Math 541
Problem Set 5

2.4.1. Recall that

$$\langle H\rangle = \bigcap_{H\subseteq K\leq G} K.$$

H is contained in each intersectand, hence in the intersection, that is, $H \subseteq \langle H\rangle$. But $H \subseteq H \leq G$, so H is one of the intersectands, so $\langle H\rangle \subseteq H$.

2.4.3. We will argue that the generators commute with each other, so the subgroup they generate is abelian.

Let $x, y \in H \cup Z(G)$. If $x \in Z(G)$ or $y \in Z(G)$ then $xy = yx$ by definition of the center. If neither is in $Z(G)$ then $x, y \in H$, so again $xy = yx$ since H is abelian. Thus any two elements of $H \cup Z(G)$ commute.

Observe that if $x \in H \cup Z(G)$ then $x^{-1} \in H \cup Z(G)$ since H and $Z(G)$ are subgroups. Now an arbitrary element of $\langle H, Z(G)\rangle = \overline{H \cup Z(G)}$ is of the form $x_1 \cdots x_m$, where $x_1 \cdots x_m \in H \cup Z(G)$. Let $y_1 \cdots y_n$ be another element. Then the x_i's commute with the y_i's, so $x_1 \cdots x_m \cdot y_1 \cdots y_n = y_1 \cdots y_n \cdot x_1 \cdots x_m$.

For the counterexample, let G be any nonabelian group and $H = 1$, so $C_G(H) = G$, which is not ablian. For a more satisfying counterexample, take $G = D_8$ and $H = \{1, r^2\}$. Then $r \in C_G(H)$, since $r^2 r = r r^2$, and $s \in C_G(H)$, since $r^2 s = rsr^{-1} = sr^{-2} = sr^2$ (since $r^4 = 1$). Thus $C_G(H) = G$, which is not abelian.

2.4.7. First, we figure out what this subgroup looks like. Let $x = (1\ 2)$ and $y = (1\ 3)(2\ 4)$. Since $x^2 = y^2 = 1$, the elements of $\langle x, y\rangle$ are strings of x's and y's such as yxy and $xyxyx$; that is, we don't need powers of x and y in our strings. Moreover, $xy = (1\ 3\ 2\ 4)$, so $(xy)^4 = 1$, so $y = xyxyxyx$, so we can assume that our strings start with x. These are the possible strings:

$$\begin{aligned} x &= (1\ 2) & xyxyx &= (3\ 4) \\ xy &= (1\ 3\ 2\ 4) & xyxyxy &= (1\ 4\ 3\ 2) \\ xyx &= (1\ 4)(2\ 3) & xyxyxyx &= (1\ 3)(2\ 4) \\ xyxy &= (1\ 2)(3\ 4) & xyxyxyxy &= 1. \end{aligned}$$

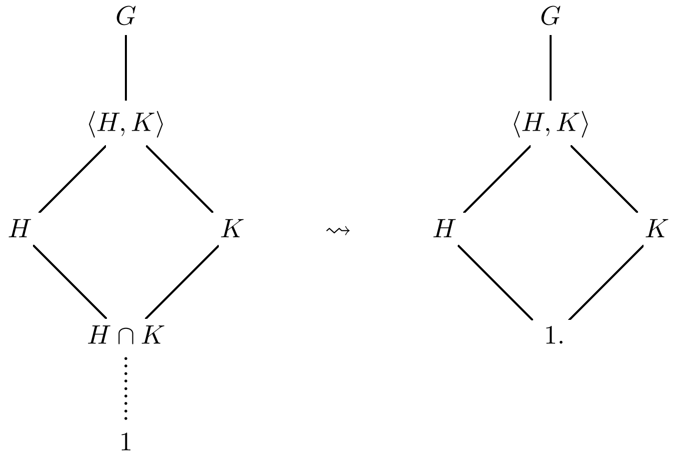
In particular, $|\langle x, y\rangle| = 8$.

