Algebraic Geometry of the Ring $C(X)$

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October 2007

Abstract

If X is compact, the maximal ideals of $C(X)$ correspond to the points of X. For an algebraic geometer, this is exciting. But if we try to go further than this, we find that $C(X)$ was just leading us on.

1

Let X be a compact Hausdorff space. We work throughout with the ring $C(X) = C(X, \mathbb{R})$ of continuous, real-valued functions on X. If $A \subseteq X$, let $I(A) = \{f \in C(X) : f(x) = 0\ \forall x \in A\}$ be the ideal of functions that vanish on A. If $\mathfrak{a} \subseteq C(X)$ is an ideal, let $V(\mathfrak{a}) = \{x \in X : f(x) = 0\ \forall f \in \mathfrak{a}\}$ be the common vanishing locus of the functions in $\mathfrak{a}$. As usual,

(1) (a) If $\mathfrak{a} \subseteq \mathfrak{b}$ then $V(\mathfrak{b}) \subseteq V(\mathfrak{a})$.
(b) $V(0) = X$ and $V(1) = \varnothing$.
(c) $V(\sum \mathfrak{a}_i) = \bigcap V(\mathfrak{a}_i)$.
(d) $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})$.

(2) (a) If $A \subseteq B$ then $I(B) \subseteq I(A)$.
(b) $I(X) = 0$ and $I(\varnothing) = (1)$.
(c) $I(\bigcup A_i) = \bigcap I(A_i)$.

(3) $A \subseteq V(I(A))$ and $\mathfrak{a} \subseteq I(V(\mathfrak{a}))$.

(4) $V(I(V(\mathfrak{a}))) = V(\mathfrak{a})$ and $I(V(I(A))) = I(A)$.

2

If $x \in X$ is a point then $I(x)$ is a maximal ideal, since it is the kernel of the surjective map $C(X) \to \mathbb{R}$ sending f to $f(x)$. Conversely,

Proposition. *Every maximal ideal is the ideal of a point.*[1]

Proof. Let $\mathfrak{m}$ be maximal, and suppose that $V(\mathfrak{m})$ is empty. For each $x \in X$, choose $f_x \in \mathfrak{m}$ such that $f_x(x) \neq 0$ and let U_x be the set where f_x does not vanish. Since $x \in U_x$, the collection of all U_x cover X, so we can extract a finite subcover $U_{x_1}, \ldots, U_{x_k}$. Now $f_{x_1}^2 + \cdots + f_{x_k}^2$ never vanishes, hence is a unit, but this is impossible since $\mathfrak{m}$ is a proper ideal.

Thus there is a point $x \in V(\mathfrak{m})$, so $\mathfrak{m} \subseteq I(V(\mathfrak{m})) \subseteq I(x)$, and since $\mathfrak{m}$ is maximal, $\mathfrak{m} = I(x)$. □

[1] If X is not compact, there are more maximal ideals: let $\mathfrak{a} = C_c(X)$ be the ideal of functions with compact support. Then $\mathfrak{a}$ is proper, hence is contained in a maximal ideal, but $V(\mathfrak{a}) = \varnothing$. We might try to remedy this by taking $C_c(X)$ as our ring to begin with, but no one wants to work in a ring without unit. In fact, the maximal ideals of $C(X)$ are in bijection with the points of $\beta(X)$, the Stone-Čech compactification of X, as Dan Turetsky and I show in some other notes.

3

Since maximal ideals of $C(X)$ correspond to points just as in algebraic geometry, we would like prime ideals to correspond to subvarieties in some sense: maybe subspaces, or closed subspaces, or submanifolds if X is a manifold—really, we would like the prime ideals to tell us what we should mean by "subvariety". We are sorely disappointed:

Proposition. *If $\mathfrak{p}$ is prime then $V(\mathfrak{p})$ consists of a single point.*

Proof. Since $\mathfrak{p}$ is proper, $V(\mathfrak{p})$ consists of at least one point as we saw above. Suppose that $x, y \in V(\mathfrak{p})$ and $x \neq y$. Let U and V be neighborhoods of x and y, respectively, with $U \cap V = \varnothing$. Since X is compact and Hausdorff, it is completely regular, so there are bump functions f and g supported in U and V, respectively, with $f(x) = g(y) = 1$. Now $f \notin \mathfrak{p}$ and $g \notin \mathfrak{p}$, but $fg = 0 \in \mathfrak{p}$, which is a contradiction. □

Observe in particular that every prime ideal is contained in a unique maximal ideal.

