

DERIVED CATEGORIES OF TWISTED SHEAVES ON ELLIPTIC THREEFOLDS

ANDREI CĂLDĂRARU

ABSTRACT. We construct an equivalence between the derived category of sheaves on an elliptic threefold without a section and a derived category of twisted sheaves (modules over an Azumaya algebra) on any small resolution of its relative Jacobian.

1. INTRODUCTION

Equivalences of derived categories have attracted much interest in the past few years, generated in part by applications to the study of moduli spaces ([26], [25], [4]), and in part by the conjectured relationship between derived categories and mirror symmetry ([22]). Recently there have been suggestions ([6, 6.8], [21]) that in order to obtain a good description of physical phenomena, one should study not only the derived category of usual sheaves but also the derived categories of sheaves of modules over Azumaya algebras on the spaces involved. From a physical point of view, one needs B -fields in the construction of conformal field theories, and these fields often have a component which is an element of the Brauer group of the Calabi-Yau manifold used. Here we present an example of a Fourier-Mukai transform involving a derived category of sheaves over an Azumaya algebra. Few such examples are known: a symplectic case was studied in [28], and intersections of quadrics were studied in [20], but none of these relate to the Calabi-Yau property.

The simplest case of a Fourier-Mukai equivalence occurs between the derived category of an elliptic curve E and that of its dual \hat{E} , and is induced by the Poincaré bundle on $E \times \hat{E}$. This has been generalized to a relative situation ([4], [5]) as follows: starting with an elliptic fibration $X \rightarrow S$, one considers a relative moduli space $Y \rightarrow S$ of semistable sheaves on the fibers of $X \rightarrow S$. Under the assumptions that $\dim X = \dim Y$ and that the moduli problem is fine, one proves that Y is smooth and that the universal sheaf induces an equivalence of derived categories $\mathbf{D}_{\text{coh}}^b(X) \cong \mathbf{D}_{\text{coh}}^b(Y)$.

In this paper we study what happens when one removes the assumption that the moduli problem is fine. We work this out in the particular case when $X \rightarrow S$ is an elliptic fibration which does not possess a section, and the moduli problem is the one that gives the *relative Jacobian*, the moduli space of semistable sheaves of rank 1, degree 0 along the fibers of $X \rightarrow S$. We analyze the ways in which this moduli problem fails to be fine, and we obtain an equivalence of derived categories (which involves sheaves of modules over an Azumaya algebra) after making some necessary changes.

1.1. Let $f : X \rightarrow S$ be an elliptic fibration, with X and S smooth complex manifolds of dimensions 3 and 2, respectively, and satisfying the following extra properties:

- (1) f is flat (i.e. all fibers are 1-dimensional) and projective;
- (2) f does not have any multiple fibers;
- (3) f admits a multisection;
- (4) the discriminant locus Δ is a reduced, irreducible curve in S , having only nodes and cusps as singularities;
- (5) the fiber of f over a general point of Δ is a rational curve with one node.

We call such a map a *generic elliptic threefold*. (For the definition of elliptic fibration, multiple fibers, multisection, discriminant locus, the reader is referred to [11, 2.1].) The most interesting applications of our results are for Calabi-Yau threefolds, but the Calabi-Yau condition is not needed for the results in this paper to hold.

The reason for calling such an elliptic fibration “generic” is the fact that in many families of elliptic threefolds the above properties are shared by the general members of the family (especially when X is Calabi-Yau, see 3.1 and 3.2). For technical reasons, we shall only restrict our attention to generic elliptic fibrations.

From here on fix a generic elliptic threefold $f : X \rightarrow S$, and a relatively ample line bundle $\mathcal{O}_{X/S}(1)$ for f .

1.2. Define the *relative Jacobian* $p : J \rightarrow S$ of f to be the relative moduli space of semistable sheaves of rank 1, degree 0 on the fibers of f . (To be precise, rank 1, degree 0 is defined as having the same Hilbert polynomial as the trivial line bundle.) It is a flat, projective elliptic fibration which has a natural section $s : S \rightarrow J$, obtained by associating to a point $t \in S$ the point $[\mathcal{O}_{X_t}]$ corresponding to the semistable sheaf \mathcal{O}_{X_t} on the fiber X_t . (It is not hard to see that \mathcal{O}_{X_t} is semistable for all the fibers of a generic elliptic threefold, Section 2.)

The above description of the relative Jacobian is based on Simpson’s construction [30] of relative moduli spaces of semistable sheaves, a fact which allows us to analyze the geometry of J by studying moduli spaces of semistable sheaves on various degenerations of the fibers of f . These degenerations are well understood by work of Miranda [24], and moduli spaces of semistable sheaves on these singular fibers have been studied in [27]. Putting together these results will give us a good understanding of the geometry of J (Section 3).

1.3. In order to obtain an equivalence of derived categories between X and J one needs to find a good replacement for the notion of universal sheaf. This problem arises because in our situation a universal sheaf does not exist on $X \times_S J$. The main contribution of this paper consists in the proposed solution to this problem, as well as the consequences deduced from it.

There are two main obstructions to the existence problem: one is the fact that there are properly semistable sheaves in the moduli problem (on reducible fibers). They are responsible for the apparition of singularities in J (a whole S-equivalence class of sheaves gets contracted to a point). This situation should be contrasted with Bridgeland’s result [5] that when the moduli problem under consideration is fine, the moduli space is smooth. The approach we take for solving this problem is to

replace J by an analytic small resolution \bar{J} of its singularities. (In general, one can not expect to be able to find an algebraic small resolution, 5.3. If one wanted to stay in the algebraic realm, one would replace schemes by algebraic spaces, in the sense of Artin. We stick to the analytic situation for ease of exposition.)

The other obstruction to the existence of a universal sheaf is the fact that although one can find universal sheaves on $X \times_S U$ for small enough open sets U in \bar{J} , the lack of uniqueness of these universal sheaves may prevent them from gluing together. (More precisely, these are pseudo-universal sheaves, parametrizing all the stable sheaves and some of the semistable sheaves in the moduli problem.) In the particular situation under consideration this is indeed the case, and the obstruction to this gluing is naturally an element α of $\text{Br}(\bar{J})$, the Brauer group of \bar{J} . We resolve this problem replacing sheaves by α -twisted sheaves in the resulting equivalence of derived categories. (A quick introduction to derived categories of twisted sheaves is provided in Section 4.)

Theorem 5.1. *Let $X \rightarrow S$ be a generic elliptic threefold, let $J \rightarrow S$ be its relative Jacobian, and let $\bar{J} \rightarrow J$ be an analytic small resolution of the singularities of J , with exceptional locus E . Let $\alpha \in \text{Br}(\bar{J})$ be the unique extension to \bar{J} of the obstruction to the existence of a universal sheaf on $X \times_S (\bar{J} \setminus E)$. Then there exists a $\pi_{\bar{J}}^* \alpha^{-1}$ -twisted pseudo-universal sheaf on $X \times_S \bar{J}$ whose extension by zero to $X \times \bar{J}$ induces an equivalence of derived categories*

$$\mathbf{D}_{\text{coh}}^b(X) \cong \mathbf{D}_{\text{coh}}^b(\bar{J}, \alpha).$$

(By extending by zero we mean pushing forward by the natural inclusion $X \times_S \bar{J} \hookrightarrow X \times \bar{J}$. $\pi_{\bar{J}} : X \times_S \bar{J} \rightarrow \bar{J}$ is the projection, and $\mathbf{D}_{\text{coh}}^b(\bar{J}, \alpha)$ denotes the derived category of α -twisted sheaves on \bar{J} .)

On a side note, it is worthwhile observing that there is a classical construction which provides an element $\alpha \in \text{Br}(\bar{J})$ – the Ogg-Shafarevich theory of elliptic fibrations without a section. It is not hard to trace through the definitions and to see that the element constructed by Ogg-Shafarevich theory agrees with the element α we use in Theorem 5.1. Therefore the above result can be seen as a generalization of Ogg-Shafarevich theory via derived categories.

