# INTERSECTIONS OF TWO GRASSMANNIANS IN $\mathbf{P}^9$

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Abstract. We study the intersection of two copies of $\mathrm{Gr}(2,5)$ embedded in $\mathbf{P}^9$, and the intersection of the two projectively dual Grassmannians in the dual projective space. These intersections are deformation equivalent, derived equivalent Calabi–Yau threefolds. We prove that generically they are not birational. As a consequence, we obtain a counterexample to the birational Torelli problem for Calabi–Yau threefolds. We also show that these threefolds give a new pair of varieties whose classes in the Grothendieck ring of varieties are not equal, but whose difference is annihilated by a power of the class of the affine line. Our proof of non-birationality involves a detailed study of the moduli stack of Calabi–Yau threefolds of the above type, which may be of independent interest.

## 1. Introduction

Let $k$ be an algebraically closed field of characteristic 0. Let $W$ be a 10-dimensional vector space over $k$, whose projectivization we denote by $\mathbf{P} = \mathbf{P}(W)$. Let $V$ be a 5-dimensional vector space over $k$, together with isomorphisms

$$\phi_i \colon \wedge^2 V \xrightarrow{\sim} W, \ i = 1, 2.$$

By composing the Plücker embedding with the resulting isomorphisms $\mathbf{P}(\wedge^2 V) \cong \mathbf{P}$, we obtain two embeddings $\mathrm{Gr}(2,V) \hookrightarrow \mathbf{P}$, whose images we denote by $\mathrm{Gr}_i \subset \mathbf{P}$. For generic $\phi_i$, the intersection

$$X = \mathrm{Gr}_1 \cap \mathrm{Gr}_2 \subset \mathbf{P} \tag{1.1}$$

is a smooth Calabi–Yau threefold (i.e. $\omega_X \cong \mathcal{O}_X$ and $\mathrm{H}^j(X, \mathcal{O}_X) = 0$ for $j = 1, 2$) with Hodge numbers

$$h^{1,1}(X) = 1, \quad h^{1,2}(X) = 51.$$

These varieties first appeared in work of Gross and Popescu [7]. Later G. Kapustka [14] used geometric transitions to construct Calabi–Yau threefolds with the above Hodge numbers, which were shown by M. Kapustka [15] to be isomorphic to Grassmannian intersections of the above form. Independently, Kanazawa [13] gave a direct computation of the Hodge numbers of such Grassmannian intersections. After these authors, we call $X$ as above a *GPK$^3$ threefold*.

The isomorphisms $\phi_i$ naturally determine another GPK$^3$ threefold, as follows. We write $\mathbf{P}^\vee = \mathbf{P}(W^\vee)$ for the dual projective space. Then the induced isomorphisms

$$(\phi_i^{-1})^* \colon \wedge^2 V^\vee \xrightarrow{\sim} W^\vee, \ i = 1, 2,$$

correspond to two embeddings $\mathrm{Gr}(2, V^\vee) \hookrightarrow \mathbf{P}^\vee$, whose images we denote by $\mathrm{Gr}_i^\vee \subset \mathbf{P}^\vee$. As the notation suggests, $\mathrm{Gr}_i^\vee$ is the projective dual of $\mathrm{Gr}_i$ (see Remark 5.4). We consider the intersection

$$Y = \mathrm{Gr}_1^\vee \cap \mathrm{Gr}_2^\vee \subset \mathbf{P}^\vee. \tag{1.2}$$

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If $X$ is a smooth threefold, then $Y$ is too (Lemma 5.1). In this case, $X$ and $Y$ are thus smooth deformation equivalent Calabi–Yau threefolds, which we call *GPK$^3$ double mirrors*. This terminology is justified by the following result, which appears as an example in the forthcoming work [19] (and can also be deduced from [12]). It should be thought of as saying $X$ and $Y$ "have the same mirror".

