Formality of derived intersections and the orbifold HKR isomorphism

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Abstract

We study when the derived intersection of two smooth subvarieties of a smooth variety is formal. As a consequence we obtain a derived base change theorem for non-transversal intersections. We also obtain applications to the study of the derived fixed locus of a finite group action and argue that for a global quotient orbifold the exponential map is an isomorphism between the Lie algebra of the free loop space and the loop space itself. This allows us to give new proofs of the HKR decomposition of orbifold Hochschild (co)homology into twisted sectors.

1. Introduction

1.1. Let S be a smooth variety over a field of characteristic zero and let X and Y be smooth subvarieties of S. We shall assume that X and Y intersect cleanly (meaning that their scheme theoretic intersection $W = X \times_S Y$ is smooth) but not necessarily transversely. Derived algebraic geometry associates to this data a geometric object, the *derived intersection* of X and Y,

$$W' = X \times_S^R Y.$$

It is a differential graded (dg) scheme whose structure complex is constructed by taking the derived tensor product of the structure sheaves of X and Y. (The reader unfamiliar with the subject of dg schemes is referred to Section 2.) The underived intersection W naturally sits inside W' as a closed subscheme.

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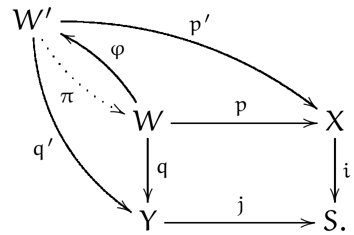
We organize these spaces and the maps between them in the diagram



The purpose of this note is to understand when W' is as simple as possible. Our main result (Theorem 1.8) makes this precise in two ways. In algebraic terms it describes when W' is *formal* in the sense of [7]. In geometric terms it gives a necessary and sufficient condition for the existence of a map $\pi : W' \to W$ exhibiting W' as the total space of a (shift of a) vector bundle over W. When this holds we gain a geometric understanding of the structure of the maps φ, p' and q': φ is the inclusion of the zero section of the bundle, and p' and q' factor through the bundle map π.

1.2. The problem we study originates in classical intersection theory. While the scheme-theoretic intersection W is determined algebraically by the *underived* tensor product

$$\mathscr{O}_W = \mathscr{O}_X \otimes_{\mathscr{O}_S} \mathscr{O}_Y,$$

Serre [11] argued that in order to obtain a theory with good formal properties we need to use instead the *derived* tensor product

$$\mathscr{O}_{W'} = \mathscr{O}_X \otimes^{\mathsf{L}}_{\mathscr{O}_S} \mathscr{O}_Y.$$

Since $\mathscr{O}_{W'}$ is naturally a commutative dg algebra we can regard it as the structure complex of a dg scheme W'.

1.3. For classical applications it suffices to work with the class of $\mathscr{O}_{W'}$ in K-theory. Put differently, we only need to know the sheaves

$$\mathscr{H}^{-*}(\mathscr{O}_{W'}) = \mathsf{Tor}^S_*(\mathscr{O}_X, \mathscr{O}_Y).$$

A local calculation as in [5, Proposition A.3] shows that these sheaves are the exterior powers $\wedge^* E^\vee$ of the *excess bundle* E, the vector bundle on W defined by

$$E = \frac{T_S}{T_X + T_Y}.$$

(We omit writing the restrictions from X, Y, S to W in formulas like the one above. So when we write T_X we mean $T_X|_W$, the restriction of the tangent bundle of X to W.) The excess bundle E is a vector bundle on W of rank $\dim S + \dim W - \dim X - \dim Y$ which measures the failure of the intersection to be transversal.

1.4. For certain problems it is not enough to know just the Tor sheaves $\mathscr{H}^k(\mathscr{O}_{W'})$; we need to understand the full dg algebra $\mathscr{O}_{W'}$. For example there is considerable interest in computing $\mathrm{Ext}^*_S(i_*F, j_*G)$ for vector bundles F and G on X and on Y. These groups can be computed using the spectral sequence

$$^2E^{pq} = H^p(W, F^\vee \otimes G \otimes \omega_{W/Y} \otimes \wedge^{q-m} E) \Rightarrow \mathrm{Ext}^{p+q}_S(i_*F, j_*G),$$

from [5, Theorem A.1], where $\omega_{W/Y}$ denotes the relative dualizing sheaf of the embedding $W \to Y$ and m denotes the codimension of W inside Y. (Again, we omit the restrictions of F and G to W.) The differentials in this spectral sequence arise as obstructions to splitting the canonical filtration on $\mathscr{O}_{W'}$, that is, they vanish if there is an isomorphism

$$\mathscr{O}_{W'} \cong \bigoplus_k \mathscr{H}^k(\mathscr{O}_{W'})[-k].$$

1.5. In the above discussion we have skated over an important detail. The splitting of $\mathscr{O}_{W'}$ is not an intrinsic property of the dg scheme W'; rather, the concept only makes sense for a morphism from W' to to a base scheme. We discuss its relation to a more general notion, that of formality.

Consider a dg scheme Z' which is affine over an ordinary scheme Z, i.e., the dg scheme Z' is endowed with a structure morphism $s : Z' \to Z$ such that s is affine. We shall consider two related dg algebras over $\mathscr{O}_Z$. One is $s_*\mathscr{O}_{Z'}$; the other is its associated graded counterpart

$$\mathscr{O}_{\hat{Z}'} = \bigoplus_k \mathscr{H}^k(s_*\mathscr{O}_{Z'})[-k].$$

Note that the right hand side inherits an associative commutative product structure from that of $\mathscr{O}_{Z'}$, so it can be regarded as the structure complex of a dg scheme $\hat{Z}'$ over Z.

We shall say that Z' is formal over Z if there exists an isomorphism $Z' \cong \hat{Z}'$ in the category of derived dg schemes over Z. (See Section 2 for further details.) This is equivalent to saying that the dg algebra $s_*\mathscr{O}_{Z'}$ is a formal $\mathscr{O}_Z$-dg algebra, that is, there exists an isomorphism of *dg algebras* over $\mathscr{O}_Z$

$$s_*\mathscr{O}_{Z'} \cong \mathscr{O}_{\hat{Z}'}.$$

(The terminology is inspired by the work [7] of Deligne, Griffiths, Morgan, and Sullivan, where the authors define a smooth manifold to be formal if its de Rham dg algebra is formal in the above sense.)

Note that in particular Z' being formal over Z implies that the complex $s_* \mathcal{O}_{Z'}$ is split in $\mathbf{D}(Z)$ (it is isomorphic, as an $\mathcal{O}_Z$-module, to the sum of its cohomology sheaves). In our context this is a non-trivial condition, unlike the situation in [7].