We note here a particularly striking consequence of Theorem 5.1:

Theorem 6.1. *Assume we are in the setup of Theorem 5.1, and let n be the order of α in $\text{Br}(\bar{J})$. Then we have*

$$\mathbf{D}_{\text{coh}}^b(\bar{J}, \alpha) \cong \mathbf{D}_{\text{coh}}^b(\bar{J}, \alpha^k),$$

for any k coprime to n .

The paper is structured as follows: in Section 2 we give a brief overview of moduli spaces of semistable sheaves on the singular fibers that occur in generic elliptic threefolds. Section 3 is devoted to a geometric study of the relative Jacobian and of its small resolutions. The next section provides a short introduction to the topic of twisted sheaves and their derived categories (for complete details the reader should consult [6, Part 1]). In Section 5 we prove Theorem 5.1, and in a final section we discuss the relationship of our result to Ogg-Shafarevich theory and Theorem 6.1.

Conventions. We work over the field of complex numbers, and all the spaces considered are analytic spaces. The topology used is the analytic topology, unless otherwise specified. The same results hold in the algebraic category, by replacing analytic spaces by algebraic spaces and using the étale topology instead.

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2. THE FIBERWISE PICTURE

In this section we consider a generic elliptic threefold $f : X \rightarrow S$, and we review Miranda's results [24] about the kinds of degenerate fibers of f that can occur. We then apply the results of Oda and Seshadri [27], which describe what the corresponding degenerations are in the relative Jacobian. For clarity, we also sketch a proof of the fact that the moduli space of rank 1, degree 0 semistable sheaves on an I_2 curve is a nodal curve.

2.1. The conditions that we have imposed on the elliptic fibration $f : X \rightarrow S$ highly restrict the possibilities of what singular curves can occur as fibers of f . The results in [24] imply that the fibers of f can be classified as follows:

- (0) over $s \in S \setminus \Delta$, X_s is a smooth elliptic curve (Δ is the discriminant locus of f);
- (I_1) over a smooth point s of Δ , X_s is a rational curve with one node;
- (I_2) over a node s of Δ , X_s is a reducible curve of type I_2 , i.e., two smooth \mathbf{P}^1 's meeting transversely at two points;
- (II) over a cusp s of Δ , X_s is a rational curve with one cusp.

Furthermore, in the case of the I_2 fiber, each component C_i of the fiber has normal bundle

$$\mathcal{N}_{C_i/X} \cong \mathcal{O}_{C_i}(-1) \oplus \mathcal{O}_{C_i}(-1).$$

2.2. This classification allows us to do a case by case analysis of what the moduli space of rank 1, degree 0 semistable sheaves on each one of the four types of curves looks like. We use Gieseker's notion of stability, in its generalized form for pure sheaves used by Simpson in [30]. For a quick account of these notions, see [19, Section 1.2].

Let C be one of the four types of curves in the list in 2.1, arbitrarily polarized, and let M be the moduli space of semistable sheaves on C whose Hilbert polynomial (with respect to the polarization) equals that of \mathcal{O}_C . Then M is isomorphic to C in cases (0), (I_1), (II), while in case (I_2), M is isomorphic to a rational curve with one node. This is an immediate consequence of the results in [27], but for clarity of the exposition we'll give a brief description of this result, placing the emphasis on the geometry of case (I_2).

2.3. First recall the description of degree 0 line bundles on an elliptic curve C . All these line bundles are of the form $\mathcal{O}_C(P - Q)$, with Q fixed and P sweeping out C . One can consider a universal family \mathcal{E} on $C \times C$, whose restriction to $C \times \{P\}$ is isomorphic to $\mathcal{O}_C(P - Q)$ for any $P \in C$ (the second component of the product). One way to describe \mathcal{E} is

$$\mathcal{E} = \mathcal{O}_{C \times C}(\Delta) \otimes \pi_1^* \mathcal{O}_C(-Q),$$

where $\pi_1 : C \times C \rightarrow C$ is the projection onto the first factor and Δ is the diagonal in $C \times C$. Indeed, \mathcal{E} is obviously flat over the second component, and its restriction $\mathcal{E}(P, Q) = \mathcal{E}|_{C \times \{P\}}$ is isomorphic to

$$\mathcal{O}(\Delta)|_{C \times \{P\}} \otimes \mathcal{O}_C(-Q) = \mathcal{O}_C(P) \otimes \mathcal{O}_C(-Q) = \mathcal{O}_C(P - Q).$$

2.4. In the above discussion we made use of the fact that C was smooth, and thus we were able to speak about $\mathcal{O}_{C \times C}(\Delta)$. We want to generalize this to the other possible cases in 2.1.

Let C be any curve occurring in the classification 2.1, let Q be a fixed smooth point of C , and let Δ be the diagonal in $C \times C$. Define $\mathcal{O}_{C \times C}(\Delta) = \underline{\text{Hom}}(\mathcal{I}_\Delta, \mathcal{O}_{C \times C})$, the dual of the ideal sheaf of Δ .

Dualizing the exact sequence

$$0 \rightarrow \mathcal{I}_\Delta \rightarrow \mathcal{O}_{C \times C} \rightarrow \mathcal{O}_\Delta \rightarrow 0,$$

one gets

$$0 \rightarrow \mathcal{O}_{C \times C} \rightarrow \mathcal{O}_{C \times C}(\Delta) \rightarrow \underline{\text{Ext}}^1(\mathcal{O}_\Delta, \mathcal{O}_{C \times C}) \rightarrow 0.$$

A local computation shows that $\underline{\text{Ext}}^1(\mathcal{O}_\Delta, \mathcal{O}_{C \times C})$ is a line bundle on Δ , and a deformation argument shows that this line bundle is trivial. We conclude that we have an exact sequence

$$0 \rightarrow \mathcal{O}_{C \times C} \rightarrow \mathcal{O}_{C \times C}(\Delta) \rightarrow \mathcal{O}_\Delta \rightarrow 0,$$

as expected.

Define $\mathcal{E} = \mathcal{O}_{C \times C}(\Delta) \otimes \pi_1^* \mathcal{O}_C(-Q)$. The remainder of this section is devoted to the study of the properties of \mathcal{E} .

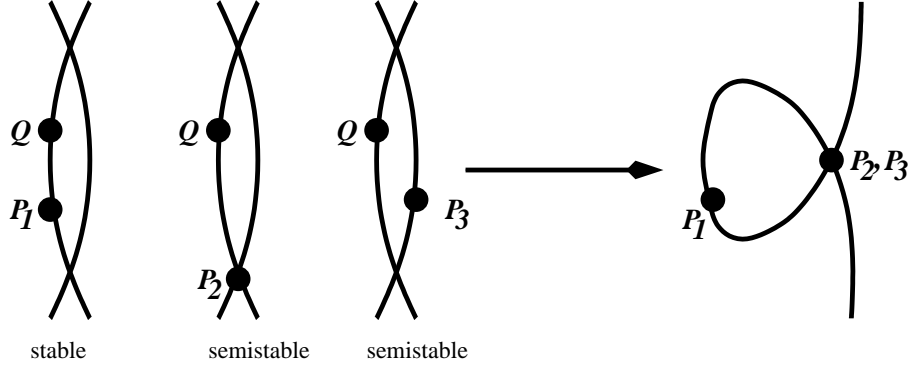
2.5. The restriction $\mathcal{E}(P)$ of \mathcal{E} to a fiber $C \times \{P\}$ is the unique non-trivial extension

$$0 \rightarrow \mathcal{O}_C(-Q) \rightarrow \mathcal{E}(P) \rightarrow \mathcal{O}_P \rightarrow 0.$$

It is a torsion free sheaf for all $P \in C$, and for smooth P it equals $\mathcal{O}_C(P - Q)$. Thus it makes sense to consider the question of stability for the sheaves $\mathcal{E}(P)$.

This is the point where the analysis changes between the irreducible fibers (cases (0), (I_1) , (II)) and the reducible fibers (case (I_2)). In the first three situations, all the sheaves $\mathcal{E}(P)$ are stable. However, in the I_2 case the stability of $\mathcal{E}(P)$ depends on the relative position of P and Q . This explains why in the first three cases the moduli space is isomorphic to the curve itself, while for the I_2 fibers the moduli space has a component contracted. Since these results are known (see [27] or [6, 6.3]), from here on we only sketch what happens in the I_2 case.

