Consider the following second-order linear difference equation
\[ f(n) = af(n - 1) + bf(n + 1), \quad K < n < N, \] (1)
where \( f(n) \) is a function defined on the integers \( K \leq n \leq N \), the value \( N \) can be chosen to be infinity, and \( a \) and \( b \) are nonzero real numbers. Note that if \( f \) satisfies (1) and if the values \( f(K), f(K + 1) \) are known then \( f(n) \) can be determined for all \( K \leq n \leq N \) recursively via the formula
\[ f(n + 1) = \frac{1}{b} [f(n) - af(n - 1)]. \]

Note also that if \( f_1 \) and \( f_2 \) are two solutions of (1), then \( c_1 f_1 + c_2 f_2 \) is a solution for any real numbers \( c_1, c_2 \). Therefore, the solution space of (1) is a two-dimensional vector space and one basis for the space is \( \{f_1, f_2\} \) with \( f_1(K) = 1, f_1(K + 1) = 0 \) and \( f_2(K) = 0, f_2(K + 1) = 1 \).

We will solve (1) by looking for solutions of the form \( f(n) = \alpha^n \), for some \( \alpha \neq 0 \). Plugging \( \alpha^n \) into equation (1) yields
\[ \alpha^n = a\alpha^{n-1} + b\alpha^{n+1}, \quad K < n < N, \]
or
\[ \alpha = a + ba^2 \iff ba^2 - \alpha + a = 0. \]
Solving this quadratic gives
\[ \alpha = \frac{1 \pm \sqrt{1 - 4ba}}{2b}. \] (2)

There are two cases that need handling based upon whether or not the discriminant, \( 1 - 4ba \), is zero.

**Case 1:** If \( 1 - 4ba \neq 0 \), we find two solutions, \( \alpha_1 \) and \( \alpha_2 \), and see that the general solution to the difference equation (1) is
\[ c_1 \alpha_1^n + c_2 \alpha_2^n, \]
with \( c_1, c_2 \) found depending upon the boundary conditions. If \( 1 - 4ba < 0 \), then the roots are complex and the general solution is found by switching to polar coordinates. That is, we let \( \alpha = re^{i\theta} \), and find
\[ f(n) = r^n e^{in\theta} = r^n \cos(n\theta) \pm ir^n \sin(n\theta), \]
are solutions, implying both the real and imaginary parts are solutions. Therefore, the general solution is
\[ c_1 r^n \cos(n\theta) + c_2 r^n \sin(n\theta), \]
with \( c_1, c_2 \) found depending upon the boundary conditions.

**Case 2:** If \( 1 - 4ba = 0 \), we only find the one solution, \( f_1(n) = (1/2b)^n \) by solving the quadratic.
However, let $f_2(n) = n(2b)^{-n}$. We have that 

$$af_2(n - 1) + bf_2(n + 1) = a(n - 1)(2b)^{-(n-1)} + b(n + 1)(2b)^{-(n+1)}$$

$$= \left(\frac{1}{2b}\right)^n \left[a(n - 1)2b + b(n + 1)\frac{1}{2b}\right]$$

$$= \left(\frac{1}{2b}\right)^n \left[(n - 1)\frac{1}{2} + (n + 1)\frac{1}{2}\right]$$

(remember, $4ab = 1$)

$$= \left(\frac{1}{2b}\right)^n n$$

$$= f_2(n).$$

Note that $f_2$ is obviously linearly independent from $f_1$. Thus, when $4ab = 1$, the general form of the solution is

$$f(n) = c_1 \left(\frac{1}{2b}\right)^n + c_2 n \left(\frac{1}{2b}\right)^n,$$

with $c_1, c_2$ found depending upon the boundary conditions.

**Example 1.** Find a function $f(n)$ satisfying

$$f(n) = 2f(n - 1) + \frac{1}{10} f(n + 1), \quad 0 < n < \infty,$$

with $f(0) = 8, f(1) = 2$.

**Solution.** Here, $a = 2$ and $b = 1/10$. Therefore, plugging into (2) gives

$$\alpha = 5 \pm \sqrt{5},$$

and the general solution is

$$f(n) = c_1 \left(5 + \sqrt{5}\right)^n + c_2 \left(5 - \sqrt{5}\right)^n.$$

Using the boundary conditions yields

$$8 = f(0) = c_1 + c_2$$

$$2 = f(1) = c_1 (5 + \sqrt{5}) + c_2 (5 - \sqrt{5}),$$

which has solution $c_1 = 4 - 19\sqrt{5}/5$, $c_2 = 4 + 19\sqrt{5}/5$. Thus, the solution to the problem is

$$f(n) = \left(4 - \frac{19\sqrt{5}}{5}\right) \left(5 + \sqrt{5}\right)^n + \left(4 + \frac{19\sqrt{5}}{5}\right) \left(5 - \sqrt{5}\right)^n.$$ 

Some of the most important difference equations we will see in this course are those of the form

$$f(n) = pf(n - 1) + qf(n + 1), \quad \text{with } p + q = 1, \quad p, q \geq 0.$$ 

These will arise when studying random walks with $p$ and $q$ interpreted as the associated probabilities of moving right or left. Supposing that $p \neq q$, the roots of the quadratic formula (2) can be found:

$$\frac{1 \pm \sqrt{1 - 4(1-p)p}}{2q} = \frac{1 \pm \sqrt{(1-2p)^2}}{2q} = \frac{1 \pm |q-p|}{2q} = \left\{1, \frac{p}{q}\right\}.$$
Thus, the general solution when $p \neq 1/2$ is

$$f(n) = c_1 + c_2 \left( \frac{p}{q} \right)^n.$$  

For the case that $p = q = 1/2$, the only root is 1, hence the general solution is

$$f(n) = c_1 + c_2 n.$$  

We analyzed only second-order linear difference equations above. However, and similar to the study of differential equations, higher order difference equations can be studied in the same manner. Consider the general $k$th order, homogeneous linear difference equation:

$$f(n + k) = a_0 f(n) + a_1 f(n + 1) + \cdots + a_{k-1} f(n + k - 1), \quad (3)$$

where we are given $f(0), f(1), \ldots, f(k-1)$. Then, again, we may solve for the general $f(n)$ recursively using (3). We look for solutions of the form $f(n) = \alpha^n$, which is a solution if and only if

$$\alpha^k = a_0 + a_1 \alpha + \cdots + a_{k-1} \alpha^{k-1}.$$  

If there are $k$ distinct roots of the above equation, then we automatically get $k$ linearly independent solutions to (3). However, if a given root $\alpha$ is a root with a multiplicity of $j$, then

$$\alpha^n, n\alpha^n, \ldots, n^{j-1} \alpha^n,$$

are linearly independent solutions. We can then use the given initial conditions to find the desired particular solution.