Notes on Spin and Pin

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In some other notes, I claim that the second Stiefel-Whitney class

$$\check{H}^1(X;\mathrm{O}(n)) \xrightarrow{w_2} \check{H}^2(X;\mathbb{Z}/2)$$

is the connecting homomorphism coming from the central extension

$$1 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathrm{Pin}(n) \longrightarrow \mathrm{O}(n) \longrightarrow 1$$

and that one should prove this by showing that it satisfies the axioms given in [1]. In §§1 and 2 we work out an essential ingredient for checking the direct sum axiom $w_2(E \oplus E') = w_2(E) + w_1(E) \smile w_1(E') + w_2(E')$. We avoid Clifford algebras and work instead with covering spaces. In §3 we define $\mathrm{Spin}^{\mathbb{C}}(n)$ and show that a vector bundle admits a $\mathrm{Spin}^{\mathbb{C}}(n)$-structure if and only if w_2 has an integral lift. In §4 we express the first Chern class as a connecting homomorphism. In §5 we describe explicitly the "accidental isomorphisms" between $\mathrm{Spin}(n)$ and better-known Lie groups for $3 \le n \le 6$.

1 Spin

The fiber bundle $\mathrm{SO}(n) \hookrightarrow \mathrm{SO}(n+1) \to S^n$ yields an exact sequence of homotopy groups

$$\pi_2(S^n) \longrightarrow \pi_1(\mathrm{SO}(n)) \longrightarrow \pi_1(\mathrm{SO}(n+1)) \longrightarrow \pi_1(S^n).$$

If $n \ge 2$ then $\pi_1(S^n) = 0$, so the embedding of $\mathrm{SO}(n)$ in $\mathrm{SO}(n+1)$ induces a surjection on fundamental groups. If $n \ge 3$ then $\pi_2(S^n) = 0$, so the induced map is in fact an isomorphism. We do not claim that this is true of *every* embedding $\mathrm{SO}(n) \hookrightarrow \mathrm{SO}(n+1)$, just the block diagonal embedding

$$A \longmapsto \begin{pmatrix} A & \\ & 1 \end{pmatrix}.$$

Since $\mathrm{SO}(2) = S^1$ and $\mathrm{SO}(3) = \mathbb{RP}^3$ we have $\pi_1(\mathrm{SO}(2)) = \mathbb{Z}$ and $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}/2$ for $n \ge 3$. Thus $\mathrm{SO}(n)$ has a unique double cover, which we call $\mathrm{Spin}(n)$, and if $n \ge 3$ this is the universal cover.

The procedure for making a covering space into a group is well-known, but we will go through it in detail for later reference. We follow [2], pp. 43–45. Let $p : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ be the covering map, $m : \mathrm{SO}(n) \times \mathrm{SO}(n) \to \mathrm{SO}(n)$ the group operation, and $i : \mathrm{SO}(n) \to \mathrm{SO}(n)$ the inversion. Observe that $m_* : \pi_1(\mathrm{SO}(n)) \times \pi_1(\mathrm{SO}(n)) \to \pi_1(\mathrm{SO}(n))$ is the group operation on π_1, and similarly with i_*. Consider the diagram

$$\begin{array}{ccc} \mathrm{Spin}(n) \times \mathrm{Spin}(n) & & \mathrm{Spin}(n) \\ \big\downarrow{\scriptstyle p \times p} & & \big\downarrow{\scriptstyle p} \\ \mathrm{SO}(n) \times \mathrm{SO}(n) & \xrightarrow{m} & \mathrm{SO}(n). \end{array}$$