Let C_1 be the component of C that Q lies in, and let C_2 be the other component. If $P \in C_1 \setminus C_2$, then $\mathcal{E}(P)$ is stable. (This parallels the standard picture for degree 0 line bundles on an elliptic curve.) However, if $P \in C_2$, then $\mathcal{E}(P)$ is properly

FIGURE 1. The stability of the sheaves $\mathcal{E}(P)$

semistable, with composition factors $\mathcal{O}_{C_1}(-1)$ and $\mathcal{O}_{C_2}(-1)$. We can easily see how this happens when P is not a singular point of C , since then $\mathcal{E}(P)$ is the line bundle $\mathcal{O}_C(P - Q)$. One has the exact sequence

$$0 \rightarrow \mathcal{O}_{C_2}(-S_1 - S_2) \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_{C_1} \rightarrow 0,$$

where S_1, S_2 are the two singular points of C . Tensoring it with $\mathcal{O}_C(P - Q)$ one obtains

$$0 \rightarrow \mathcal{O}_{C_2}(P - S_1 - S_2) \rightarrow \mathcal{O}_C(P - Q) \rightarrow \mathcal{O}_{C_1}(-Q) \rightarrow 0,$$

i.e.,

$$0 \rightarrow \mathcal{O}_{C_2}(-1) \rightarrow \mathcal{E}(P) \rightarrow \mathcal{O}_{C_1}(-1) \rightarrow 0$$

because C_1 and C_2 are both isomorphic to \mathbf{P}^1 . The reduced Hilbert polynomials $p(\mathcal{O}_{C_2}(-1); t)$ and $p(\mathcal{E}(P); t)$ both equal t , independent of the polarization of C . (For a definition of the reduced Hilbert polynomial, see [19, 1.2.3].) Hence $\mathcal{O}_{C_2}(-1)$ is a destabilizing subsheaf of $\mathcal{E}(P)$, and the composition factors are $\mathcal{O}_{C_1}(-1)$ and $\mathcal{O}_{C_2}(-1)$. The details of the same analysis for P singular can be found in [6, 6.3.5].

2.6. We conclude that for $P \in C_2$, all $\mathcal{E}(P)$ are in the same S-equivalence class. If one denotes by M the moduli space of rank 1, degree 0 semistable sheaves on C , then the family \mathcal{E} gives by the universal property of M a map $C \rightarrow M$ which contracts the component C_2 of C to an image which is a rational curve with one node (Figure 1). Since it is known that M is a rational curve with one node, we conclude that the map $C \rightarrow M$ described above is onto.

The way to think about this situation is that on $C \times M$ there is no universal sheaf, because of the existence of properly semistable sheaves. There is not even a natural sheaf defined over all of $C \times M$ and universal on $C \times M^{\text{smooth}}$, because there are many choices for the sheaf that would lie over the singular point of M . The solution is to replace M by a “blow-up” of M which is isomorphic to C , and which naturally parametrizes the stable sheaves plus *some* semistable sheaves on C (those of the form $\mathcal{E}(P)$ with $P \in C_2$).

Proposition 2.1. *If C is of any one of the four types described in 2.1, then the sheaves $\mathcal{E}(P)$ satisfy*

$$\mathrm{Ext}^i(\mathcal{E}(P_1), \mathcal{E}(P_2)) = \begin{cases} \mathbf{C} & \text{if } P_1 = P_2 \text{ and } i = 0 \\ 0 & \text{if } P_1 \neq P_2 \text{ and any } i, \end{cases}$$

for $P_1, P_2 \in C$.

In other words, in the sense of Fourier-Mukai transforms, the sheaves in the family $\{\mathcal{E}(P)\}_{P \in C}$ are “almost” mutually orthogonal. The “almost” refers to the fact that we may have $\mathrm{Ext}^i(\mathcal{E}(P), \mathcal{E}(P)) \neq 0$ for $i > 1$; this fact, however, is purely a singular space phenomenon: we’ll see later (Proposition 3.1) that if we embed C in a smooth space, the extensions by zero of the sheaves $\mathcal{E}(P)$ are mutually orthogonal in the proper sense.

This result shows that, from the point of view of derived categories, using all the sheaves $\mathcal{E}(P)$ (and not just the stable ones) is the right thing to do. Although we cannot speak of an equivalence of derived categories induced by \mathcal{E} (because of the singularities of C), morally \mathcal{E} should induce a Fourier-Mukai transform from C to C .

Proof. Let P be any point of C , and apply $\mathrm{Hom}(\cdot, \mathcal{O}_C(-Q))$ to the exact sequence

$$0 \rightarrow \mathcal{O}_C(-Q) \rightarrow \mathcal{E}(P) \rightarrow \mathcal{O}_P \rightarrow 0,$$

to get

$$0 \rightarrow \mathrm{Hom}(\mathcal{E}(P), \mathcal{O}_C(-Q)) \rightarrow \mathbf{C} \xrightarrow{\delta} \mathrm{Ext}^1(\mathcal{O}_P, \mathcal{O}_C(-Q)).$$

Since the extension is non-split, $\delta \neq 0$, and therefore

$$\mathrm{Hom}(\mathcal{E}(P), \mathcal{O}_C(-Q)) = 0.$$

By Serre duality, $H^1(\mathcal{E}(P) \otimes \mathcal{O}_C(Q)) = 0$. Since

$$\chi(\mathcal{E}(P) \otimes \mathcal{O}_C(Q)) = 1$$

we conclude that $\mathrm{Hom}(\mathcal{O}_C(-Q), \mathcal{E}(P)) = \mathbf{C}$. We read this as saying that, up to multiplication by a constant, there is a unique morphism of $\mathcal{O}_C(-Q)$ into $\mathcal{E}(P)$.

Assume $P_1 \neq P_2$, and let $f : \mathcal{E}(P_1) \rightarrow \mathcal{E}(P_2)$ be any homomorphism. Compose f with the inclusion $\mathcal{O}_C(-Q) \hookrightarrow \mathcal{E}(P_1)$ to get a homomorphism $h : \mathcal{O}_C(-Q) \rightarrow \mathcal{E}(P_2)$. Since there is a unique map (up to scalars) $\mathcal{O}_C(-Q) \rightarrow \mathcal{E}(P_2)$, f can be restricted to a map g that fits in the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_C(-Q) & \longrightarrow & \mathcal{E}(P_1) & \longrightarrow & \mathcal{O}_{P_1} \longrightarrow 0 \\ & & \downarrow g & & \downarrow f & & \downarrow 0 \\ 0 & \longrightarrow & \mathcal{O}_C(-Q) & \longrightarrow & \mathcal{E}(P_2) & \longrightarrow & \mathcal{O}_{P_2} \longrightarrow 0. \end{array}$$

Since $\mathcal{O}_C(-Q)$ is a line bundle, g can be either an isomorphism or zero. If g is an isomorphism, then the snake lemma shows that $\ker f = \mathcal{O}_{P_1}$, which is impossible because $\mathcal{E}(P_1)$ is torsion-free. Therefore g must be zero, and then f is zero as well. We conclude that for $P_1 \neq P_2$ we have

$$\mathrm{Hom}(\mathcal{E}(P_1), \mathcal{E}(P_2)) = 0.$$

Note that $\mathcal{E}(P_2)$ is locally free at P_1 , so a local computation shows that

$$\mathrm{Ext}^i(\mathcal{O}_{P_1}, \mathcal{E}(P_2)) = \begin{cases} \mathbf{C} & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Applying $\mathrm{Hom}(\cdot, \mathcal{E}(P_2))$ to the exact sequence

$$0 \rightarrow \mathcal{O}_C(-Q) \rightarrow \mathcal{E}(P_1) \rightarrow \mathcal{O}_{P_1} \rightarrow 0$$

gives

$$\mathrm{Ext}^i(\mathcal{E}(P_1), \mathcal{E}(P_2)) = \mathrm{Ext}^i(\mathcal{O}_C(-Q), \mathcal{E}(P_2))$$

for all $i \geq 1$. Since C is Cohen-Macaulay with trivial dualizing sheaf, and $\mathcal{O}_C(-Q)$ is locally free, Serre duality shows that this last group is 0 for $i \geq 2$, and is isomorphic to $\mathrm{Hom}(\mathcal{E}(P_2), \mathcal{O}_C(-Q))^\vee$ for $i = 1$. Any non-trivial homomorphism $\mathcal{E}(P_2) \rightarrow \mathcal{O}_C(-Q)$ would give by composition a non-trivial one $\mathcal{E}(P_2) \rightarrow \mathcal{E}(P_1)$, contradicting our earlier result. We conclude that for $P_1 \neq P_2$, $\mathrm{Ext}^i(\mathcal{E}(P_1), \mathcal{E}(P_2)) = 0$ for all i .