1.6. The derived intersection $W' = X \times^R_S Y$ can be viewed as a dg scheme over several base schemes: either one of X, Y, S, or $X \times Y$ can naturally serve as an underlying scheme for W'. (However, note that in general we can not present W' as a dg scheme over W.) Our primary motivation for studying derived intersections comes from our desire to understand the degeneration of the spectral sequence in (1.4). For this purpose it is most useful to regard W' as a dg scheme over $X \times Y$. Indeed, in this approach the structure sheaf $\mathcal{O}_{W'}$ of W' is the kernel of the functor $j^* i_* : \mathbf{D}(X) \to \mathbf{D}(Y)$ between dg enhancements $\mathbf{D}(X)$, $\mathbf{D}(Y)$ of the derived categories of X and Y. (We omit the $\mathbf{R}$'s and $\mathbf{L}$'s in front of derived functors for simplicity.)

1.7. Formality of $W'/X \times Y$ turns out to be closely related to properties of the inclusion $W \hookrightarrow W'$. We shall say that a map of spaces $W \to W'$ *splits* if it admits a left inverse. (The term "splitting" may be misleading: it might help the reader to think of split embeddings as algebro-geometric analogues of deformation retractions in topology.) If $W \to W'$ is a closed embedding we shall say that it *splits to first order* if the induced map $W \to W'^{(1)}$ splits, where $W'^{(1)}$ is the first infinitesimal neighborhood of W inside W'.

The above concepts also make perfect sense for spaces (schemes, dg schemes) over a fixed base scheme, in which case we require the inverse map to be a map over the base scheme.

We are now ready to state the main theorem of the paper [1].

1.8. Theorem. *The following statements are equivalent.*

(1) *There exists an isomorphism of dg functors* $\mathbf{D}(X) \to \mathbf{D}(Y)$

$$j^* i_*(-) \cong q_*(p^*(-) \otimes \mathrm{Sym}(E^\vee[1])).$$

(2) *There exists an isomorphism* $W' \cong \mathbb{E}[-1]$ *of dg schemes over* $X \times Y$.

(3) W' *is formal as a dg scheme over* $X \times Y$.

[1] While working on the final draft of this paper the authors became aware that a closely related result was obtained independently and at about the same time by Grivaux [9].

(4) *The inclusion* $W \to W'$ *splits over* $X \times Y$.

(5) *The inclusion* $W \to W'$ *splits to first order over* $X \times Y$.

(6) *The short exact sequence*

$$0 \to T_X + T_Y \to T_S \to E \to 0$$

of vector bundles on W *splits.*

1.9. The above theorem can be seen as a generalization of several classical results: base change for flat morphisms or, more generally, for Tor-independent morphisms; the Hochschild-Kostant-Rosenberg isomorphism for schemes [12]; and the formality theorem for derived self-intersections of the first two authors [1]. A slightly modified version of this theorem was used in a twisted context in [2] to prove a theorem on the formality of the twisted de Rham complex.

1.10. However, the main application we have in mind for Theorem 1.8 is in the study of derived fixed loci. Let φ be a finite-order automorphism of a smooth variety Z. We are interested in the fixed locus W of φ,

$$W = Z^{\varphi} = \{z \in Z \mid \varphi(z) = z\}.$$

This fixed locus can be studied using intersection theory, as we can view W as the intersection (inside $Z \times Z$) of the diagonal Δ and the graph Δ^{φ} of φ,

$$W = \Delta \times_{Z \times Z} \Delta^{\varphi}.$$

1.11. This description makes it clear that the expected dimension of the fixed locus is zero. Whenever W is positive dimensional the cause is a failure of transversality of Δ and Δ^{φ}. It then makes sense to study the *derived fixed locus* of φ, W', which we define as the derived intersection

$$W' = \Delta \times^{R}_{Z \times Z} \Delta^{\varphi}.$$

The excess intersection bundle E for this problem is easily seen to be precisely T_W, the tangent bundle of the underived fixed locus W.

In this setup Theorem 1.8 allows us to get the following geometric characterization of the derived fixed locus W'.

1.12. Corollary. *The derived fixed locus* W' *is isomorphic, as a dg scheme over* $Z \times Z$*, to the total space over* W *of the dg vector bundle* $T_W[-1]$,

$$W' \cong \mathbb{T}_W[-1].$$

1.13. We apply the above result to the study of loop spaces of orbifolds. Recall that for a space X one defines the *free loop space* of X as

$$\mathsf{LX} = \mathsf{X} \times^{\mathsf{R}}_{\mathsf{X}\times\mathsf{X}} \mathsf{X}.$$

It is naturally a formal derived group scheme over X.

When X is a smooth scheme, the relative Lie algebra of LX/X can be identified with $\mathbb{T}_{\mathsf{X}}[-1]$, the total space of the shifted tangent bundle $\mathsf{T}_{\mathsf{X}}[-1]$. In this case the exponential map is an isomorphism (commonly called the Hochschild-Kostant-Rosenberg isomorphism, or HKR)

$$\exp : \mathbb{T}_{\mathsf{X}}[-1] \xrightarrow{\sim} \mathsf{LX}.$$

The non-trivial part of the above statement is the fact that the exponential map is an isomorphism not only in a formal neighborhood of the origin, but in fact extends to the whole group. (This follows from the fact that the loop space LX is a nilpotent extension of X.)

1.14. The above statement is known to fail for Artin stacks. See for example Ben-Zvi and Nadler [4], where the authors prove that in this case only the formal version of the HKR isomorphism holds.

By contrast, as an application of the formality of derived fixed loci, we prove that the HKR isomorphism theorem still holds for global quotient orbifolds (global quotient Deligne-Mumford stacks). The setting we will work with is as follows. Let G be a finite group acting on a smooth variety Z, and let $\mathscr{X}$ be the quotient stack $[\mathsf{Z}/\mathsf{G}]$. Denote by $\mathrm{I}\mathscr{X}$ the (underived) inertia stack

$$\mathrm{I}\mathscr{X} = \mathscr{X} \times_{\mathscr{X}\times\mathscr{X}} \mathscr{X},$$

by $\mathbb{T}_{\mathrm{I}\mathscr{X}}[-1]$ its shifted tangent bundle, and by $\mathsf{L}\mathscr{X}$ the free loop space of $\mathscr{X}$

$$\mathsf{L}\mathscr{X} = \mathscr{X} \times^{\mathsf{R}}_{\mathscr{X}\times\mathscr{X}} \mathscr{X}.$$

1.15. Theorem (Orbifold HKR isomorphism). *Let $\mathscr{X}$ be a global quotient orbifold. Then there exists a canonical isomorphism*

$$\exp : \mathbb{T}_{\mathrm{I}\mathscr{X}}[-1] \xrightarrow{\sim} \mathsf{L}\mathscr{X}$$

between the shifted tangent bundle $\mathbb{T}_{\mathrm{I}\mathscr{X}}[-1]$ of its inertia orbifold and its free loop space $\mathsf{L}\mathscr{X}$.