4

Thus disappointed, we might hope that every prime ideal is maximal, that is, that there are no strange prime ideals properly contained in $I(x)$. Again we are disappointed: let $X = [-1, 1]$, and

$$\mathfrak{r} = \left\{ f \in C(X) : \lim_{x \to 0} \frac{f(x)}{x^k} = 0 \ \forall k \geq 0 \right\}.$$

This $\mathfrak{r}$ is not prime, for let $f(x) = \max\{x, 0\}$ and $g(x) = \max\{-x, 0\}$; then $f \notin \mathfrak{r}$ and $g \notin \mathfrak{r}$, but $fg = 0 \in \mathfrak{r}$. But $\mathfrak{r}$ is radical, for if $f^n \in \mathfrak{r}$ then

$$\lim_{x \to 0} \left| \frac{f(x)}{x^k} \right| = \lim_{x \to 0} \sqrt[n]{\left| \frac{f(x)^n}{x^{kn}} \right|} = 0$$

so $f \in \mathfrak{r}$. Now $\mathfrak{r}$ is a radical ideal properly contained in $I(0)$, and since a radical ideal is the intersection of all primes that contain it, this implies that there are prime ideals properly contained in $I(0)$.

We remark that while $\mathfrak{r}$ is not prime, its contraction to $C^\infty(X) \subseteq C(X)$ is prime, as follows. If $f \in \mathfrak{r}$ is C^∞ then by l'Hôpital's rule,

$$0 = \lim_{x \to 0} \frac{f(x)}{x^{k+1}} = \lim_{x \to 0} \frac{f'(x)}{(k+1)x^k}$$

for all $k \geq 0$, so $f' \in \mathfrak{r}$. Thus $f^{(k)}(0) = 0$ for all $k \geq 0$, so $\mathfrak{r} \cap C^\infty(X)$ is the kernel of the map $C^\infty(X) \to \mathbb{R}[[x]]$ sending a function to its Taylor series

$$f \mapsto f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

and $\mathbb{R}[[x]]$ is an integral domain.

The pathological example just described comes in an infinite family: given any function g that vanishes only at 0, let

$$\mathfrak{r}_g = \left\{ f \in C(X) : \lim_{x \to 0} \frac{f(x)}{g(x)^k} = 0 \ \forall k \geq 0 \right\}.$$

As before, $\mathfrak{r}_g$ as radical. It is not zero since $e^{-1/g^2} \in \mathfrak{r}_g$. If $h \in \mathfrak{r}_g$ then $\mathfrak{r}_h$ is properly contained in $\mathfrak{r}_g$ (since $h \notin \mathfrak{r}_h$). Thus there is a huge, messy lattice of radical ideals, and hence of primes, living under $I(x)$.

5

A discussion of the Nullstellensatz will shed light the preceding example.

Proposition.

(1) $V(\mathfrak{a})$ *is closed.*

(2) $V(I(A)) = \bar{A}$.

(3) $I(A)$ *is radical.*

Proof.

(1) Each $f \in S$ is continuous and $V(\mathfrak{a}) = \bigcap_{f \in \mathfrak{a}} f^{-1}(0)$.

(2) $V(I(A))$ is a closed set containing A, so $\bar{A} \subseteq V(I(A))$. Since X is compact and Hausdorff, it is completely regular, so if $x \notin \bar{A}$ there is a function f with $f(x) = 1$ and $f|_{\bar{A}} = 0$. This $f \in I(A)$, so $x \notin V(I(A))$.

(3) If $f(x)^n = 0$ then $f(x) = 0$. □

Parts 1 and 2 of the proposition imply that the Zariski topology on X is the same as the original topology. Part 3 says that the ideal of a subset of X is radical. If there were a fourth part, it would be the Nullstellensatz: $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$, or equivalently, every radical ideal is the ideal of some subset of X. But the radical ideals are nice if and only if the prime ideals are nice.

Proposition. *The following are equivalent:*

(1) Every radical ideal is $I(A)$ *for some* $A \subseteq X$.

(2) Every prime ideal is $I(A)$ *for some* $A \subseteq X$.

Proof. Every prime ideal is radical, so (1) $\Rightarrow$ (2). Every radical ideal is an intersection of primes, so (2) $\Rightarrow$ (1). □

And we have seen that (2) fails, even for very nice spaces like $[-1, 1]$.

The reader may object that we should not expect a Nullstellensatz since we are working with real-valued functions and $\mathbb{R}$ is not algebraically closed. But this is not the problem: everything we have said goes through for complex-valued functions with only slight modifications.