Now let P be any point of C , and apply $\mathrm{Hom}(\cdot, \mathcal{E}(P))$ to the exact sequence

$$0 \rightarrow \mathcal{O}_C(-Q) \rightarrow \mathcal{E}(P) \rightarrow \mathcal{O}_P \rightarrow 0,$$

to get

$$0 \rightarrow \mathrm{Hom}(\mathcal{E}(P), \mathcal{E}(P)) \rightarrow \mathrm{Hom}(\mathcal{O}_C(-Q), \mathcal{E}(P)) = \mathbf{C}.$$

Since the first term is obviously non-trivial, we conclude that

$$\mathrm{Hom}(\mathcal{E}(P), \mathcal{E}(P)) = \mathbf{C}.$$

□

3. THE RELATIVE JACOBIAN

In 1.2 we defined the relative Jacobian of a generic elliptic threefold $X \rightarrow S$ as the relative moduli space of semistable sheaves on the fibers. In this section we use the results in Section 2 to analyze the geometry of the relative Jacobian.

For completeness, we start by providing two examples of generic elliptic threefolds without a section (in both these cases X is also Calabi-Yau).

3.1. 3-section. Let X be a general bidegree (3,3) hypersurface in $\mathbf{P}^2 \times \mathbf{P}^2$, considered with the projection $f : X \rightarrow \mathbf{P}^2$ to one of the two factors of the product $\mathbf{P}^2 \times \mathbf{P}^2$. It is a Calabi-Yau threefold, and the fibers of f are degree 3 curves in \mathbf{P}^2 , generically smooth, so that f is an elliptic fibration. The discriminant locus is a reduced curve of degree 36, with 216 cusps and 189 nodes. (This can be checked directly, using the software package Macaulay [1], or by using Euler characteristic calculations.) Finally, it admits a 3-section (which can be taken to be a general hyperplane section), and it does not admit a section (because, by the Lefschetz theorem, $\mathrm{Pic}(X)$ is the restriction of $\mathrm{Pic}(\mathbf{P}^2 \times \mathbf{P}^2)$, and every divisor in $\mathbf{P}^2 \times \mathbf{P}^2$ meets a fiber of f with multiplicity divisible by 3).

3.2. 5-section. In this example we construct a generic elliptic Calabi-Yau threefold $X \rightarrow \mathbf{P}^2$, embedded in $\mathbf{P}^2 \times \mathbf{P}^4$. Take coordinates $x_0, \dots, x_2, y_0, \dots, y_4$ on $\mathbf{P}^2 \times \mathbf{P}^4$, and let M be a generic 5×5 skew-symmetric matrix whose (i, j) -th entry is a polynomial of bidegree $(1, 1)$ everywhere, except the last row and column, where it is $(0, 1)$. According to [12, 0.1], the 4×4 Pfaffians of this matrix define a degeneracy locus X , which has a symmetric locally free resolution

$$0 \rightarrow \mathcal{L} \rightarrow \mathcal{E} \xrightarrow{\varphi} \mathcal{E}^\vee(\mathcal{L}) \rightarrow \mathcal{O}_{\mathbf{P}^2 \times \mathbf{P}^4} \rightarrow \mathcal{O}_X \rightarrow 0,$$

where

$$\begin{aligned} \mathcal{L} &= \omega_{\mathbf{P}^2 \times \mathbf{P}^4} = \mathcal{O}(-3, -5), \\ \mathcal{E} &= \bigoplus_{i=1}^5 \mathcal{O}(a_i, -3), \\ (a_i) &= (-2, -2, -2, -2, -1), \end{aligned}$$

and the map φ is given by the matrix M . Then it can be easily checked using the results in [12] that X is a smooth Calabi-Yau threefold.

The projection of X to \mathbf{P}^2 is surjective and flat, and the fibers are degree 5 curves in \mathbf{P}^4 given by Pfaffians of a skew-symmetric 5×5 matrix. Therefore the fibers are (generically) elliptic curves, and it can be checked by computer that this exhibits $X \rightarrow \mathbf{P}^2$ as a generic elliptic fibration.

The projection of X to \mathbf{P}^4 maps to a quintic threefold Q in \mathbf{P}^4 , contracting 52 lines and a conic, to 53 ordinary double points in Q . It can now be checked using standard techniques that the Picard number of Q (and therefore that of X) is 2. Let D and H be pull-backs of hyperplane sections from \mathbf{P}^2 and \mathbf{P}^4 , respectively. It is easy to compute intersection numbers. They are:

$$D^3 = 0, \quad D^2H = 5, \quad DH^2 = 9, \quad H^3 = 5.$$

Since D^2H and DH^2 are coprime, D and H must be primitive in $\text{NS}(X)$, so $\mathcal{O}_X(D)$ and $\mathcal{O}_X(H)$ generate $\text{Pic}(X)$. If $F = D^2$ is a fiber of $X \rightarrow \mathbf{P}^2$, then we have $DF = 0$ and $HF = 5$, so we conclude that $n = 5$ (smallest degree of a multi-section). One can take H for a multisection.

3.3. Let $f : X \rightarrow S$ be a generic elliptic fibration, and let $J \rightarrow S$ be the relative Jacobian. Since we have assumed that f has no multiple fibers, we can find a covering $\{U_i\}_{i \in I}$ of S such that the restriction $f_i : X_i = X \times_S U_i \rightarrow U_i$ admits a section s_i for every $i \in I$. Fix such a section s_i for every $i \in I$.

On $X_i \times_S X_i$ we can consider the sheaf \mathcal{E}_i defined as

$$\mathcal{E}_i = \mathcal{I}_\Delta^\vee \otimes \pi_1^* \mathcal{O}_{X_i}(-s_i),$$

where Δ is the diagonal in $X_i \times_S X_i$, \mathcal{I}_Δ^\vee is the dual

$$\mathcal{I}_\Delta^\vee = \underline{\text{Hom}}_{X_i \times_S X_i}(\mathcal{I}_\Delta, \mathcal{O}_{X_i \times_S X_i}),$$

and $\pi_1 : X_i \times_S X_i \rightarrow X_i$, is the projection onto the first factor. Then, as in Section 2, we find that \mathcal{E}_i is a torsion-free sheaf, flat over the second component of the product $X_i \times_S X_i$.

3.4. For $P \in X_i$, let X_P be the unique fiber of the elliptic fibration in which P lies, and let Q be the point of X_P that is in the image of the section s_i . Then the restriction $\mathcal{E}_i(P) = \mathcal{E}_i|_{X_i \times_S \{P\}}$ is isomorphic to one of the sheaves $\mathcal{E}(P)$ considered in Section 2, and therefore it is semistable on the fiber X_P .

This allows us to consider X_i as a base space, parametrizing sheaves on the fibers of f . All the sheaves $\mathcal{E}_i(P)$ are semistable of rank 1, degree 0, so we get a natural map $\varphi_i : X_i \rightarrow J_i = J \times_S U_i$, which only depends on the choice of the section s_i .

Using our knowledge of the geometry of the fibers of J we see that the map φ_i is an isomorphism away from the I_2 fibers of f , where it contracts the component of the fiber that is not hit by the section s_i . (One must remark here that since s_i is a section, it can not pass through any of the singular points of the singular fibers of $X_i \rightarrow U_i$.) Since the contracted components have normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ by 2.1, the contraction of such a component gives an ordinary double point (ODP) in J . Reversing the point of view, X_i is a small resolution of the singularities of J_i .