**Theorem 1.1** ([19])**.** *If $X$ and $Y$ are of the expected dimension* 3 *(but possibly singular), then there is an equivalence*

$$\mathrm{D^b}(X) \simeq \mathrm{D^b}(Y)$$

*of bounded derived categories of coherent sheaves.*

Our main result says that, nonetheless, $X$ and $Y$ are typically not birational.

**Theorem 1.2.** *For generic isomorphisms $\phi_i$, the varieties $X$ and $Y$ are not birational.*

**Remark 1.3.** John Ottem and Jørgen Rennemo [23] have also independently proved Theorem 1.2.

Since $X$ and $Y$ have Picard number 1, the conclusion of Theorem 1.2 is equivalent to $X$ and $Y$ being non-isomorphic. We prove this by an infinitesimal argument, summarized at the end of §1.3 below.

Generic GPK$^3$ double mirrors appear to give the first example of deformation equivalent, derived equivalent, but non-birational Calabi–Yau threefolds. We note that there are several previously known examples of derived equivalent but non-birational Calabi–Yau threefolds: the Pfaffian–Grassmannian pair [5, 17], the Gross–Popescu pair [3, 24], the Reye congruence and double quintic symmetroid pair [9], and the $G_2$-Grassmannian pair [18]. In these examples, the varieties in question are not deformation equivalent and are easily seen to be non-birational.

1.1. **The birational Torelli problem.** One of our motivations for proving Theorem 1.2 was the birational Torelli problem for Calabi–Yau threefolds, which asks the following.

**Question 1.4.** If $M_1$ and $M_2$ are smooth deformation equivalent complex Calabi–Yau threefolds such that there is an isomorphism $\mathrm{H}^3(M_1, \mathbf{Z})_{\mathrm{tf}} \cong \mathrm{H}^3(M_2, \mathbf{Z})_{\mathrm{tf}}$ of polarized Hodge structures, then are $M_1$ and $M_2$ birational?

Here, for an abelian group $A$, we write $A_{\mathrm{tf}}$ for the quotient by its torsion subgroup. As observed in [1, Page 857, footnote], if $M_1$ and $M_2$ are derived equivalent Calabi–Yau threefolds, then there is an isomorphism $\mathrm{H}^3(M_1, \mathbf{Z})_{\mathrm{tf}} \cong \mathrm{H}^3(M_2, \mathbf{Z})_{\mathrm{tf}}$ of polarized Hodge structures. (See also [6, Proposition 3.1] where up to inverting 2 such an isomorphism is shown.) In particular, any pair of deformation equivalent, derived equivalent, but non-birational complex Calabi–Yau threefolds gives a negative answer to Question 1.4. Hence together Theorems 1.1 and 1.2 give the following.

**Corollary 1.5.** *Generic complex GPK$^3$ double mirrors give a counterexample to the birational Torelli problem for Calabi–Yau threefolds.*

Previously, Szendrői [25] showed the usual Torelli problem fails for Calabi–Yau threefolds, i.e. the answer to Question 1.4 is negative if "birational" is replaced with "isomorphic". As far as we know, the birational version was open until now. For earlier work on this problem, see [26, 6].

1.2. **The Grothendieck ring of varieties.** A second motivation for this work was the problem of producing classes in the Grothendieck ring $\mathrm{K}_0(\mathrm{Var}/k)$ of $k$-varieties which are annihilated by a power of the class $\mathbf{L} = [\mathbf{A}^1]$ of the affine line. Recall that $\mathrm{K}_0(\mathrm{Var}/k)$ is defined as the free abelian group on isomorphism classes $[Z]$ of algebraic varieties $Z$ over $k$ modulo the relations

$$[Z] = [U] + [Z \setminus U] \quad \text{for all open subvarieties } U \subset Z,$$

with product induced by products of varieties. In [4] the first author used the Pfaffian–Grassmannian pair of Calabi–Yau 3-folds to show that $\mathbf{L}$ is a zero divisor in $\mathrm{K}_0(\mathrm{Var}/k)$. This sparked a flurry of results, which show that for a number of pairs of derived equivalent Calabi–Yau varieties $(M_1, M_2)$, we have $[M_1] \neq [M_2]$ but $([M_1] - [M_2])\mathbf{L}^r = 0$ for some positive integer $r$. Namely, this holds for the Pfaffian–Grassmannian pair [21] (refining [4]), the $G_2$-Grassmannian pair [10], certain pairs of degree 12 K3 surfaces [8, 11], and certain pairs $(M_1, M_2)$ where $M_1$ is a degree 8 K3 surface and $M_2$ a degree 2 K3 surface [20]. We prove that GPK$^3$ double mirrors give another such example.