The maps $m_* \circ (p_* \times p_*)$ and p_* have the same image in $\pi_1(\mathrm{SO}(n))$, so if we choose a point $\tilde{1} \in \mathrm{Spin}(n)$ in the fiber over $1 \in \mathrm{SO}(n)$, there is a unique $\tilde{m} : \mathrm{Spin}(n) \times \mathrm{Spin}(n) \to \mathrm{Spin}(n)$ that lifts $m \circ (p \times p)$ and satisfies $\tilde{m}(\tilde{1},\tilde{1}) = \tilde{1}$. Similarly,

$$\begin{array}{ccc} \mathrm{Spin}(n) & & \mathrm{Spin}(n) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ \mathrm{SO}(n) & \xrightarrow{i} & \mathrm{SO}(n) \end{array}$$

there is a unique $\tilde{\imath} : \mathrm{Spin}(n) \to \mathrm{Spin}(n)$ that lifts $i \circ p$ and satisfies $\tilde{\imath}(\tilde{1}) = \tilde{1}$. We verify that $\tilde{m}$ and $\tilde{\imath}$ make $\mathrm{Spin}(n)$ into a group:

- Associativity:

$$\begin{array}{ccc} \mathrm{Spin}(n) \times \mathrm{Spin}(n) \times \mathrm{Spin}(n) & \xrightarrow{\tilde{m} \times \mathrm{id}} & \mathrm{Spin}(n) \times \mathrm{Spin}(n) \\ \downarrow{\scriptstyle \mathrm{id} \times \tilde{m}} & & \downarrow{\scriptstyle \tilde{m}} \\ \mathrm{Spin}(n) \times \mathrm{Spin}(n) & \xrightarrow{\tilde{m}} & \mathrm{Spin}(n) \end{array}$$

  $\tilde{m} \circ (\tilde{m} \times \mathrm{id})$ and $\tilde{m} \circ (\mathrm{id} \times \tilde{m})$ lift $m \circ (m \times \mathrm{id})$ and $m \circ (\mathrm{id} \times m)$, which are equal, and agree at $(\tilde{1},\tilde{1},\tilde{1})$, hence are equal.

- Identity:

$$\mathrm{Spin}(n) \xrightarrow{\mathrm{id} \times \tilde{1}} \mathrm{Spin}(n) \times \mathrm{Spin}(n) \xrightarrow{m} \mathrm{Spin}(n)$$

  If $\tilde{1}$ and 1 denote constant maps then $\tilde{m} \circ (\mathrm{id} \times \tilde{1})$ and $\mathrm{id} : \mathrm{Spin}(n) \to \mathrm{Spin}(n)$ lift $m(\mathrm{id} \times 1)$ and $\mathrm{id} : \mathrm{SO}(n) \to \mathrm{SO}(n)$, which are equal, and agree at $\tilde{1}$, hence are equal. Similarly, $\tilde{m} \circ (\tilde{1} \times \mathrm{id}) = \mathrm{id}$.

- Inverse:

$$\mathrm{Spin}(n) \xrightarrow{\mathrm{id} \times \tilde{\imath}} \mathrm{Spin}(n) \times \mathrm{Spin}(n) \xrightarrow{m} \mathrm{Spin}(n)$$

  $\tilde{m} \circ (\mathrm{id} \times \tilde{\imath})$ and $\tilde{1}$ lift $m \circ (\mathrm{id} \times i)$ and 1, which are equal, and agree at $\tilde{1}$, hence are equal. Similarly, $\tilde{m} \circ (\tilde{\imath} \times id) = \tilde{1}$.

The fact that $\tilde{m}$ lifts $m \circ (p \times p)$, i.e. that $p \circ \tilde{m} = m \circ (p \times p)$, means that $p : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is a group homomorphism.