3.5. Each ODP naturally has *two* small resolutions (remember that we are working in the analytic category) which are related by a flop. As long as U_i is small enough so that J_i contains at most one ODP (which we can achieve since the I_2 fibers are isolated), both resolutions of J_i are isomorphic over U_i to X_i . Indeed, we can replace the section s_i by another section s'_i , which meets the component of the I_2 fiber that was contracted by φ_i . Using s'_i instead of s_i gives a new map $\varphi'_i : X_i \rightarrow J_i$, which is easily seen to be the flop of φ_i .

3.6. The sheaves $\mathcal{E}_i(P)$ remain mutually orthogonal in the global setting, as shown by the next proposition:

Proposition 3.1. *For $P \in X_i$, let $j_P : X_P \rightarrow X$ be the inclusion of the fiber X_P into X , and let $\mathcal{E}_i^0(P) = j_{P,*}\mathcal{E}_i(P)$. Then the sheaves $\mathcal{E}_i^0(P)$ are mutually orthogonal, in the sense that*

$$\mathrm{Ext}_X^j(\mathcal{E}_i^0(P_1), \mathcal{E}_i^0(P_2)) = \begin{cases} \mathbf{C} & \text{if } P_1 = P_2 \text{ and } j = 0, \\ 0 & \text{if } P_1 \neq P_2 \text{ or } j > 3. \end{cases}$$

Proof. The result follows at once from Proposition 2.1, the fact that the projective dimension of X is 3, and the following lemma. \square

Lemma 3.2. *Let $f : X \rightarrow S$ be a morphism of schemes or analytic spaces, with S of the form $\mathrm{Spec} R$ for a regular local ring R . If s is the closed point of S , let $i : X_s \rightarrow X$ be the inclusion into X of the fiber X_s over s , and let \mathcal{F}, \mathcal{G} be sheaves on X_s . If $\mathrm{Ext}_{X_s}^j(\mathcal{F}, \mathcal{G}) = 0$ for all j then $\mathrm{Ext}_X^j(i_*\mathcal{F}, i_*\mathcal{G}) = 0$ for all j .*

Proof. (An adaptation of the proof of [5, 7.2].) We have

$$\mathbf{R}\mathrm{Hom}_X(i_*\mathcal{F}, i_*\mathcal{G}) = \mathbf{R}\mathrm{Hom}_{X_s}(\mathbf{L}i^*i_*\mathcal{F}, \mathcal{G})$$

by the adjunction of $\mathbf{L}i^*$ and i_* . Furthermore,

$$\mathbf{L}i^*i_*\mathcal{F} = \mathcal{F} \otimes_{X_s}^{\mathbf{L}} \mathbf{L}i^*i_*\mathcal{O}_{X_s}$$

by the projection formula. Since S is smooth at s , writing down the Koszul resolution for \mathcal{O}_s and pulling back via f we get a free resolution of \mathcal{O}_{X_s} on X which can be used to compute $\mathbf{L}i^*i_*\mathcal{F}$. This gives

$$H^q(\mathbf{L}i^*i_*\mathcal{F}) = \mathcal{F} \otimes \bigwedge^q \mathcal{O}_{X_s}^{\oplus m}$$

where $m = \dim S$. Now the hypercohomology spectral sequence

$$\begin{aligned} E_2^{p,q} = \text{Ext}_{X_s}^p(H^q(\mathbf{L}i^*i_*\mathcal{F}), \mathcal{G}) &\implies H^{p+q}(\mathbf{R}\text{Hom}_{X_s}(\mathbf{L}i^*i_*\mathcal{F}, \mathcal{G})) \\ &= H^{p+q}(\mathbf{R}\text{Hom}_X(i_*\mathcal{F}, i_*\mathcal{G})) \\ &= \text{Ext}_X^{p+q}(i_*\mathcal{F}, i_*\mathcal{G}) \end{aligned}$$

proves the result. \square

4. TWISTED SHEAVES AND DERIVED CATEGORIES

We sketch here the definition and main properties of twisted sheaves. The reader unfamiliar with the subject is referred to [6, Chapters 1 and 2] or [7].

4.1. Let X be a scheme or analytic space, and let $\alpha \in \check{H}^2(X, \mathcal{O}_X^*)$ be represented by a Čech 2-cocycle, given along a fixed open cover $\{U_i\}_{i \in I}$ by sections

$$\alpha_{ijk} \in \Gamma(U_i \cap U_j \cap U_k, \mathcal{O}_X^*).$$

An α -twisted sheaf \mathcal{F} (along the fixed cover) consists of a pair

$$(\{\mathcal{F}_i\}_{i \in I}, \{\varphi_{ij}\}_{i,j \in I}),$$

where \mathcal{F}_i is a sheaf on U_i for all $i \in I$ and

$$\varphi_{ij} : \mathcal{F}_j|_{U_i \cap U_j} \rightarrow \mathcal{F}_i|_{U_i \cap U_j}$$

is an isomorphism for all $i, j \in I$, subject to the conditions:

- (1) $\varphi_{ii} = \text{id}$;
- (2) $\varphi_{ij} = \varphi_{ji}^{-1}$;
- (3) $\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ki} = \alpha_{ijk} \cdot \text{id}$.

The class of twisted sheaves together with the obvious notion of homomorphism is an abelian category, denoted by $\mathfrak{Mod}(X, \alpha)$, the category of α -twisted sheaves. If one requires all the sheaves \mathcal{F}_i to be coherent, one obtains the category of coherent α -twisted sheaves, denoted by $\mathfrak{Coh}(X, \alpha)$.

This notation is consistent, since one can prove that these categories are independent of the choice of the covering $\{U_i\}$ ([6, 1.2.3]) or of the particular cocycle $\{\alpha_{ijk}\}$ ([6, 1.2.8]) (all the resulting categories are equivalent to one another).

4.2. For \mathcal{F} an α -twisted sheaf, and \mathcal{G} an α' -twisted sheaf, one can define $\mathcal{F} \otimes \mathcal{G}$ (which is an $\alpha\alpha'$ -twisted sheaf), as well as $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ (which is $\alpha^{-1}\alpha'$ -twisted), by gluing together the corresponding sheaves. If $f : Y \rightarrow X$ is any morphism, $f^*\mathcal{F}$ is an $f^*\alpha$ -twisted sheaf on Y . Finally, if $\mathcal{F} \in \mathbf{Mod}(Y, f^*\alpha)$, one can define $f_*\mathcal{F}$, which is α -twisted on X . It is important to note here that one can not define arbitrary push-forwards of twisted sheaves.

These operations satisfy all the usual relations (adjointness of f_* and f^* , relations between $\underline{\mathrm{Hom}}$ and \otimes , etc.)

The category $\mathbf{Mod}(X, \alpha)$ has enough injectives, and enough \mathcal{O}_X -flats ([6, 2.1.1, 2.1.2]).

4.3. In the particular case when α can be represented by a sheaf \mathcal{A} of Azumaya algebras over X (in other words $\alpha \in \mathrm{Br}(X)$, see [15] and [23, Chapter IV]) we can give a more intrinsic description of the categories $\mathbf{Mod}(X, \alpha)$ and $\mathbf{Coh}(X, \alpha)$: they are equivalent to $\mathbf{Mod}(\mathcal{A})$ and $\mathbf{Coh}(\mathcal{A})$, the categories of sheaves of modules (respectively of coherent sheaves of modules) over \mathcal{A} . This equivalence is obtained by first remarking that there is a natural α -twisted locally free sheaf \mathcal{E} such that $\underline{\mathrm{End}}(\mathcal{E}) \cong \mathcal{A}$ (obtained by writing locally $\mathcal{A} \cong \underline{\mathrm{End}}(\mathcal{E}_i)$ for some vector bundle \mathcal{E}_i , and gluing the \mathcal{E}_i 's together into \mathcal{E}) and thus the functors

$$\begin{aligned} F : \mathbf{Mod}(X, \alpha) &\rightarrow \mathbf{Mod}(\mathcal{A}) & F(\cdot) &= \cdot \otimes_{\mathcal{O}_X} \mathcal{E}^\vee, \\ G : \mathbf{Mod}(\mathcal{A}) &\rightarrow \mathbf{Mod}(X, \alpha) & G(\cdot) &= \cdot \otimes_{\mathcal{A}} \mathcal{E} \end{aligned}$$

are inverse to one another by standard Morita theory results.