1.16. As in the case of the usual HKR isomorphism for smooth schemes, the above theorem allows us to give a decomposition of the Hochschild (co)homology of the orbifold $\mathscr{X}$. In order to state this decomposition in more concrete terms we need the following notations for $g \in G$:

- Z^g is the (underived) fixed locus of g in Z;
- i_g is the closed embedding of Z^g in Z;
- c_g is the codimension of Z^g in Z;
- ω_g is the relative dualizing bundle of the embedding i_g, that is, the top exterior power of the normal bundle $N_{Z^g/Z}$ of Z^g in Z,

$$\omega_g = \wedge^{c_g} N_{Z^g/Z};$$

- T_g is the vector bundle on Z^g obtained by taking coinvariants of $T_Z|_{Z^g}$ with respect to the action of g;
- Ω_g^j is the dual, along Z^g, of $\wedge^j T_g$;
- $\mathsf{Sym}(\Omega_g^1[1])$ is the symmetric algebra of $\Omega_g^1[1])$, i.e., the object of $\mathbf{D}(Z^g)$

$$\mathsf{Sym}(\Omega_g^1[1]) = \oplus \Omega_g^j[j].$$

With these notations we can phrase the following consequence of Theorem 1.15.

1.17. Corollary. *The two projections $p', q' : L\mathscr{X} \to \mathscr{X}'$ are homotopic (and hence equal in the derived category of dg stacks). There are natural isomorphisms of dg functors $\mathbf{D}(\mathscr{X}) \to \mathbf{D}(\mathscr{X})$*

$$\Delta^* \Delta_*(-) \cong q'_* p'^*(-) \cong - \otimes q'_* \mathscr{O}_{L\mathscr{X}}.$$

The object $q'_* \mathscr{O}_{L\mathscr{X}} \in \mathbf{D}(\mathscr{X})$ is represented by the G-equivariant object of $\mathbf{D}(Z)$

$$\bigoplus_{g \in G} i_{g,*} \mathsf{Sym}(\Omega_g^1[1]).$$

Therefore

(1) $\Delta^*\Delta_*\mathscr{O}_{\mathscr{X}} = \bigoplus_{g\in G} i_{g,*}\mathsf{Sym}(\Omega_Z^g[1])$.

(2) $\mathsf{HH}_*(\mathscr{X}) = \left(\bigoplus_{g\in G}\bigoplus_{q-p=*} H^p(Z^g,\Omega_g^q)\right)^G$.

(3) $\mathsf{HH}^*(\mathscr{X}) = \left(\bigoplus_{g\in G}\bigoplus_{p+q=*} H^{p-c_g}(Z^g,\wedge^q T_g\otimes\omega_g)\right)_G$.

This result generalizes known Hochschild-Kostant-Rosenberg isomorphisms for orbifolds, for example those of Baranovsky [3] and Ganter [8].

1.18. The paper is organized as follows. In Section 2 we collect some general results about dg schemes in the sense of Ciocan-Fontanine and Kapranov. In particular we discuss the concept of dg schemes relative to a base scheme and the concept of formality. We construct presentations of the derived intersection W' over X, Y, $X\times Y$, and S. In Section 3 we present the proof of Theorem 1.8. In the final section of the paper we discuss applications to orbifolds, and present proofs of Corollary 1.12, Theorem 1.15, and Corollary 1.17.

1.19. Conventions. We work over a field of characteristic zero. The same results also hold when the characteristic of the ground field is sufficiently large; we shall make it explicit in the statement of each theorem how large the characteristic needs to be for the results to hold. All schemes are assumed to be smooth, quasi-projective over this field.

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2. Background on dg schemes

In this section we review some facts from the basic theory of differential graded schemes, following the work of Ciocan-Fontanine and Kapranov [6]. We emphasize the point of view that a dg scheme $Z' = (Z,\mathscr{O}_{Z'})$ should be thought of as a dg scheme *over* Z, and explain how the derived intersection $W' = X\times_S^R Y$ can be viewed in a natural way as a dg scheme over X, Y, $X\times Y$, or S.

2.1. Following Ciocan-Fontanine and Kapranov [6], a *differential graded scheme* Z' is a pair $(\mathsf{Z}, \mathscr{O}_{\mathsf{Z}'})$ consisting of an ordinary scheme Z, the *base scheme* of Z', and a complex of quasi-coherent sheaves $\mathscr{O}^{\cdot}_{\mathsf{Z}'}$ on Z, the *structure complex* of Z'. The complex $\mathscr{O}_{\mathsf{Z}'}$ is assumed to be endowed with the structure of a commutative dg algebra over $\mathscr{O}_{\mathsf{Z}}$, and must satisfy

1. $\mathscr{O}^{i}_{\mathsf{Z}'} = 0$ for $i > 0$;
2. $\mathscr{O}^{0}_{\mathsf{Z}'} = \mathscr{O}_{\mathsf{Z}}$.

Maps between dg schemes are obtained by a localization procedure similar to the one that leads to the construction of derived categories. In a first stage morphisms of dg schemes are considered as maps of ringed spaces. For dg schemes $\mathsf{Z}' = (\mathsf{Z}, \mathscr{O}_{\mathsf{Z}'})$ and $\mathsf{W}' = (\mathsf{W}, \mathscr{O}_{\mathsf{W}'})$ a morphism $\mathsf{Z}' \to \mathsf{W}'$ consists of a map of schemes $f : \mathsf{Z} \to \mathsf{W}$ along with a map of dg algebras $f^{\#} : f^{*}\mathscr{O}_{\mathsf{W}'} \to \mathscr{O}_{\mathsf{Z}'}$. In the resulting category we have a natural notion of quasi-isomorphisms of dg schemes – those morphisms $(f, f^{\#})$ for which $f^{\#}$ is a quasi-isomorphism of complexes of sheaves. Formally inverting those quasi-isomorphisms produces a category $\mathfrak{DSch}$, the right derived category of schemes.

2.2. Because quasi-isomorphisms become isomorphisms in $\mathfrak{DSch}$, isomorphic dg schemes can be presented over different base schemes. Thus the base scheme is not an intrinsic part of a dg scheme in $\mathfrak{DSch}$. For certain purposes, however, it is useful to be able to refer to the base scheme of a dg scheme. Instead of carrying over this additional data, we give an alternative way of looking at the relationship between a dg scheme Z' and its supporting scheme Z.