If $x \in \mathrm{Spin}(n)$, let $-x$ denote the other point in the fiber over $p(x)$. We will show that $x \cdot -\tilde{1} = -\tilde{1} \cdot x = -x$. Let γ be a path in $\mathrm{SO}(n)$ that starts and ends at 1 and generates $\pi_1(\mathrm{SO}(n), 1)$. Then the lift $\tilde{\gamma}$ of γ that begins at $\tilde{1}$ ends at $-\tilde{1}$. Left multiplication by $p(x)$ is a homeomorphism $\mathrm{SO}(n) \to \mathrm{SO}(n)$, so $p(x) \cdot \gamma$ generates $\pi_1(\mathrm{SO}(n), p(x))$, so the lift of $p(x) \cdot \gamma$ that begins at x ends at $-x$. But $x \cdot \tilde{\gamma}$ is such a lift and ends at $x \cdot -\tilde{1}$, so $x \cdot -\tilde{1} = -x$. Similarly, $-\tilde{1} \cdot x = -x$.

Let $n < N$ and embed $\mathrm{SO}(n)$ in $\mathrm{SO}(N)$ by identifying $A \in \mathrm{SO}(n)$ with

$$\begin{pmatrix} A & \\ & 1 \end{pmatrix} \in \mathrm{SO}(N).$$

Let G be the subgroup $p^{-1}(\mathrm{SO}(n)) \subset \mathrm{Spin}(N)$:

$$\begin{array}{ccc} G & \longrightarrow & \mathrm{Spin}(N) \\ \downarrow & & \downarrow \\ \mathrm{SO}(n) & \longrightarrow & \mathrm{SO}(N). \end{array}$$

Then G is a double cover of $\mathrm{SO}(n)$, hence is either $\mathrm{Spin}(n)$ or is homeomorphic to two disjoint copies of $\mathrm{SO}(n)$. We will show that the latter is not possible because the two sheets of G are connected. Let γ be a loop in $\mathrm{SO}(n)$ that starts and ends at 1 and generates $\pi_1(\mathrm{SO}(n))$. Since the embedding of $\mathrm{SO}(n)$ in $\mathrm{SO}(N)$ induces a surjection on fundamental groups, γ also generates $\pi_1(\mathrm{SO}(N))$. Thus γ has a lift $\tilde{\gamma}$ that starts at $\tilde{1}$ and ends at $-\tilde{1}$. Since γ stays in $\mathrm{SO}(n)$, $\tilde{\gamma}$ stays in G, so the two sheets of G are connected.

Now embed SO(m) and SO(n) in SO($m+n$) by identifying $A \in \mathrm{SO}(m)$ and $B \in \mathrm{SO}(n)$ with

$$\begin{pmatrix} A & \\ & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & \\ & B \end{pmatrix}.$$

Observe that $AB = BA$. A copy of Spin(m) sits over SO(m), and a copy of Spin(n) sits over SO(n). Let $\tilde{A} \in \mathrm{Spin}(m)$ and $\tilde{B} \in \mathrm{Spin}(n)$ lift A and B respectively. We will show that $\tilde{A}\tilde{B} = \tilde{B}\tilde{A}$. Choose a path γ in Spin(m) from $\tilde{1}$ to $\tilde{A}$. Then $p \circ \gamma$ is a path in SO(m) from 1 to A. Now $\gamma \cdot \tilde{B}$ and $\tilde{B} \cdot \gamma$ lift $(p \circ \gamma) \cdot B$ and $B \cdot (p \circ \gamma)$, which are equal, and the paths upstairs both start at $\tilde{B}$, so their endpoints are the same: $\tilde{A}\tilde{B} = \tilde{B}\tilde{A}$.

2 Pin

As a space, O(n) is the disjoint union of two copies of SO(n). Let Pin(n) be the disjoint union of two copies of Spin(n), $p : \mathrm{Pin}(n) \to \mathrm{O}(n)$ the obvious map, $m : \mathrm{O}(n) \times \mathrm{O}(n) \to \mathrm{O}(n)$ the group operation, and $i : \mathrm{O}(n) \to \mathrm{O}(n)$ the inversion. Fix a reflection $r \in \mathrm{O}(n)$, and choose lifts $\tilde{1}$ of 1 and and $\tilde{r}$ of r.