4.4. If R is a commutative ring, A and B are Azumaya algebras over R , then the R -linear categories $\mathbf{Mod}\text{-}A$ and $\mathbf{Mod}\text{-}B$ are equivalent if and only if $[A] = [B]$ as elements of $\mathrm{Br}(R)$. On the other hand, if R is a k -algebra for some subring k of R , any automorphism $\rho : R \rightarrow R$ over k induces an equivalence of k -linear categories $\rho^* : \mathbf{Mod}\text{-}A \rightarrow \mathbf{Mod}\text{-}\rho^*A$ for any Azumaya algebra A over R . Since the action of $\mathrm{Aut}_k(R)$ on $\mathrm{Br}(R)$ may be non-trivial, one sees at once that there may exist Azumaya algebras A and B over R which are not equal in the Brauer group, but which are k -linearly Morita equivalent. However, the results in [29] suggest that any k -linear Morita equivalence can be made into an R -linear one by pulling back by an automorphism. Therefore it makes sense to make the following conjecture:

Conjecture 4.1. *Let X be a scheme or complex analytic space, and let \mathcal{A}, \mathcal{B} be sheaves of Azumaya algebras over X . Then the \mathbf{C} -linear categories $\mathbf{Coh}(\mathcal{A})$ and $\mathbf{Coh}(\mathcal{B})$ are equivalent if and only if there exists an automorphism ρ of X such that $\rho^*[\mathcal{A}] = [\mathcal{B}]$, where $[\mathcal{A}], [\mathcal{B}]$ denote the classes of \mathcal{A}, \mathcal{B} in $\mathrm{Br}(X)$.*

In other words, the conjecture claims that the set of Azumaya algebras on X , modulo Morita equivalence, is precisely the quotient of $\mathrm{Br}(X)$ by the action of the automorphism group of X .

4.5. We are mainly interested in $\mathbf{D}_{\mathrm{coh}}^b(\mathbf{Mod}(X, \alpha))$, the derived category of complexes of α -twisted sheaves on X with coherent cohomology. For brevity, we'll denote it by $\mathbf{D}_{\mathrm{coh}}^b(X, \alpha)$. Since the category $\mathbf{Coh}(X, \alpha)$ does not have locally free sheaves of finite rank if $\alpha \notin \mathrm{Br}(X)$, from here on we'll only consider the case $\alpha \in \mathrm{Br}(X)$.

The technical details of the inner workings of $\mathbf{D}_{\text{coh}}^b(X, \alpha)$ can be found in [7] or [6, Chapter 2]. The important facts are that one can define derived functors for all the functors considered in 4.2, and they satisfy the same relations as the untwisted ones (see for example [17, II.5]). One can prove duality for a smooth morphism $f : X \rightarrow Y$, which provides a right adjoint

$$f^!(\cdot) = \mathbf{L}f^*(\cdot) \otimes_X \omega_{X/Y}[n]$$

to $\mathbf{R}f_*(\cdot)$, as functors between $\mathbf{D}_{\text{coh}}^b(Y, \alpha)$ and $\mathbf{D}_{\text{coh}}^b(X, f^*\alpha)$.

4.6. If X and Y are smooth schemes or analytic spaces, $\alpha \in \text{Br}(Y)$, and $\mathcal{E} \in \mathbf{D}_{\text{coh}}^b(X \times Y, \pi_Y^*\alpha^{-1})$ (where π_X and π_Y are the projections from $X \times Y$ to X and Y respectively), we define the integral functor

$$\Phi_{Y \rightarrow X}^{\mathcal{E}} : \mathbf{D}_{\text{coh}}^b(Y, \alpha) \rightarrow \mathbf{D}_{\text{coh}}^b(X),$$

given by

$$\Phi_{Y \rightarrow X}^{\mathcal{E}}(\cdot) = \pi_{X,*}(\pi_Y^*(\cdot) \otimes^{\mathbf{L}} \mathcal{E}).$$

The following criterion for determining when $\Phi_{Y \rightarrow X}^{\mathcal{E}}$ is an equivalence (whose proof can be found in [7] or [6, 3.2.1]) is entirely similar to the corresponding ones for untwisted derived categories due to Mukai [25], Bondal-Orlov [2] and Bridgeland [3].

Theorem 4.2. *The functor $F = \Phi_{Y \rightarrow X}^{\mathcal{E}}$ is fully faithful if and only if for each point $y \in Y$,*

$$\text{Hom}_{\mathbf{D}_{\text{coh}}^b(X)}(F\mathcal{O}_y, F\mathcal{O}_y) = \mathbf{C},$$

and for each pair of points $y_1, y_2 \in Y$, and each integer i ,

$$\text{Ext}_{\mathbf{D}_{\text{coh}}^b(X)}^i(F\mathcal{O}_{y_1}, F\mathcal{O}_{y_2}) = 0$$

unless $y_1 = y_2$ and $0 \leq i \leq \dim Y$. (Here \mathcal{O}_y is the skyscraper sheaf \mathbf{C} on y , which is naturally an α -sheaf.)

Assuming the above conditions satisfied, then F is an equivalence of categories if and only if for every point $y \in Y$,

$$F\mathcal{O}_y \otimes^{\mathbf{L}} \omega_X \cong F\mathcal{O}_y.$$

5. THE DERIVED EQUIVALENCE

5.1. The goal of this section is to prove the main theorem:

Theorem 5.1. *Let $X \rightarrow S$ be a generic elliptic threefold, let $J \rightarrow S$ be its relative Jacobian, and let $\bar{J} \rightarrow J$ be an analytic small resolution of the singularities of J , with exceptional locus E . Let $\alpha \in \text{Br}(\bar{J})$ be the unique extension to \bar{J} of the obstruction to the existence of a universal sheaf on $X \times_S (\bar{J} \setminus E)$. Then there exists a $\pi_{\bar{J}}^*\alpha^{-1}$ -twisted pseudo-universal sheaf on $X \times_S \bar{J}$ whose extension by zero to $X \times \bar{J}$ induces an equivalence of derived categories*

$$\mathbf{D}_{\text{coh}}^b(X) \cong \mathbf{D}_{\text{coh}}^b(\bar{J}, \alpha).$$

(By extending by zero we mean pushing forward by the natural inclusion $X \times_S \bar{J} \hookrightarrow X \times \bar{J}$.)

5.2. If X is Calabi-Yau, then an immediate consequence of this theorem is the fact that $K_{\bar{J}} = 0$. Indeed, this follows from the uniqueness of the Serre functor, which is constructed categorically and is thus preserved by an equivalence of categories.

It is also worthwhile remarking that, through the conjectural translation of physics into derived categories, one should interpret this theorem as saying that the conformal field theory built on X (with no discrete torsion in the B -field) is equivalent to the one built on \bar{J} , with discrete torsion α turned on.

5.3. Let us start by explaining the statement of the theorem. In Section 3 we saw that the relative Jacobian J is singular, having an ODP for each I_2 fiber in X , caused by the contraction of a whole S-equivalence class of properly semistable sheaves on X . The local picture we have developed suggests that in order to obtain a well-behaved replacement for the universal sheaf, one should consider a small resolution of these ODP's.

Unfortunately, we can not expect such a small resolution to exist in the algebraic category: in order to obtain one, we need to blow up a Weil divisor which is not even \mathbf{Q} -Cartier, and therefore we need to have $\text{rk Cl}(J) > \text{rk Pic}(J)$. It is not hard to prove that $\text{rk Cl}(J) = \text{rk Cl}(X) = \text{rk Pic}(X)$, and hence if $\text{rk Pic}(X) = 2$ (as is the case in both examples 3.1 and 3.2) J can not have an algebraic small resolution (because $\text{Pic}(J)$ has rank at least 2 having a section and a projective base).