The definition of dg schemes implies that the structure complex $\mathscr{O}_{\mathsf{Z}'}$ of a dg scheme $\mathsf{Z}' = (\mathsf{Z}, \mathscr{O}_{\mathsf{Z}'})$ admits a natural morphism of dg algebras $\mathscr{O}_{\mathsf{Z}} \to \mathscr{O}_{\mathsf{Z}'}$ (where $\mathscr{O}_{\mathsf{Z}}$ is regarded as a complex concentrated in degree zero). This shows that a dg scheme Z' presented over a base scheme Z comes with a canonical morphism $\mathsf{Z}' \to \mathsf{Z}$.

2.3. This observation motivates us to study dg schemes *over a fixed scheme* Z instead of arbitrary dg schemes. These are dg schemes Z' endowed with a morphism $\mathsf{Z}' \to \mathsf{Z}$. (We shall mostly be concerned with the situation when this morphism is *affine* – this is the case when the dg scheme Z' is presented over Z. But the concept makes sense in general.) As in the theory of schemes, morphisms of dg schemes over Z are morphisms between dg schemes which commute with the structure morphisms.

2.4. We now turn to discussing the construction of derived intersections over various bases. We place ourselves in the context described in the introduction, with X and Y subschemes of S. The structure complex of the derived intersection $W' = X \times^{\mathbf{R}}_{S} Y$ is obtained by taking the derived tensor product $\mathscr{O}_{W'} = \mathscr{O}_X \otimes^{\mathbf{L}}_{\mathscr{O}_S} \mathscr{O}_Y$.

The main question we want to address is over what base scheme should the complex $\mathscr{O}_{W'}$ be considered. If the schemes were affine, this would be equivalent to deciding whether to consider this tensor product as an algebra over $\mathscr{O}_X$, $\mathscr{O}_Y$, $\mathscr{O}_S$, etc. Likewise, in the general case there is no canonical choice of base scheme for the dg scheme W', and either one of X, Y, S, or $X \times Y$ can serve for this purpose. For example, it is easy to see W' as a dg scheme over X by resolving $\mathscr{O}_Y$ by a flat commutative dg algebra over S and pulling back the resolution to X. Similarly, in order to obtain a model over S resolve both $\mathscr{O}_X$ and $\mathscr{O}_Y$ over S and tensor them over $\mathscr{O}_S$.

It is essential to emphasize that in general it is not possible to present W' as a dg scheme over W, the underived intersection.

2.5. For the purpose of this article we are most interested in a model of W' whose base scheme is $X \times Y$. To obtain such a presentation define

$$\mathscr{O}_{W'} = \mathscr{O}_{\Gamma_i} \circ \mathscr{O}_{\Gamma_j} = \pi_{XY,*}(\pi^*_{XS}\mathscr{O}_{\Gamma_i} \otimes_{X\times S\times Y} \pi^*_{SY}\mathscr{O}_{\Gamma_j}),$$

the convolution of the kernels $\mathscr{O}_{\Gamma_i} \in \mathbf{D}(X \times S)$ and $\mathscr{O}_{\Gamma_j} \in \mathbf{D}(S \times Y)$. Here $\Gamma_i \subset X \times S$, $\Gamma_j \subset S \times Y$ are the graphs of the inclusions $i : X \hookrightarrow S$, $j : Y \hookrightarrow S$, and π_{XS}, π_{SY} and π_{XY} are the projections from $X \times S \times Y$ to $X \times S$, $S \times Y$, and $X \times Y$, respectively. (We omit the $\mathbf{R}$'s and $\mathbf{L}$'s in front of derived functors for simplicity.) The reader can easily supply the required equality of tensor products of rings which shows that this definition of W' is quasi-isomorphic to the previous ones.

Note that the kernels $\mathscr{O}_{\Gamma_i}$ and $\mathscr{O}_{\Gamma_j}$ induce the functors $i_* : \mathbf{D}(X) \to \mathbf{D}(S)$ and $j^* : \mathbf{D}(S) \to \mathbf{D}(Y)$. Since $\mathscr{O}_{W'}$ is the convolution of these kernels, we conclude that $\mathscr{O}_{W'}$ is the kernel of the dg functor $j^* i_* : \mathbf{D}(X) \to \mathbf{D}(Y)$.

This fact allows us to connect with our earlier discussion in (1.4). Indeed, in order to guarantee the degeneration of the spectral sequence computing $\mathrm{Ext}^*_S(i_*F, j_*G)$ we need to understand the functor $j^* i_*$. Since this functor is controlled by W' as presented over $X \times Y$, this explains why we want to understand formality properties of $W'/X \times Y$ and not over other bases.

2.6. There is another description of $\mathscr{O}_{W'}$ as an object in $\mathbf{D}(X \times Y)$ which is useful in the proof of Theorem 1.8. The original problem of studying the intersection of X and Y into S can be reformulated to study the intersection of $X \times Y$ with the diagonal in $S \times S$. Let $\bar{\imath}$ and $\bar{\jmath}$ be the embeddings of S and $X \times Y$ into $S \times S$.

The derived and underived intersections in the new problem are the same as in the old one. The excess bundle is also the same. However, by replacing the original problem with the new one we have simplified the initial situation in two ways. First, the embedding $\bar{\imath} : S \hookrightarrow S \times S$ is now split. Second, since the object $\bar{\jmath}^* \bar{\imath}_* \mathscr{O}_S$ realizes $\mathscr{O}_{W'}$ as an object of $\mathbf{D}(X \times Y)$, the problem of understanding the functor $j^* i_*$ is replaced by the problem of understanding the single object $\bar{\jmath}^* \bar{\imath}_* \mathscr{O}_S$. We have replaced the functor $j^* i_*$ by the more complicated functor $\bar{\jmath}^* \bar{\imath}_*$, but we only apply it to a single object $\mathscr{O}_S$ which is well behaved.

2.7. We now turn to questions of formality. Given a dg scheme Z' over a scheme Z, with structure morphism $f : Z' \to Z$ being affine, we shall say that Z' is *formal* over Z if $f_* \mathscr{O}_{Z'}$ is a formal $\mathscr{O}_Z$-dg algebra, that is, if there exists an isomorphism

$$f_* \mathscr{O}_{Z'} \cong \bigoplus_j \mathscr{H}^j(f_* \mathscr{O}_{Z'})[-j]$$

of $\mathscr{O}_Z$-dg algebras. This is equivalent to the dg schemes Z' and $\hat{Z}'$ being isomorphic in the derived category of dg schemes over Z. (Here $\hat{Z}'$ is the dg scheme whose structure complex is $\bigoplus_j \mathscr{H}^j(f_* \mathscr{O}_{Z'})[-j]$.)