As before, consider the diagram

$$\begin{array}{ccc} \mathrm{Pin}(n) \times \mathrm{Pin}(n) & & \mathrm{Pin}(n) \\ \Big\downarrow{\scriptstyle p \times p} & & \Big\downarrow{\scriptstyle p} \\ \mathrm{O}(n) \times \mathrm{O}(n) & \xrightarrow{\ m\ } & \mathrm{O}(n). \end{array}$$

To obtain a lift $\tilde{m} : \mathrm{Pin}(n) \times \mathrm{Pin}(n) \to \mathrm{Pin}(n)$ of $m \circ (p \times p)$, we must choose $\tilde{m}(\tilde{1}, \tilde{1})$, $\tilde{m}(\tilde{1}, \tilde{r})$, $\tilde{m}(\tilde{r}, \tilde{1})$, and $\tilde{m}(\tilde{r}, \tilde{r})$. Clearly the first three must be $\tilde{1}$, $\tilde{r}$, and $\tilde{r}$. The last must lift $r^2 = 1$, so we can have $\tilde{m}(\tilde{r}, \tilde{r}) = \tilde{1}$ or $\tilde{m}(\tilde{r}, \tilde{r}) = -\tilde{1}$. These two choices give us two groups, which we will call $\mathrm{Pin}_+(n)$ and $\mathrm{Pin}_-(n)$. For $\mathrm{Pin}_+(n)$ we should take $\tilde{\imath}(\tilde{1}) = \tilde{1}$ and $\tilde{\imath}(\tilde{r}) = \tilde{r}$, and for $\mathrm{Pin}_-(n)$, $\tilde{\imath}(\tilde{r}) = -\tilde{r}$.

We verify that Pin $+(n)$ and $\mathrm{Pin}_-(n)$ are groups, following our argument in the previous section. Since we are dealing with disconnected spaces, there are more points to check.

- Associativity: $\tilde{m} \circ (\tilde{m} \times \mathrm{id})$ and $\tilde{m} \circ (\mathrm{id} \times \tilde{m})$ lift equal maps and agree at each of the following points:

  $$(\tilde{1}, \tilde{1}, \tilde{1}),\ (\tilde{1}, \tilde{1}, \tilde{r}),\ (\tilde{1}, \tilde{r}, \tilde{1}),\ (\tilde{1}, \tilde{r}, \tilde{r}),\ (\tilde{r}, \tilde{1}, \tilde{1}),\ (\tilde{r}, \tilde{1}, \tilde{r}),\ (\tilde{r}, \tilde{r}, \tilde{1}),\ (\tilde{r}, \tilde{r}, \tilde{r}).$$

  The only point of interest is $(\tilde{r}, \tilde{r}, \tilde{r})$.

- Identity: $\tilde{m} \circ (\mathrm{id} \times \tilde{1})$ and $\mathrm{id} : \mathrm{Pin}_\pm(n) \to \mathrm{Pin}_\pm(n)$ lift equal maps and agree at $\tilde{1}$ and $\tilde{r}$, hence are equal. Similarly, $\tilde{m} \circ (\tilde{1} \times \mathrm{id}) = \mathrm{id}$.

- Inverse: $\tilde{m} \circ (\mathrm{id} \times \tilde{\imath})$ and $\tilde{1}$ lift equal maps and agree at $\tilde{1}$ and $\tilde{r}$, hence are equal. Similarly, $\tilde{m} \circ (\tilde{\imath} \times \mathrm{id}) = \tilde{1}$.

Let $n < N$ and embed O(n) in O(N) as before, and let $G = p^{-1}(\mathrm{O}(n)) \subset \mathrm{Pin}_\pm(N)$. Then the component of G lying over $\mathrm{SO}(n) \subset \mathrm{O}(n)$ is Spin(n) as we showed above. If $\tilde{r} \in G$ lifts a reflection $r \in O(n)$ then $\tilde{r}^2 = \pm 1$ since multiplication in G is just the restriction of multiplication in $\mathrm{Pin}_\pm(N)$. Thus $G = \mathrm{Pin}_\pm(n)$.