5.4. Therefore we are led to considering an analytic small resolution \bar{J} of the singularities of J (we can take any such resolution). Since the singular points of J coincide with its properly semistable points, we'll call the points in the exceptional locus of the map $\bar{J} \rightarrow J$ semistable as well; the other points of \bar{J} (which are in a 1-1 correspondence with the stable points of J) will be called stable.

Cover S with open sets U_i small enough that there is at most one I_2 fiber in $X_i = X \times_S U_i$, and such that a section $s_i : U_i \rightarrow X_i$ exists. Let $J_i = J \times_S U_i$ and $\bar{J}_i = \bar{J} \times_S U_i$. Then, by 3.5, we can find an isomorphism (of spaces over S) $\varphi_i : \bar{J}_i \rightarrow X_i$. In 3.3 we constructed a sheaf \mathcal{E}_i on $X_i \times_S X_i$ which was a good replacement for the universal sheaf. Pulling back \mathcal{E}_i via the isomorphism $\text{id} \times_S \varphi_i : X_i \times_S \bar{J}_i \rightarrow X_i \times_S X_i$ we obtain a sheaf \mathcal{U}_i on $X_i \times_S \bar{J}_i$ for each i .

We'll call such a sheaf \mathcal{U}_i a local *pseudo-universal* sheaf for the moduli problem under consideration, because it parametrizes the stable sheaves (just like a good universal sheaf should) but over the semistable points of \bar{J} it parametrizes some of the semistable sheaves in the corresponding S-equivalence class.

5.5. We want to show that the local pseudo-universal sheaves \mathcal{U}_i can be put together into a twisted (global) pseudo-universal sheaf \mathcal{U} on $X \times_S \bar{J}$, and to understand the meaning of the actual twisting.

It is a general fact that, for any moduli problem of semistable sheaves on a space X , if the stable part of the moduli space is M^s , there exists a unique $\alpha \in \text{Br}(M^s)$ such that a $\pi_M^* \alpha^{-1}$ -twisted sheaf exists on $X \times M^s$ (see, for example, [6, 3.3.2 and 3.3.4],

as well as [26, Appendix 2]). An easy explanation of this fact goes as follows: cover M^s with open sets U_i small enough to have a local universal sheaf \mathcal{U}_i over $X \times U_i$, and such that $\text{Pic}(U_i \cap U_j) = 0$. Then, from the universal property of the \mathcal{U}_i 's, one concludes that the restrictions $\mathcal{U}_i|_{U_i \cap U_j}$ and $\mathcal{U}_j|_{U_i \cap U_j}$ are isomorphic. Choosing isomorphisms φ_{ij} between them, and using the fact that stable sheaves are simple, one finds that there exists a Čech 2-cocycle $\{\alpha_{ijk}\}$ such that

$$\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ki} = \alpha_{ijk}^{-1} \text{id},$$

therefore $(\{\mathcal{U}_i\}, \{\varphi_{ij}\})$ is an α^{-1} -twisted sheaf. (The inverse is taken for notational convenience.) The uniqueness of α as a cohomology class is an easy consequence of the universality of the \mathcal{U}_i 's, and the fact that $\alpha \in \text{Br}(M^s)$ follows from their flatness. The class α is called the *obstruction to the existence of a universal sheaf* for the moduli problem under consideration.

The sheaves \mathcal{U}_i constructed in 5.4 restrict to universal sheaves along the stable part of \bar{J} , so there exists a unique $\alpha \in \text{Br}(\bar{J}^s)$ which makes the collection $\{\mathcal{U}_i|_{\bar{J}^s}\}$ into a $\pi_{\bar{J}}^* \alpha^{-1}$ -twisted sheaf on \bar{J}^s . But the condition that $\{\mathcal{U}_i\}$ be a twisted sheaf only concerns restrictions of these sheaves to intersections $(X_i \times_S \bar{J}_i) \cap (X_j \times_S \bar{J}_j)$, and these are all in the stable part of \bar{J} . We conclude that there is a unique extension of α to $\check{H}^2(\bar{J}, \mathcal{O}_{\bar{J}}^*)$ that makes $\{\mathcal{U}_i\}$ into a $\pi_{\bar{J}}^* \alpha^{-1}$ -twisted sheaf \mathcal{U} , and from the flatness of \mathcal{U}_i over \bar{J}_i we conclude that $\alpha \in \text{Br}(\bar{J})$ by [6, 3.3.4]. (To extend α from \bar{J}^s to \bar{J} we could also have used the standard purity theorem for cohomological Brauer groups, [15, III, 6.2].)

5.6. In our particular case there is a more down-to-earth description of the twisting. First, note that the open sets $\bar{J}_{ij} = \bar{J}_i \cap \bar{J}_j$ are not small enough to have trivial Picard group, but we can force the Picard group to be generated by sections of the map $\bar{J}_{ij} \rightarrow U_{ij} = U_i \cap U_j$. On $X_{ij} = X_i \cap X_j$ we have the line bundle $\mathcal{O}_{X_{ij}}(s_i - s_j)$, and we can consider its pull-back \mathcal{L}_{ij} to \bar{J}_{ij} via the isomorphism φ_i . The collection $\{\mathcal{L}_{ij}\}$ satisfies the following properties:

- (1) \mathcal{L}_{ii} is trivial;
- (2) $\mathcal{L}_{ij} \cong \mathcal{L}_{ji}^{-1}$;
- (3) $\mathcal{L}_{ijk} = \mathcal{L}_{ij} \otimes \mathcal{L}_{jk} \otimes \mathcal{L}_{ki}$ is trivial, but no canonical trivialization exists;
- (4) $\mathcal{L}_{ijk} \otimes \mathcal{L}_{ijl}^{-1} \otimes \mathcal{L}_{ikl} \otimes \mathcal{L}_{jkl}^{-1}$ is canonically trivial.

These are precisely the necessary properties to make the collection $\{\mathcal{L}_{ij}\}$ represent an element $\alpha \in \check{H}^2(\bar{J}, \mathcal{O}_{\bar{J}}^*)$, which is viewed as a gerbe, i.e., an object obtained by gluing together trivial gerbes via “transition functions” which are line bundles (see [13], [18] and [9] for details). The fact that the collection $\{\mathcal{U}_i\}$ can be made into an α^{-1} -twisted sheaf \mathcal{U} for this α now follows from the definitions.

5.7. Having α and \mathcal{U} we can apply the criterion of Theorem 4.2 to the extension by zero \mathcal{U}^0 of \mathcal{U} to $X \times \bar{J}$. (One can push forward, since both the twisting on $X \times_S \bar{J}$ and on $X \times \bar{J}$ is pulled back from \bar{J} .) Note that the sheaves $F\mathcal{O}_x$ for $x \in \bar{J}$ are precisely $\mathcal{E}_i^0(P)$ of Proposition 3.1, and they have the orthogonality property required. Therefore the integral transform induced by \mathcal{U}^0 is full and faithful. Finally, since the canonical class of the fibers of X is trivial, the last condition of Theorem 4.2

is vacuously true, and hence $\Phi_{\bar{J} \rightarrow X}^{\mathcal{Z}^0}$ is an equivalence of categories between $\mathbf{D}_{\text{coh}}^b(\bar{J}, \alpha)$ and $\mathbf{D}_{\text{coh}}^b(X)$, thus finishing the proof of Theorem 5.1.

6. OGG-SHAFAREVICH THEORY; CONSEQUENCES OF THE MAIN THEOREM

6.1. We want to discuss the connection of the analysis in Section 5 to Ogg-Shafarevich theory. Recall that, in its simplest form, Ogg-Shafarevich theory starts with an elliptic fibration $J_U \rightarrow U$ which has a section and does not have any singular fibers, and establishes a 1-1 correspondence between the Tate-Shafarevich group $\text{III}_U(J_U) = \text{Br}(J_U)/\text{Br}(U)$ and isomorphism classes (as spaces over U) of elliptic fibrations $X_U \rightarrow U$ whose relative Jacobian is isomorphic to $J_U \rightarrow U$. (One needs to be a bit careful here: since $J_U \rightarrow U$ has at least one non-trivial automorphism – the negation along the fibers – one needs to rigidify the isomorphism between J_U and the relative Jacobian of X_U further; otherwise one could not distinguish between α and α^{-1} as elements of $\text{III}_U(J_U)$.) For a general discussion of Ogg-Shafarevich theory and the Tate-Shafarevich group, the reader is referred to [11].