Note in particular that this implies that the complex $f_* \mathscr{O}_{Z'}$ splits in the derived category $\mathbf{D}(Z)$ (it is isomorphic to the sum of its cohomology sheaves).

2.8. The notion of formality of a dg scheme depends on the scheme over which we are working. Indeed, consider a smooth subvariety X of a smooth space S, and let $X' = X \times^R_S X$ be the derived self-intersection of X inside S. Then X' is a dg scheme over X in two distinct ways (using the two projections), and hence it is also a dg scheme over $X \times X$. In [1] the first two authors introduced two classes,

$$\alpha_{HKR} \in H^2(X, N \otimes N^\vee \otimes N^\vee)$$

and

$$\eta \in H^1(X, T_X \otimes N^\vee).$$

The results of [loc. cit.] and the present paper show that

– X' is formal over X if and only if the HKR class α_{HKR} vanishes;

– X' is formal over $X \times X$ if and only if the class η, vanishes.

It is known ([1]) that $\eta = 0$ implies $\alpha_{HKR} = 0$, but not vice-versa. Thus X' being formal over $X \times X$ implies it is formal over X, but the converse can fail.

3. The proof of the main theorem

In this section we shall prove our main result, Theorem 1.8, which we restate below. Throughout this section we shall drop the index "R" in the notation of derived fiber products, and write $X \times_S Y$ for the derived fiber product $X \times_S^R Y$. If F is any vector bundle on a space X we shall let $\mathbb{F}[-1]$ denote the total space of the shifted vector bundle $F[-1]$, i.e., the dg scheme whose structure complex is $\mathsf{Sym}(F^\vee[1])$. We assume that the characteristic of the ground field $\mathbf{k}$ is either zero or large enough (in all the results below, large enough means larger than the dimension of S).

3.1. Lemma. *Let $i : X \to S$ be a closed embedding of smooth varieties with normal bundle $N_{X/S}$. A choice of splitting of i (if one exists) determines an isomorphism*

$$X \times_S X \cong \mathbb{N}_{X/S}[-1]$$

in the derived category of dg schemes over $X \times X$, commuting with the embeddings of X.

Proof. Let π_1 and π_2 denote the two projections from $X \times_S X$ to X. We regard $X \times_S X$ as a space over $X \times X$ via the map (π_1, π_2). Note that by the very construction of the fiber product the compositions $i \circ \pi_1$ and $i \circ \pi_2$ are homotopic in the $(\infty, 1)$-category of dg schemes (before deriving it).

Fix a splitting p of the embedding i. Composing a homotopy between $i \circ \pi_1$ and $i \circ \pi_2$ with p we conclude that the maps π_1 and π_2 are themselves homotopic. Thus in the derived category of dg schemes, where homotopic maps become equal, we have $\pi_1 = \pi_2$. In other words the original map

$$(\pi_1, \pi_2) : X \times_S X \to X \times X$$

is equal to the map

$$(\pi_1, \pi_1) : X \times_S X \to X \times X$$

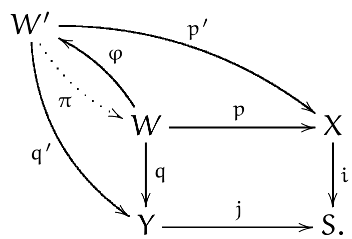
and the latter obviously factors through the diagonal map: $(\pi_1, \pi_1) = \Delta \circ \pi_1$. Thus the structure map (π_1, π_2) of $X \times_S X$ factors through Δ.

The splitting of the map i implies that we are in a situation where we can apply the main theorem of [1]. Thus (choosing one side) there exists an isomorphism of spaces over X

$$\varphi : X \times_S X \xrightarrow{\sim} \mathbb{N}_{X/S}[-1],$$

where $X \times_S X$ is regarded as a space over X via π_1. Since we have seen that the structure maps from $X \times_S X$ and $\mathbb{N}_{X/S}[-1]$ to $X \times X$ factor through the diagonal map, it follows that φ, which originally was an isomorphism over X, can be regarded as an isomorphism over $X \times X$. The compatibility with the embeddings of X is obvious from the construction of φ in [1]. □

3.2. We now place ourselves in the context of (1.1), with X and Y smooth subschemes of S, and with W' and W being their derived and underived intersections, respectively. The maps between these spaces are listed in the diagram below



The excess intersection bundle E on W is defined as

$$E = \frac{T_S}{T_X + T_Y}$$

where all the bundles above are assumed to have been restricted to W. As usual we shall denote by $\mathbb{E}[-1]$ the total space of the shift of E.

We begin by studying a special case of the main theorem, where the map i is split.

3.3. Proposition. *Assume that the map i is split, and fix a splitting of i. Then a choice of splitting of the short exact sequence*

$$0 \to N_{W/Y} \to N_{X/S}|_W \to E \to 0 \qquad (*)$$

gives rise to an isomorphism of spaces over $X \times Y$

$$\mathbb{E}[-1] \xrightarrow{\sim} X \times_S Y.$$

Conversely, existence of an isomorphism in $\mathbf{D}(Y)$

$$j^* i_* \mathcal{O}_X \cong q_*(\mathrm{Sym}(E^\vee[1])$$

implies that the short exact sequence $(*)$ *splits.*

Proof. A splitting $E \to N_{X/S}|_W$ of the short exact sequence $(*)$ gives rise to a map $\mathbb{E}[-1] \to \mathbb{N}_{X/S}[-1]|_W$. We have fixed a splitting of i; by Lemma 3.1 this gives rise to an isomorphism over $X \times X$

$$\mathbb{N}_{X/S}[-1] \xrightarrow{\sim} X \times_S X,$$

compatible with the inclusion of X. Using this isomorphism we obtain a morphism over $X \times Y$

$$\mathbb{E}[-1] \to \mathbb{N}_{X/S}[-1]|_W \cong (X \times_S X) \times_X W \cong X \times_S W \to X \times_S Y$$

where the last map comes from the inclusion $W \hookrightarrow Y$. Checking that this morphism is an isomorphism is a local computation which can be checked using Koszul resolutions.