Now embed $\mathrm{O}(m) \times \mathrm{O}(n)$ in O($m+n$) as before, and let $\tilde{A} \in \mathrm{Pin}_\pm(m)$ and $\tilde{B} \in \mathrm{Pin}_\pm(n)$ lift $A \in \mathrm{O}(m)$ and $B \in \mathrm{O}(n)$. We will show that $\tilde{A}\tilde{B} = \tilde{B}\tilde{A}$ if either of A and B is a rotation. If A is a rotation, choose a path γ in $\mathrm{Pin}_\pm(m)$ from $\tilde{1}$ to $\tilde{A}$. Then $\gamma \cdot \tilde{B}$ and $\tilde{B} \cdot \gamma$ have the same image downstairs and both start at $\tilde{B}$, so their endpoints are equal: $\tilde{A}\tilde{B} = \tilde{B}\tilde{A}$. Similarly, if B is a rotation then $\tilde{A}$ and $\tilde{B}$ commute.

If A and B are both reflections, we will show that $\tilde{A}\tilde{B} = -\tilde{B}\tilde{A}$. Let

$$r = \begin{pmatrix} -1 &&&&&&& \\ & 1 &&&&&& \\ && \ddots &&&&& \\ &&& 1 &&&& \\ &&&& 1 &&& \\ &&&&& 1 && \\ &&&&&& \ddots & \\ &&&&&&& 1 \end{pmatrix} \qquad \gamma(t) = \begin{pmatrix} \cos t &&&& -\sin t &&& \\ & 1 &&&&&& \\ && \ddots &&&&& \\ &&& 1 &&&& \\ \sin t &&&& \cos t &&& \\ &&&&& 1 && \\ &&&&&& \ddots & \\ &&&&&&& 1 \end{pmatrix}$$

where the lower $\cos t$ is in the $(m+1, m+1)$ entry. Choose a lift $\tilde{r}$ of r and a lift $\tilde{\gamma}$ of γ such that $\tilde{\gamma}(0) = \tilde{1}$. Then $\tilde{\gamma}(t)\tilde{r}$ and $\tilde{r}\tilde{\gamma}(-t)$ lift $\gamma(t)r = r\gamma(-t)$ and agree at $t = 0$, hence are equal. Since $\gamma|_{[-\pi,\pi]}$ generates $\pi_1(SO(m+n))$, we have $\tilde{\gamma}(-\pi) = -\tilde{\gamma}(\pi)$. Observe that $r \in O(m)$ and $r\gamma(\pi) \in O(n)$, and choose paths γ_A in $\mathrm{Pin}_\pm(m)$ from $\tilde{r}$ to $\tilde{A}$ and γ_B in $\mathrm{Pin}_\pm(n)$ from $\tilde{r}\tilde{\gamma}(\pi)$ to $\tilde{B}$. Then $\gamma_A \cdot \gamma_B$ and $-\gamma_B \cdot \gamma_A$ have the same image downstairs and start at $\tilde{r}^2\tilde{\gamma}(\pi) = -\tilde{r}\tilde{\gamma}(\pi)\tilde{r}$, so their endpoints are equal: $\tilde{A}\tilde{B} = -\tilde{B}\tilde{A}$.

With $\mathrm{SL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{R})$ the situation is the same as with $\mathrm{O}(n)$.