Now define a map $\varphi : D_8 \to S_4$ by $\varphi(s) = x$ and $\varphi(r) = xy$. This really does determine a group homomorphism, since it respects the relations: $\varphi(s)^2 = 1$, $\varphi(r)^4 = 1$, and $\varphi(r)\varphi(s) = \varphi(s)\varphi(r)^{-1}$. Since $\operatorname{im}\varphi$ is a subgroup containing $x = \varphi(r)$ and $y = \varphi(sr)$, $\langle x, y\rangle \subseteq \operatorname{im}\varphi$. Since $|D_8| = 8$ and $|\langle x, y\rangle| = 8$, this is the whole image, and φ is a bijection onto its image, hence an isomorphism onto its image.

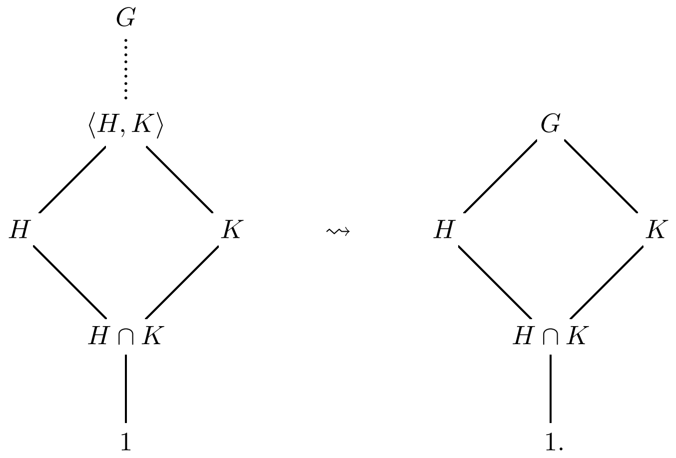
2.5.1. The most general lattice is

$$\begin{tikzcd} & G \arrow[d, no head] & \\ & \langle H, K\rangle \arrow[dl, no head] \arrow[dr, no head] & \\ H \arrow[dr, no head] & & K \arrow[dl, no head] \\ & H \cap K \arrow[d, no head] & \\ & 1. & \end{tikzcd}$$

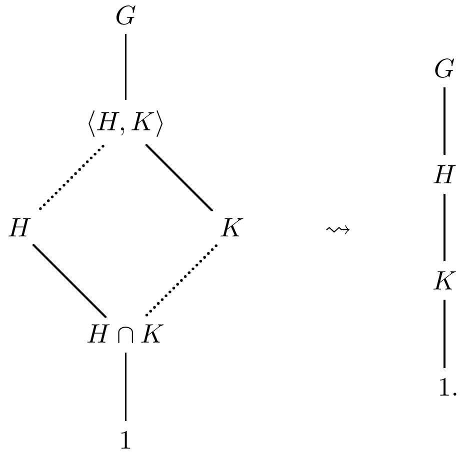
If $H \cap K = 1$ then we contract the edge from $H \cap K$ to 1:



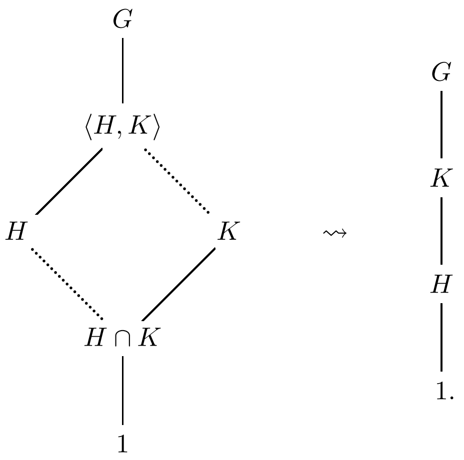
If $\langle H, K\rangle = G$ then we contract the edge from G to $\langle H, K\rangle$:



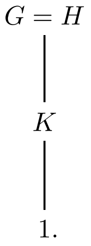
If $K \subseteq H$ then $H \cap K = K$ and $\langle H, K\rangle = H$, so we contract two edges:



Similarly, if $H \subseteq K$,



Any combination of these four contractions could occur, making $2^4 = 16$ possible lattices. We will not draw them all, but for example, we could have $\langle H, K\rangle = G$ and $K \subseteq H$, so



2.5.9. (b)