Math 541
Problem Set 6

3.1.1. $\varphi(1) = 1 \in E$, so $1 \in \varphi^{-1}(E)$, so $\varphi^{-1}(E) \neq \varnothing$. If $x, y \in \varphi^{-1}(E)$ then $\varphi(x), \varphi(y) \in E$, so $\varphi(xy^{-1}) = \varphi(x)\varphi(y)^{-1} \in E$ since E is a subgroup, so $xy^{-1} \in \varphi^{-1}(E)$. Thus $\varphi^{-1}(E)$ is a subgroup by the subgroup criterion.

Now suppose that E is normal. For all $g \in G$,

$$g\varphi^{-1}(E)g \subseteq \varphi^{-1}(\varphi(g\varphi^{-1}(E)g)) = \varphi^{-1}(\varphi(g)E\varphi(g)^{-1}) = \varphi^{-1}(E),$$

so $\varphi^{-1}(E)$ is normal by Theorem 6, part (5).

The trivial subgroup $1 \leq H$ is normal and $\ker\varphi = \varphi^{-1}(1)$, so $\ker\varphi$ is normal.

3.1.5. The order of gN is the smallest positive integer α such that $(gN)^\alpha$ is the identity $1N \in G/N$, but $g^\alpha N = 1N$ if and only if $g^\alpha \in N$, so the order of gN is the smallest positive integer such that $g^\alpha \in N$.

For the example, take $G = \mathbb{Z}/6\mathbb{Z}$ and $H = \{0, 3\}$; then the element 1 has order 6, but in $G/H = \mathbb{Z}/3\mathbb{Z}$, 1 has order 3.

3.2.6. Suppose that $Hg = kH$ for some $k \in G$. Observe that $g \in Hg$ and $g \in gH$, so gH intersects $kH = Hg$. But two left cosets are either disjoint or equal, so $gH = kH = Hg$. Multiplying on the right by g^{-1}, we have $gHg^{-1} = H$, so $g \in N_G(H)$.

3.2.18. Let $\varphi : G \to G/N$ be the natural projection. Then $\varphi(H) \leq G/N$, so $|\varphi(H)|$ divides $|G/N| = |G : N|$. But $\varphi(H) \cong H/(H \cap N)$, so $|\varphi(H)|$ divides $|H|$, so $|\varphi(H)| = 1$, so $\varphi(H) = 1$, so $H \subseteq \ker\varphi = N$.

3.3.3. Let $\varphi : G \to G/H$ be the natural projection, so as before $|\varphi(K)|$ divides $|G : H| = p$. Either $|\varphi(K)| = 1$, so $K \subseteq \ker\varphi = H$; or $|\varphi(K)| = p$, so $\varphi(K) = G/H$, so $G = HK$, and thus $G/H = HK/H \cong K/(K \cap H)$ by the diamond isomorphism theorem, so $|K : K \cap H| = |K/(K \cap H)| = |G/H| = |G : H| = p$.