Math 541
Problem Set 7

3.5.12. Write $\mathbb{Z}/2\mathbb{Z}$ additively, so $\epsilon(\sigma)$ is 0 if σ is even and 1 if σ is odd, and $\epsilon(\sigma\tau) = \epsilon(\sigma) + \epsilon(\tau)$. Consider S_n as a subgroup of S_{n+2}. Let $\tau = (n-1\ n) \in C_G(S_n)$. Define a map $\varphi : S_n \to S_{n+2}$ by $\varphi(\sigma) = \tau^{\epsilon(\sigma)}\sigma$. Then φ is a homomorphism:

$$\varphi(\sigma_1)\varphi(\sigma_2) = \tau^{\epsilon(\sigma_1)}\sigma_1\tau^{\epsilon(\sigma_2)}\sigma_2 = \tau^{\epsilon(\sigma_1)+\epsilon(\sigma_2)}\sigma_1\sigma_2 = \tau^{\epsilon(\sigma_1\sigma_2)}\sigma_1\sigma_2 = \varphi(\sigma_1\sigma_2).$$

Thus $\operatorname{im}\varphi$ is a subgroup. In fact $\operatorname{im}\varphi \subseteq A_{n+2}$:

$$\epsilon(\tau^{\epsilon(\sigma)}\sigma) = \epsilon(\tau^{\epsilon(\sigma)}) + \epsilon(\sigma) = \epsilon(\sigma)\cdot\epsilon(\tau) + \epsilon(\sigma) = \epsilon(\sigma)\cdot 1 + \epsilon(\sigma) = 0.$$

φ is clearly injective, hence is an isomorphism onto its image.

4.3.2. (a) The elements of D_8 are $1, r, r^2, r^3, s, sr, sr^2, sr^3$. If we conjugate these by r we get $1, r, r^2, r^3, sr^2, sr^3, s, sr$. If we conjugate by s we get $1, r^3, r^2, r, s, sr^3, s^2, sr$. Thus the conjugacy classes are $\{1\}$, $\{r^2\}$, $\{r, r^3\}$, $\{s, sr^2\}$, and $\{sr, sr^3\}$. The class equation is $8 = 2 + 2 + 2 + 2$.

(b) The elements of Q_8 are $1, -1, i, -i, j, -j, k, -k$. If we conjugate by i we get $1, -1, i, -i, -j, j, -k, k$. If we conjugate by j we get $1, -1, -i, i, j, -j, -k, k$. Since $k = ij$, conjugating by it won't give us more information. Thus the conjugacy classes are $\{1\}$, $\{-1\}$, $\{i, -i\}$, $\{j, -j\}$, and $\{k, -k\}$. The class equation is $8 = 2 + 2 + 2 + 2$.

(c) The elements of A_4 are the identity, the three products of disjoint transpositions $(1\ 2)(3\ 4)$, $(1\ 3)(2\ 4)$, and $(1\ 4)(2\ 3)$, and eight 3-cycles: for each $n \in \{1, 2, 3, 4\}$, there are two 3-cycles that leave n fixed.

The products of transpositions are all conjugate to one another: $(1\ 3)(2\ 4) = (1\ 3\ 2) \circ (1\ 2)(3\ 4) \circ (1\ 2\ 3)$ and $(1\ 4)(2\ 3) = (2\ 4\ 3) \circ (1\ 2)(3\ 4) \circ (2\ 3\ 4)$. They have order 2, hence are not conjugate to the 3-cycles, which have order 3.

The order of a conjugacy class divides the order of the group, so the eight 3-cycles do not form a single conjugacy class. But if σ is a 3-cycle fixing n and $\tau \in A_4$ then $\tau \circ \sigma \circ \tau^{-1}$ is a 3-cycle fixing $\tau(n)$, so each 3-cycle is conjugate to at least three others, so in fact the 3-cycles split into two conjugacy classes of four elements each.

Thus the conjugacy classes of A_4 are $\{1\}$, $\{(1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}$, $\{(1\,2\,3), (2\,4\,3), (1\,3\,4), (1\,4\,2)\}$, and $\{(1\ 3\ 2), (2\ 3\ 4), (1\ 4\ 3), (1\ 2\ 4)\}$. The class equation is $12 = 1 + 3 + 4 + 4$.

4.3.4. To show that $N_G(gSg^{-1}) = gN_G(S)g^{-1}$, observe that the following are equivalent:

$$\begin{gathered} x \in N_G(gSg^{-1}) \\ xgSg^{-1}x^{-1} = gSg^{-1} \\ g^{-1}xgSg^{-1}x^{-1}g = S \\ g^{-1}xg \in N_G(S) \\ x \in gN_G(S)g^{-1}. \end{gathered}$$

To show that $C_G(gSg^{-1}) = gC_G(S)g^{-1}$, observe that the following are equivalent:

$$\begin{gathered} x \in C_G(gSg^{-1}) \\ xgsg^{-1}x^{-1} = gsg^{-1} \quad \forall s \in S \\ g^{-1}xgsg^{-1}x^{-1}g = s \quad \forall s \in S \\ g^{-1}xg \in C_G(S) \\ x \in gC_G(S)g^{-1}. \end{gathered}$$

4.4.1. For all $x \in G$ we have

$$\sigma(\varphi_g(\sigma^{-1}(x))) = \sigma(g\sigma^{-1}(x)g^{-1}) = \sigma(g)x\sigma(g)^{-1} = \varphi_{\sigma(g)}(x),$$

as desired. Thus $\sigma \operatorname{Inn}(G)\sigma^{-1} \subseteq \operatorname{Inn}(G)$, so by Theorem 3.6(5), $\operatorname{Inn}(G) \trianglelefteq \operatorname{Aut}(G)$. (That $\operatorname{Inn}(G) \leq \operatorname{Aut}(G)$ is shown on page 134).

4.4.13. Observe that the conjugacy class of an element $g \in G$ has order 1 if and only if $g \in Z(G)$. The order of a conjugacy class divides the order of the group, so a conjugacy class may have order 1, 7, 29, or 203. Since H is normal, it is a union of conjugacy classes, so either it consists of 7 conjugacy classes of order 1, hence is contained in $Z(G)$, or it consists of a single conjugacy class of order 7, but this is impossible since $1 \in H$.

Now choose $x \notin H$ and let $K = \langle H, x\rangle$. Since $H \subseteq Z(G)$, K is abelian, as we showed two weeks ago in Exercise 2.4.3. Now $|K| > 7$, $|K|$ divides 203, and 7 divides $|K|$ since $H \leq K$, so $|K| = 203$, so $K = G$. Thus G is abelian.