MATH 751: Homework III

*Due Date: Thursday, November 20**

1. Identify the vector space of all $n \times n$ real matrices with the smooth manifold $\mathbb{R}^{n^2}$. Let $Y_n = SO(n)$ denote the *topological subspace* of all $n \times n$ orthogonal matrices of determinant one. Using the natural embedding

$$A \longmapsto \begin{pmatrix} & & & 0 \\ & A & & \vdots \\ & & & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}$$

we obtain an inclusion $Y_n \subset Y_{n+1}$.

(a) Prove that Y_n is a compact smooth manifold for all $n \geq 1$. What is its dimension?

(b) Show that Y_{n-1} is a closed subgroup of Y_n, and that for $n \geq 2$, the space of left cosets Y_n/Y_{n-1} is diffeomorphic to $\mathbb{S}^{n-1}$.

(c) Using induction and part (b), show that Y_n is connected for all values of $n \geq 1$.

2. Prove that $SO(3)$ is diffeomorphic to $\mathbb{R}P^3$.

3. Show that for each $n \geq 0$ there is a diffeomorphism $T\mathbb{S}^n \times \mathbb{R} \simeq \mathbb{S}^n \times \mathbb{R}^{n+1}$.

4. Show that a product of two spheres can be embedded in Euclidean space of one dimension higher.

5. Let M denote a smooth compact manifold of dimension n and let $f : M \to \mathbb{R}^n$ denote a smooth map. Show that f cannot be everywhere nonsingular.

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* *Problems 3 & 4 are from Differential Topology by M.Hirsch.*