \notin H$. Similarly, $hk \notin K$. Thus $hk \notin H \cup K$, so $H \cup K$ is not a subgroup.

2.2.3. Suppose $g \in C_G(B)$. For all $a \in A$, $a \in B$, so $gag^{-1} = a$. Thus $g \in C_G(A)$. Now $C_G(B)$ is a subset of $C_G(A)$, and it is shown in the text that $C_G(B)$ is a subgroup of G, so it is nonempty and closed under products and inverses, hence is a subgroup of $C_G(A)$.

2.2.6. (a) Fix $h \in H$. Then $hHh^{-1} \subset H$, since for all $x \in H$, $hxh^{-1} \in H$; and $H \subset hHh^{-1}$, since for all $x \in H$, $h^{-1}xh \in H$ and $x = h(h^{-1}xh)h^{-1}$, so $x \in hHh^{-1}$. Thus $hHh^{-1} = H$, so $h \in N_G(H)$.

For a counterexample when H is not a subgroup, consider the transpositions $a = (1\ 2)$ and $b = (2\ 3)$ in S_3, and take $H = \{a, b\}$. Then $aba^{-1} = (1\ 2)(2\ 3)(1\ 2) = (1\ 3) \notin H$, so $aHa^{-1} \not\subset H$, so $a \notin N_G(H)$.

(b) The following are equivalent: $H \subset C_G(H)$; for all $h \in H$, $h \in C_G(H)$; for all $h \in H$ and all $x \in H$, $hxh^{-1} = x$; for all $h, x \in H$, $hx = xh$; H is abelian.