6

As wild as the primes of $C(X)$ are, at least we can study them locally. If $x \in X$, let

$$J(x) = \{f \in C(X) : f|_U = 0 \text{ for some open } U \ni x\}$$

be the ideal of functions that vanish in a neighborhood of x, so $C(X)/J(x)$ is the ring of germs of continuous functions at x.

Proposition. *The prime ideals contained in* $I(x)$ *are in bijection with the prime ideals of* $C(X)/J(x)$.

Proof. It suffices to show that $J(x) \subseteq \mathfrak{p}$ for all primes $\mathfrak{p} \subseteq I(x)$. If $f \in J(x)$ then f vanishes in a neighborhood U of x. Let g be a function supported in U with $g(x) = 1$. Then $g \notin I(x)$, so $g \notin \mathfrak{p}$, but $fg = 0 \in \mathfrak{p}$, so $f \in \mathfrak{p}$. □

We give a second proof, based on the following fact:

Proposition. *The ring of germs $C(X)/J(x)$ is isomorphic to the localization $C(X)_{I(x)}$.*

Proof. First we show that any $f \notin I(x)$ becomes a unit in $C(X)/J(x)$. It suffices to show that f^2 becomes a unit. Let $y = f(x)^2$ and $g = \max\{f^2, y/2\}$. Then g never vanishes, hence is a unit, and $g - f^2 \in J(x)$.

Second we show that the natural map $C(X) \to C(X)_{I(x)}$ sends any $f \in J(x)$ to 0. Since $f \in J(x)$, f vanishes on a neighborhood U of x. Let g be a function supported in U with $g(x) = 1$. Then $g \notin I(x)$ and $fg = 0$, so $f/1 = 0/g$. □

Now for any ring R and prime $\mathfrak{p}$, the primes of $R_\mathfrak{p}$ are in bijection with the primes contained in $\mathfrak{p}$.

7

We conclude with some remarks about finite generation.

Proposition. *If a radical ideal $\mathfrak{r} \subseteq C(X)$ is finitely generated then $V(\mathfrak{r})$ is open.*

Proof. Suppose that $\mathfrak{r} = (f_1, \ldots, f_k)$. For each i, $|f_i|^2 = f_i^2 \in \mathfrak{r}$, so $|f_i| \in \mathfrak{r}$. Let $f = \sqrt{|f_1| + \cdots + |f_k|}$. Since $f \in \mathfrak{r}$, there are $g_1, \ldots, g_k \in C(X)$ such that $f = g_1 f_1 + \cdots + g_k f_k$, so

$$\begin{aligned} f &= g_1 f_1 + \cdots + g_k f_k \\ &\leq |g_1||f_1| + \cdots + |g_k||f_k| \\ &\leq (|g_1| + \cdots + |g_k|)(|f_1| + \cdots + |f_k|) \\ &= (|g_1| + \cdots + |g_k|)f^2 \end{aligned}$$

Outside of $V(\mathfrak{r}) = V(f)$, we can divide by f to get

$$1 \leq (|g_1| + \cdots + |g_k|)f,$$

but if $f(x) = 0$ this inequality fails. Thus the complement of $V(\mathfrak{r})$ is $((|g_1| + \cdots + |g_k|)f)^{-1}([1, \infty))$, which is closed. □

Since $V(\mathfrak{r})$ is also closed, when X is connected the only finitely generated radical ideals are 0 and the whole ring. In particular, if $X = [-1, 1]$ then $I(0)$ is not finitely generated. We contrast this with the C^∞ situation:

Proposition. *In $C^\infty(\mathbb{R}^n)$, $I(0) = (x_1, \ldots, x_n)$.*

Proof. This proof is essentially from Milnor's *Morse Theory*. Let $f \in I(0)$. Then

$$f(x_1, \ldots, x_n) = \int_0^1 \frac{d}{dt} f(tx_1, \ldots, tx_n)\, dt = \int_0^1 \left(\frac{\partial f}{\partial x_1}(tx_1, \ldots, tx_n)x_1 + \cdots + \frac{\partial f}{\partial x_n}(tx_1, \ldots, tx_n)x_n \right) dt.$$

Taking $f_i = \int_0^1 \frac{\partial f}{\partial x_i}(tx_1, \ldots, tx_n)\, dt$, we have $f = x_1 f_1 + \ldots + x_n f_n$. □

Neither $C(X)$, nor $C^\infty(X)$ if X is a smooth manifold, is typically Noetherian: to produce an ascending chain of ideals, we need only produce a descending chain of closed sets, which is easy.

8

In conclusion, it is well-known that algebraic geometry produces some pretty terrible topological spaces. Here we have shown that topology produces some pretty terrible rings.

I thank Dan Turetsky, Matt Davis, and Sam Eckles for helpful discussions.