In the other direction assume that there exists an isomorphism in $\mathbf{D}(Y)$

$$\varphi : j^* i_* \mathscr{O}_X \xrightarrow{\sim} q_*(\mathrm{Sym}(E^\vee[1]).$$

Without loss of generality we can assume that φ commutes with the natural maps of the two sides to $q_* \mathscr{O}_W$. To see this consider the map φ^0 induced by φ on $\mathscr{H}^0$ of the two complexes. It is an automorphism of the $\mathscr{O}_Y$-module $q_* \mathscr{O}_W$. As $q_* \mathscr{O}_W$ is a quotient algebra of $\mathscr{O}_Y$, φ^0 is in fact an automorphism of the ring $\mathscr{O}_W$, given by multiplication by an invertible element s of $\mathscr{O}_W$. Since $\mathrm{Sym}(E^\vee[1])$ is an $\mathscr{O}_W$-algebra, multiplication by s^{-1} gives an automorphism ψ of it. The composition $q_* \psi \circ \varphi$ is a new isomorphism

$$j^* i_* \mathscr{O}_X \cong q_*(\mathrm{Sym}(E^\vee[1]))$$

which commutes with the maps to $q_* \mathscr{O}_W$, as desired. We shall call this new map φ.

The map φ induces an isomorphism

$$p^* i^* i_* \mathscr{O}_X \cong q^* j^* i_* \mathscr{O}_X \cong q^* q_*(\mathrm{Sym}(E^\vee[1])).$$

Applying $\mathscr{H}^{-1}$ to both sides gives an isomorphism

$$N^\vee_{X/S}|_W \cong N^\vee_{W/Y} \oplus E^\vee$$

where the component $N^\vee_{W/Y}$ comes from $\mathscr{H}^{-1}$ of the summand $q^* q_* \mathscr{O}_W$ of $q^* q_*(\mathrm{Sym}(E^\vee[1]))$. The fact that φ is compatible with the map to $q_* \mathscr{O}_W$ shows that the map

$$N^\vee_{X/S} \to N^\vee_{W/Y}$$

in the above decomposition is the same as the map obtained by applying $\mathscr{H}^{-1}$ to the morphism

$$p^*i^*i_*\mathscr{O}_X \cong q^*j^*i_*\mathscr{O}_X \to q^*q_*\mathscr{O}_W.$$

A straightforward calculation with the Koszul complex shows that this map is precisely the dual of the inclusion map

$$N_{W/Y} \to N_{X/S}|_W$$

from the short exact sequence $(*)$. Thus the direct sum decomposition above is compatible with the maps in $(*)$, and hence this short exact sequence must split. □

3.4. Theorem. *The following statements are equivalent.*

(1) *There exists an isomorphism of dg functors* $\mathbf{D}(X) \to \mathbf{D}(Y)$

$$j^*i_*(-) \cong q_*(p^*(-) \otimes \mathrm{Sym}(E^\vee[1]))$$

(2) *There exists an isomorphism* $W' \cong \mathbb{E}[-1]$ *of dg schemes over* $X \times Y$.

(3) W' *is formal as a dg scheme over* $X \times Y$.

(4) *The inclusion* $W \to W'$ *splits over* $X \times Y$.

(5) *The inclusion* $W \to W'$ *splits to first order over* $X \times Y$.

(6) *The short exact sequence*

$$0 \to T_X + T_Y \to T_S \to E \to 0$$

of vector bundles on W *splits.*

Proof. We shall prove the following chains of implications and equivalences

$$(2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6) \Rightarrow (2),$$

and

$$(1) \Rightarrow (6) \Rightarrow (2) \Rightarrow (1).$$

The implications (2) ⇒ (3) ⇒ (4) ⇒ (5) are obvious. The implication (5) ⇒ (6) is a dg version of [10, 20.5.12 (iv)] (which we restate as Lemma 3.5

below), once one notes that the derived relative tangent bundles involved are

$$\begin{aligned} T_{W/W'} &= E[-2] \\ T_{W/X\times Y} &= (T_X + T_Y)[-1] \\ T_{W'/X\times Y} &= T_S[-1]. \end{aligned}$$

The implication (2) ⇒ (1) follows from the considerations in (2.5). Indeed, the kernel giving the functor j^*i_* is $\mathscr{O}'_W$, while the kernel giving the functor $q_*(p^*(-)\otimes \mathsf{Sym}(E^\vee[1]))$ is $\mathscr{O}_{\mathbb{E}[-1]}$, and an isomorphism between these objects gives rise to an isomorphism between the corresponding functors.

In the other direction there is a subtle point. Work of Toën [13] does imply that the equivalence of functors in (1) guarantees an isomorphism of $\mathscr{O}_{X\times Y}$-*modules*

$$\mathscr{O}_{W'} \cong \mathscr{O}_{\mathbb{E}[-1]};$$

while condition (2) is the strongest requirement that the two be isomorphic as *algebras*. Hence the implication (1) ⇒ (2) is not automatic and will only follow indirectly from the rest of the proof.

The implications that we still need to prove are (6) ⇒ (2) and (1) ⇒ (6). We replace the initial intersection problem with the problem of intersecting $X \times Y$ with the diagonal in $S \times S$, as in (2.6). We keep denoting the new spaces and embeddings by X, Y, and S, i, j, etc. Thus the new S is the old $S \times S$, the new X is the diagonal in the old $S \times S$, and the new Y is the old $X \times Y$. Note that now i is split (it is the old diagonal map, hence it is split by either projection).

We reformulate (1), (2), and (6) of the theorem in the new setting. Statements (1) and (2) become the statements that there exist isomorphisms

$$j^*i_*\mathscr{O}_X \cong q_*\mathsf{Sym}(E^\vee[1])$$

as objects of $\mathbf{D}(Y)$ and as commutative dg algebra objects in $\mathbf{D}(Y)$, respectively. The short exact sequence of (6) becomes the sequence

$$0 \to N_{W/Y} \to N_{X/S}|_W \to E \to 0.$$

We are now in the situation of Proposition 3.3: indeed, the main property we need is that the map i splits, and this is true because now i is the old diagonal map. The conclusions of this proposition are exactly the implications (6) ⇒ (2) and (1) ⇒ (6) that we still needed to prove. □

3.5. Lemma. *Let $i : X \hookrightarrow Y$ be a closed embedding of dg schemes over a fixed scheme S. Then i is split to first order over S if and only if the natural map*

$$\mathbb{T}_{X/Y} \to \mathbb{T}_{X/S}$$

is the zero map, where $\mathbb{T}$ denotes the tangent complex of the corresponding morphism.

Proof. The proof is nothing but a restating in dg language of [10, 20.5.12 (iv)].

4. Applications to orbifolds

In this section we discuss how Theorem 1.8 can be used to understand the structure of derived fixed loci. In turn this allows us to understand the structure of the free loop spaces of orbifolds.