3 $\mathrm{Spin}^{\mathbb{C}}$

We define $\mathrm{Spin}^{\mathbb{C}}(n) = \mathrm{U}(1) \times \mathrm{Spin}(n)/\pm(1,1)$. Some authors write this as $\mathrm{Spin}(n) \times_{\mathbb{Z}/2} \mathrm{U}(1)$ to underscore the fact that it is an associated bundle. It is a central extension of $\mathrm{SO}(n)$ by $\mathrm{U}(1)$:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{U}(1) & \longrightarrow & \mathrm{Spin}^{\mathbb{C}}(n) & \longrightarrow & \mathrm{SO}(n) & \longrightarrow & 1 \\ & & z & \longmapsto & (z,1) & & & & \\ & & & & (z,A) & \longmapsto & p(A). & & \end{array}$$

A vector bundle E on X admits a $\mathrm{Spin}^{\mathbb{C}}(n)$-structure if and only if it maps to zero under the connecting homomorphism

$$\check{H}^1(X; \mathrm{SO}(n)) \longrightarrow \check{H}^2(X; \mathrm{U}(1)).$$

But from the short exact sequence

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{R} & \longrightarrow & \mathrm{U}(1) & \longrightarrow & 1 \\ & & & & x & \longmapsto & e^{2\pi i x} & & \end{array}$$

we know that $\check{H}^2(X; \mathrm{U}(1)) = \check{H}^3(X; \mathbb{Z})$, so it is natural to ask what the image of E in $\check{H}^3(X;\mathbb{Z})$ is in terms of the characteristic classes of E. The answer is the third integral Stiefel-Whitney class $W_3 = \beta(w_2(E))$, where β is the Bockstein homomorphism coming from the short exact sequence

$$1 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 1.$$

We can see this by considering the short exact sequences

$$\begin{array}{ccccc} \mathbb{Z} & \longrightarrow & \mathbb{R} & \longrightarrow & \mathrm{U}(1) \\ \downarrow 2 & & \downarrow 2 & & \downarrow \\ \mathbb{Z} & \xrightarrow{m \mapsto (m,(-1)^m)} & \mathbb{R} \times \mathrm{Spin}(n) & \xrightarrow{(x,A) \mapsto (e^{i\pi x}, A)} & \mathrm{Spin}^{\mathbb{C}}(n) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Z}/2 & \longrightarrow & \mathrm{Spin}(n) & \longrightarrow & \mathrm{SO}(n). \end{array}$$

Thus E admits a $\mathrm{Spin}^{\mathbb{C}}(n)$-structure if and only if $w_2(E)$ is the reduction mod 2 of an integral class.

We could have worked just as well with $\mathrm{Pin}_+^{\mathbb{C}}(n)$.

It seems very difficult to find an exposition of the integral Stiefel-Whitney classes, so let us say a word here. We define $W_i = \beta(w_{i-1})$. Some authors seem to suggest that W_i is an integral lift of w_i, that is, $\rho(W_i) = w_i$, where ρ is reduction mod 2. This is not true: $\rho \circ \beta = \mathrm{Sq}^1$, and $\mathrm{Sq}^1(w_{i-1}) = w_1 \smile w_{i-1} - (i-2)w_i$.

It is well-known that

$$H^*(B\,\mathrm{O}(n); \mathbb{Z}/2) = \mathbb{Z}/2[w_1, \ldots, w_n],$$

so any mod-2 characteristic class is a polynomial in the Stiefel-Whitney classes. It is also true that

$$H^*(B\,\mathrm{O}(n); \mathbb{Z}) = \mathbb{Z}[p_1, \ldots, p_{\lfloor n/2 \rfloor}] \oplus \operatorname{im} \beta$$

as groups, so any integral characteristic class is a polynomial in the the Pontryagin classes and the integral Stiefel-Whitney classes. For more detail see Appendix B of [3] or the exercises in Chapter 15 of [1].

4 First Chern Class

It is well-known that the first Chern class of a complex line bundle is the connecting homomorphism

$$\check{H}^1(X;\mathbb{C}^\times) \xrightarrow{c_1} \check{H}^2(X;\mathbb{Z})$$

coming from the short exact sequence

$$1 \longrightarrow \mathbb{Z} \xrightarrow{2\pi i} \mathbb{C} \xrightarrow{\exp} \mathbb{C}^\times \longrightarrow 1.$$

It is also well-known that if E is any complex vector bundle then $c_1(E) = c_1(\det E)$, where $\det E = \bigwedge^{\text{top}} E$.