6.2. Let $X \rightarrow S$ be a generic elliptic threefold without a section. By looking at its restriction X_U over $U = S \setminus \Delta$, we obtain an element $\alpha' \in \text{III}_U(J_U)$. Standard results (see [11]) show that there is a natural inclusion $\text{III}_U(J_U) \subseteq \text{Br}'(\bar{J})$ (the cohomological Brauer group of \bar{J}), and thus α' can be regarded as an element in $\text{Br}'(\bar{J})$. Tracing through the definition of α' given by Ogg-Shafarevich theory we find that it coincides precisely with the one we gave for α as a gerbe, in 5.6. We conclude that the obstruction α to the existence of a universal sheaf on $X \times_S \bar{J}$ matches the element α' constructed by Ogg-Shafarevich theory.

6.3. This partly answers an old question of Michael Artin: although one can construct J and α from X in a purely algebraic way (using the étale topology instead of the analytic one), there is no purely algebraic construction that would take us from J and α to X . The above analysis shows that one can take X to be the “spectrum” of the derived category $\mathbf{D}_{\text{coh}}^b(\bar{J}, \alpha)$, i.e. a space X with $\mathbf{D}_{\text{coh}}^b(X) \cong \mathbf{D}_{\text{coh}}^b(\bar{J}, \alpha)$. The unsatisfying aspect of this description is the fact that X is not uniquely determined (see below), and it is not obvious how to describe its elliptic structure just in terms of its derived category.

6.4. An interesting question one can ask now is if we can have

$$\mathbf{D}_{\text{coh}}^b(\bar{J}, \alpha) = \mathbf{D}_{\text{coh}}^b(\bar{J}, \beta)$$

for distinct $\alpha, \beta \in \text{Br}(\bar{J})$. The above analysis, combined with Theorem 5.1 allows us to find examples of this phenomenon.

Theorem 6.1. *Assume we are in the setup of Theorem 5.1, and let n be the order of α in $\text{Br}(\bar{J})$. Then we have*

$$\mathbf{D}_{\text{coh}}^b(\bar{J}, \alpha) \cong \mathbf{D}_{\text{coh}}^b(\bar{J}, \alpha^k),$$

for any k coprime to n .

6.5. Theorem 6.1 is very surprising, in view of the following result which shows that a similar phenomenon can not occur in a local situation:

Theorem 6.2. *Let R be a commutative local ring, A and B Azumaya algebras over R . Then A is R -linearly Morita equivalent to B if and only if A is derived Morita equivalent to B , i.e. $\mathbf{D}_{\text{coh}}^b(\mathfrak{Mod}(A)) \cong \mathbf{D}_{\text{coh}}^b(\mathfrak{Mod}(B))$ as R -linear triangulated categories.*

Proof. [31]. □

A reasoning similar to the one that led to Conjecture 4.1 shows that one should expect that over a local k -algebra R one has $\mathbf{D}_{\text{coh}}^b(A) \cong \mathbf{D}_{\text{coh}}^b(B)$ as \mathbf{C} -linear triangulated categories if and only if there exists an automorphism of R over k that takes $[A]$ to $[B]$ as elements of the Brauer group of R . In the global setting, Theorem 6.1 shows that this is no longer true: in Example 3.2 there is no automorphism ρ of the relative Jacobian \bar{J} which takes α to α^2 . Indeed, let \bar{J} be a small resolution of the relative Jacobian of the elliptic fibration X studied in 3.2, and let $\alpha \in \text{Br}(\bar{J})$ be the element that corresponds to X . Since X has a 5-section, α has order 5 in $\text{Br}(\bar{J})$ by standard Ogg-Shafarevich theory. Assume that ρ is an automorphism of \bar{J} that takes α to α^2 . Since $\text{rk Pic}(\bar{J}) = 2$, there is a unique elliptic fibration structure on \bar{J} , so ρ must act along the fibers of $\bar{J} \rightarrow S$. Restricting it to the generic fiber \bar{J}_ξ of $\bar{J} \rightarrow S$, we obtain an automorphism ρ_ξ of the elliptic curve \bar{J}_ξ . If it fixes the origin it must have order divisible by 5 (since the map $\alpha \mapsto \alpha^2$ has order 5), which is impossible by [16, IV.4.7]. Thus ρ must correspond to translation by a non-zero section s . This section must be torsion since otherwise $\text{rk Pic}(\bar{J}) > 2$. Let H be a general line in S , and let \bar{J}_H be the pull-back of \bar{J} to H . The map $\bar{J}_H \rightarrow H$ has no multiple or reducible fibers, it is algebraic (we stayed away from the small resolutions), and is not isotrivial, so we can apply [10, Proposition 5.3.4 (ii)] to conclude that $\bar{J}_H \rightarrow H$ has no torsion sections, contradicting the existence of s . We conclude that there is no automorphism of \bar{J} taking α to α^2 , and therefore Theorem 6.1 is a purely global phenomenon.

Proof of Theorem 6.1. Let $X^k \rightarrow S$ be the relative moduli space of semistable sheaves of rank 1, degree k on the fibers of $X \rightarrow S$, computed with respect to a polarization $\mathcal{O}_{X/S}(1)$ of fiber degree n . (For the existence of such a polarization see [11, Section 1].) This moduli problem is fine (this is apparently a known fact, [4, 4.2]; or one can prove it directly, [6, 6.6.2]), and using the results in [5] one concludes that X^k is smooth, and the universal sheaf on $X \times_S X^k$ induces an equivalence of derived categories

$$\mathbf{D}_{\text{coh}}^b(X) \cong \mathbf{D}_{\text{coh}}^b(X^k).$$

It is easy to see that X^k is again a generic elliptic threefold: X and X^k are locally isomorphic over open sets in S , by an analysis similar to that of Section 3, simplified by the fact that there are no contractions caused by semistable sheaves. Thus X and X^k have the same singular fibers, and their discriminant loci are the same, therefore X^k is a generic elliptic threefold.

The Jacobian of X^k can be identified in a natural way with J , and therefore we get an isomorphism

$$\mathbf{D}_{\text{coh}}^b(X^k) \cong \mathbf{D}_{\text{coh}}^b(\bar{J}, \beta),$$

where β is the element of $\text{Br}(\bar{J})$ that corresponds to $X^k \rightarrow S$. A computation similar to the one in 5.6 shows that the restrictions of β and α^k along $\bar{J}_{S \setminus \Delta}$ are the same, so by 6.2 we must have $\beta = \alpha^k$ in $\text{Br}(\bar{J})$. Therefore, we have

$$\mathbf{D}_{\text{coh}}^b(\bar{J}, \alpha) \cong \mathbf{D}_{\text{coh}}^b(X) \cong \mathbf{D}_{\text{coh}}^b(X^k) \cong \mathbf{D}_{\text{coh}}^b(\bar{J}, \alpha^k).$$

□

6.6. Note that if X is Calabi-Yau, then X^k is Calabi-Yau as well. Indeed, the equivalence of categories $\mathbf{D}_{\text{coh}}^b(X) \cong \mathbf{D}_{\text{coh}}^b(X^k)$ implies $K_{X^k} = 0$ by the uniqueness of the Serre functor. Also, the universal sheaf induces an isomorphism

$$H^{2,0}(X) \oplus H^{4,0}(X) \cong H^{2,0}(X^k) \oplus H^{4,0}(X^k),$$

by the standard technique of Mukai, and therefore $H^{2,0}(X^k) = 0$, and we conclude that X^k is Calabi-Yau.

This fact can be used to construct counterexamples to the Torelli problem for Calabi-Yau threefolds. See [6, 6.7] and [8] for details.

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DEPARTMENT OF MATHEMATICS AND STATISTICS,
 UNIVERSITY OF MASSACHUSETTS,
 AMHERST, MA 01003-4515, USA
e-mail: andreic@math.umass.edu