4.1. We review the setup in (1.10). Let Z be a smooth variety over a field $\mathbf{k}$, and let φ be an automorphism of Z of finite order n. Since φ is of finite order its fixed locus, which we shall denote by W, is scheme-theoretically smooth. We shall assume that the characteristic of $\mathbf{k}$ is either zero or greater than $\max(n, \dim Z)$.

Note that the ordinary fixed locus W can be understood as the intersection

$$W = \Delta \times_{Z\times Z} \Delta^{\varphi},$$

where Δ and Δ^{φ} denote the diagonal in $Z \times Z$ and the graph of φ, respectively. As such the expected dimension of W is zero. Whenever $\dim W > 0$ it is important to understand the failure of the spaces in this intersection problem to meet transversally, by studying the derived intersection space

$$W' = \Delta \times^{R}_{Z\times Z} \Delta^{\varphi}.$$

We shall sometimes call this space the derived fixed locus of φ.

4.2. Theorem 1.8 shows that in order to understand the structure of W' we need to study the short exact sequence

$$0 \to T_{\Delta} + T_{\Delta^{\varphi}} \to T_{Z\times Z} \to E \to 0$$

where E is the excess bundle for this intersection problem. We shall prove that this sequence is always split, under the assumptions we made for the characteristic of $\mathbf{k}$. We begin with a lemma.

4.3. Lemma. *In the setup of Theorem 1.8, assume that the map* $X \to S$ *is split to first order. Then the short exact sequence*

$$0 \to N_{W/Y} \to N_{X/S}|_W \to E \to 0$$

splits if and only if the six equivalent statements of Theorem 1.8 are all true.

Proof. It is easy to see that the two conditions of the lemma imply that the short exact sequence of (6) of Theorem 1.8 splits. Equivalently, these two conditions are what was used in the proof of Theorem 1.8 after changing the problem to an intersection of the diagonal with $X \times Y$. □

4.4. Theorem. *Assume we are in the setup of (4.1). Then the derived fixed locus* W' *is isomorphic, as a dg scheme over* $Z \times Z$*, to the total space over* W *of the dg vector bundle* $T_W[-1]$,

$$W' \cong \mathbb{T}_W[-1].$$

Proof. It is easy to see that the excess intersection bundle E for this intersection problem is $(T_Z)_\varphi$, the bundle on W of coinvariants of the action of φ on T_Z,

$$(T_Z)_\varphi = \frac{T_Z}{\langle v - \varphi(v)\rangle}.$$

We now apply Lemma 4.3. The embedding $Z \to Z \times Z$ is split to first order (it is actually split). The map $N_{Z/Z\times Z} \to E$ is given by the natural projection

$$T_Z \to \frac{T_Z}{\langle v - \varphi(v)\rangle}.$$

If the characteristic of $\mathbf{k}$ is 0 or prime to n, the averaging map $(T_Z)_\varphi \to T_Z$ given by

$$t \mapsto \frac{1}{n}\sum_{i=1}^{n} \varphi^i(t)$$

splits the projection above. Finally, with the same assumptions on the characteristic of $\mathbf{k}$, the bundles of invariants and coinvariants are naturally isomorphic: $(T_Z)_\varphi \cong (T_Z)^\varphi$ and the latter is precisely T_W. □

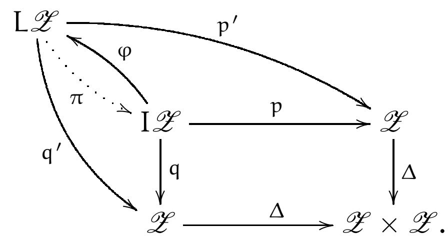
4.5. We apply the above theorem to the study of orbifolds. Let G be a finite group acting on a smooth variety Z, and denote the quotient stack $[\mathsf{Z}/\mathsf{G}]$ by $\mathscr{X}$. We are interested in understanding the relationship between the inertia stack of $\mathscr{X}$,

$$\mathrm{I}\mathscr{X} = \mathscr{X} \times_{\mathscr{X}\times\mathscr{X}} \mathscr{X},$$

and the free loop space of the corresponding derived intersection,

$$\mathsf{L}\mathscr{X} = \mathscr{X} \times^{\mathsf{R}}_{\mathscr{X}\times\mathscr{X}} \mathscr{X}.$$

We organize these spaces and the maps between them in the diagram below:



4.6. We wish to realize the above diagram of (dg) stacks as the global quotient by the fixed group $\mathsf{G}\times\mathsf{G}$ of a similar diagram of (dg) schemes. This allows us to reduce the problem of understanding the dg stack $\mathsf{L}\mathscr{X}$ and its maps to $\mathrm{I}\mathscr{X}$ and $\mathscr{X}$ to the parallel problem of understanding the corresponding dg schemes and maps.

As originally formulated the diagonal map Δ is a map between the global quotient stacks $\mathscr{X} = [\mathsf{Z}/\mathsf{G}]$ and $\mathscr{X}\times\mathscr{X} = [(\mathsf{Z}\times\mathsf{Z})/(\mathsf{G}\times\mathsf{G})]$. We wish to replace the presentation $[\mathsf{Z}/\mathsf{G}]$ of $\mathscr{X}$ by a different presentation of the same stack, but where the group we quotient by is $\mathsf{G}\times\mathsf{G}$. Consider the action of $\mathsf{G}\times\mathsf{G}$ on $\mathsf{Z}\times\mathsf{G}$ given by

$$(h,k).(z,g) \mapsto (h.z, kgh^{-1}).$$

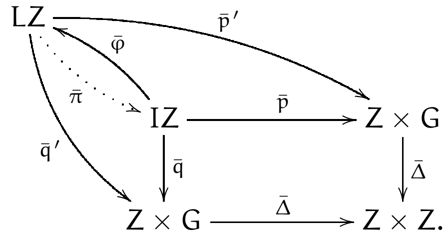
Note that the second copy of G acts freely on $\mathsf{Z}\times\mathsf{G}$, thus yielding an isomorphism

$$[(\mathsf{Z}\times\mathsf{G})/(\mathsf{G}\times\mathsf{G})] \cong [\mathsf{Z}/\mathsf{G}].$$

With this presentation the diagonal map $\mathscr{X}\to\mathscr{X}\times\mathscr{X}$ becomes the quotient by $\mathsf{G}\times\mathsf{G}$ of the equivariant map of spaces

$$\begin{aligned} \bar{\Delta} : \mathsf{Z}\times\mathsf{G} &\to \mathsf{Z}\times\mathsf{Z} \\ (z,g) &\mapsto (z, g.z). \end{aligned}$$

Summarizing the above discussion, the main diagram in (4.5) is obtained by taking the quotient by $\mathsf{G} \times \mathsf{G}$ of the spaces and maps in the diagram below:



Here we have denoted by LZ and IZ the corresponding derived and underived fiber products, respectively.