Now in general, c_1 is the connecting homomorphism

$$\check{H}^1(X;\mathrm{GL}_n(\mathbb{C})) \xrightarrow{c_1} \check{H}^2(X;\mathbb{Z})$$

coming from the short exact sequence

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z} & \xrightarrow{2\pi i/n} & \mathbb{C}\times\mathrm{SL}_n(\mathbb{C}) & \longrightarrow & \mathrm{GL}_n(\mathbb{C}) & \longrightarrow & 1 \\ & & & & (z,A) & \longmapsto & e^z A, & & \end{array}$$

as we see from the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{C}\times\mathrm{SL}_n(\mathbb{C}) & \longrightarrow & \mathrm{GL}_n(\mathbb{C}) & \longrightarrow & 1 \\ & & \| & & \downarrow n & & \downarrow \det & & \\ 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{C} & \longrightarrow & \mathbb{C}^\times & \longrightarrow & 1. \end{array}$$

It is interesting to note that c_1 is associated with the universal cover of $\mathrm{GL}_n(\mathbb{C})$, just as w_2 is associated with the universal cover of $\mathrm{GL}_n(\mathbb{R})$.

5 Accidental Isomorphisms

From their Dynkin diagrams we know that the groups $\mathrm{Spin}(n)$ are isomorphic to other classical Lie groups for small n, but if one wants to compute one needs the explicit isomorphisms, which are hard to find written down. The most economical way to describe the isomorphisms is to describe the actions on $\mathbb{R}^n$.

- $\mathrm{Spin}(3) = \mathrm{Sp}(1)$. Identify $\mathbb{R}^3$ with the imaginary quaternions and let $q \in \mathrm{Sp}(1)$ act by $x \mapsto qx\bar{q}$.
- $\mathrm{Spin}(4) = \mathrm{Sp}(1)\times\mathrm{Sp}(1)$. Identify $\mathbb{R}^4$ with $\mathbb{H}$ and let $(p,q) \in \mathrm{Sp}(1)\times\mathrm{Sp}(1)$ act by $x \mapsto px\bar{q}$.
- $\mathrm{Spin}(5) = \mathrm{Sp}(2)$. Let $\mathrm{Sp}(2)$ act on the 16-dimensional vector space $\mathfrak{gl}_2(\mathbb{H})$ by conjugation. If we endow $\mathfrak{gl}_2(\mathbb{H})$ with the (real) inner product $\langle X, Y\rangle = \mathrm{Re}(\mathrm{tr}\, XY^*)$ then $\mathrm{Sp}(2)$ acts by isometries. The 10-dimensional subspace $\mathfrak{sp}(2)$ of skew-adjoint matrices is invariant—this is the adjoint representation. The 1-dimensional subspace of real scalar matrices is also invariant—this is the trivial representation. The remaining 5-dimensional subspace of traceless self-adjoint matrices is the one we want.
- $\mathrm{Spin}(6) = \mathrm{SU}(4)$. Let $\mathbb{C}^4$ have basis $e_1, \ldots, e_4$ and let $V = \bigwedge^2 \mathbb{C}^4$. The usual Hermitian inner product on $\mathbb{C}^4$ induces one on V. If $v \in V$, define $*v \in V$ by the formula

  $$u \wedge *v = \langle u, v\rangle \cdot e_1 \wedge e_2 \wedge e_3 \wedge e_4$$

  for all $u \in V$. Then the natural action of $\mathrm{SU}(4)$ on V fixes the the subspace $\{v \in V : v = *v\}$ of self-dual forms, which has real dimension 6.

References

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[2] John M. Lee. *Introduction to Smooth Manifolds.* Springer–Verlag, New York, 2002.

[3] H. Blaine Lawson, Jr. and Marie-Louise Michelsohn. *Spin Geometry.* Princeton University Press, Princeton, 1989.