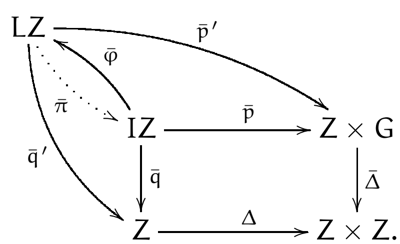
4.7. For $\mathsf{g} \in \mathsf{G}$ denote by Δ^{g} the subvariety of $\mathsf{Z} \times \mathsf{Z}$ which is the graph of the action of g on Z,

$$\Delta^{\mathsf{g}} = \{(z, \mathsf{g}.z) \mid z \in \mathsf{Z}\}.$$

The space IZ decomposes as the disjoint union

$$\mathsf{IZ} = \coprod_{\mathsf{g},\mathsf{h} \in \mathsf{G}} \Delta^{\mathsf{g}} \cap \Delta^{\mathsf{h}},$$

and similarly for LZ (where the intersection is replaced by the derived intersection). Since the second copy of G acts freely on $\mathsf{X} \times \mathsf{G}$ it follows that it also acts freely on IZ and LZ (which can be thought of as subvarieties of $\mathsf{X} \times \mathsf{G}$). This allows us to further simplify the calculation of $\mathrm{I}\mathscr{Z}$ and $\mathrm{L}\mathscr{Z}$ by first taking the quotient of IZ and LZ by the second copy of G (which still yields a space), leaving the first copy of G to quotient by later. This amounts to replacing in the above calculation of a (derived) intersection the horizontal map $\bar{\Delta}$ by just the map $\Delta : \mathsf{Z} \to \mathsf{Z} \times \mathsf{Z}$, while the vertical map $\bar{\Delta}$ stays the same. We shall abuse notation and denote the new derived and underived intersection spaces by the old names of LZ and IZ. They fit in the diagram



4.8. Observe that after this reduction the space IZ is just the disjoint union

$$\mathsf{IZ} = \coprod_{g \in \mathsf{G}} \mathsf{Z}^g,$$

while LZ has the same decomposition as a disjoint union, but the fixed loci Z^g are replaced by their derived analogues $(\mathsf{Z}^g)'$. The action of $h \in \mathsf{G}$ shuffles these fixed loci by sending Z^g to $\mathsf{Z}^{hgh^{-1}}$.

Applying Theorem 4.4 immediately yields Theorem 1.15, which we restate below.

4.9. Theorem. *Let $\mathscr{X}$ be a global quotient orbifold. Then there exists a canonical isomorphism*

$$\exp : \mathbb{T}_{\mathrm{I}\mathscr{X}}[-1] \xrightarrow{\sim} \mathsf{L}\mathscr{X}$$

between the shifted tangent bundle $\mathbb{T}_{\mathrm{I}\mathscr{X}}[-1]$ of its inertia orbifold and its free loop space $\mathsf{L}\mathscr{X}$.

Proof. Theorem 4.4 implies that the derived fixed loci $(\mathsf{Z}^g)'$ are isomorphic to the total spaces $(\mathbb{T}_\mathsf{Z})_g[-1]$, and it is immediate to see that $(\mathsf{T}_\mathsf{Z})_g \cong \mathsf{T}_{\mathsf{Z}^g}$. Thus we have an isomorphism

$$\mathbb{T}_{\mathsf{IZ}}[-1] \xrightarrow{\sim} \mathsf{LZ};$$

the isomorphism in the statement of the theorem is nothing but the quotient of this isomorphism by the action of G. $\square$

4.10. Corollary 1.17 follows easily from the above theorem once one remembers that

$$\mathsf{HH}_*(\mathscr{X}) = \mathbf{R}\Gamma(\mathscr{X}, \Delta^*\Delta_*\mathscr{O}_\mathscr{X}) = \mathbf{R}\Gamma(\mathsf{Z}, \bar{q}'_*\mathscr{O}_{\mathsf{LZ}})^\mathsf{G}$$

and

$$\begin{aligned}\mathsf{HH}^*(\mathscr{X}) &= \mathbf{R}\mathsf{Hom}^*_{\mathscr{X}\times\mathscr{X}}(\Delta_*\mathscr{O}_\mathscr{X}, \Delta_*\mathscr{O}_\mathscr{X}) = \mathbf{R}\Gamma(\mathscr{X}, \Delta^!\Delta_*\mathscr{O}_\mathscr{X}) \\ &= \mathbf{R}\Gamma(\mathsf{Z}, (\bar{q}'_*\mathscr{O}_{\mathsf{LZ}})^\vee)^\mathsf{G}.\end{aligned}$$

Here $\bar{q}'$ is the map $\mathsf{LZ} \to \mathsf{Z}$ from (4.7).

4.11. Theorem 4.9 also highlights a somewhat striking difference between the behavior of disconnected Lie groups in derived algebraic geometry versus classical geometry.

We think of the free loop space $\mathsf{L}\mathscr{X}$ of an orbifold $\mathscr{X}$ as a family of (homotopy) groups $\Omega_z\mathscr{X}$ parametrized by $z \in \mathscr{X}$. These groups are not in general connected, having components indexed by the inertia group of z. Our theorem shows that the exponential map we have defined is an isomorphism between an appropriate number of copies of the Lie algebra of the group $\Omega_z\mathscr{X}$ and the group itself.

This situation is different when one studies disconnected Lie groups in the classical setting. Let H be a disconnected Lie group, let H^0 be the connected component of the identity (a normal subgroup of H), let $\mathfrak{h}$ denote the Lie algebra of H (i.e., the Lie algebra of H^0), and let $\mathsf{G} = \mathsf{H}/\mathsf{H}^0$ denote the group of components. Suppose the homomorphism $\mathsf{H} \to \mathsf{H}/\mathsf{H}_0 = \mathsf{G}$ admits a right inverse. Once the inverse is fixed, we can view G as a subgroup of H, and H decomposes as a semi-direct product of H_0 and G.

In this setting there is no natural exponential map that covers *all* of H (or at least a formal neighborhood of G in H). Indeed, one could translate the usual exponential map around the origin of H to the other components of H, but this involves a choice – whether to use left or right translation by elements of G. The only natural map to a neighborhood of $\mathsf{g} \in \mathsf{G}$ is the restriction of the exponential map to $(\mathsf{H}^0)^{\mathsf{g}}$, which will map into the centralizer of g. This map will be far from surjective, unlike the derived case where it is an isomorphism.

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