**Theorem 1.6.** *If $X$ and $Y$ are GPK$^3$ double mirrors, then*

$$([X] - [Y])\mathbf{L}^4 = 0.$$

*If the isomorphisms $\phi_i$ defining $X$ and $Y$ are generic, then further $[X] \neq [Y]$.*

The first statement is proved by studying a certain incidence correspondence, and the second statement follows from Theorem 1.2 by an argument from [4]. We note that Theorem 1.6 verifies a case of the “D-equivalence implies L-equivalence” conjecture of [20].

1.3. **Geometry of GPK$^3$ threefolds and their moduli.** Along the way to Theorem 1.2, we prove a number of independently interesting results on the geometry of GPK$^3$ threefolds and their moduli.

For $X$ a fixed GPK$^3$ threefold as in (1.1), we prove the two Grassmannians $\mathrm{Gr}_1$ and $\mathrm{Gr}_2$ containing $X$ are unique (Proposition 2.3), and use this to explicitly describe the automorphism group of $X$ (Lemma 2.4).

In terms of moduli, we consider the open subscheme $U$ of the moduli space of pairs of embedded Grassmannians $\mathrm{Gr}_1, \mathrm{Gr}_2 \subset \mathbf{P}$ such that $X = \mathrm{Gr}_1 \cap \mathrm{Gr}_2$ is a smooth threefold. The group $\mathbf{Z}/2 \times \mathrm{PGL}(W)$ acts on $U$ (with $\mathbf{Z}/2$ swapping the Grassmannians), and we define the *moduli stack of GPK$^3$ data* as the quotient $\mathcal{N} = [(\mathbf{Z}/2 \times \mathrm{PGL}(W))\backslash U]$. Let $\mathcal{M}$ be the *moduli stack of GPK$^3$ threefolds*, defined as a $\mathrm{PGL}(W)$-quotient of an open subscheme of the appropriate Hilbert scheme. The morphism $U \to \mathcal{M}$ given pointwise by $(\mathrm{Gr}_1, \mathrm{Gr}_2) \mapsto \mathrm{Gr}_1 \cap \mathrm{Gr}_2$ descends to a morphism $f \colon \mathcal{N} \to \mathcal{M}$, which we call the PGL-*parameterization of* $\mathcal{M}$. Our main moduli-theoretic results are the following.

**Theorem 1.7.** *The* PGL*-parameterization $f \colon \mathcal{N} \to \mathcal{M}$ is an open immersion of smooth separated Deligne–Mumford stacks of finite type over $k$.*

**Theorem 1.8.** *Let $s \in \mathcal{N}$ be a geometric point. Then the automorphism group of $s$ acts faithfully on the tangent space $\mathrm{T}_s\mathcal{N}$, i.e. the homomorphism $\mathrm{Aut}_{\mathcal{N}}(s) \to \mathrm{GL}(\mathrm{T}_s\mathcal{N})$ is injective.*

**Corollary 1.9.** *A generic GPK$^3$ threefold has trivial automorphism group.*

*Proof.* The stack $\mathcal{N}$ is irreducible by construction, and smooth and Deligne–Mumford by Theorem 1.7. It is well-known that in this situation, the generic point of $\mathcal{N}$ has trivial automorphism group if and only if the automorphism groups of geometric points act faithfully on tangent spaces. □

The operation $(\mathrm{Gr}_1, \mathrm{Gr}_2) \mapsto (\mathrm{Gr}_1^\vee, \mathrm{Gr}_2^\vee)$ descends to an involution $\tau\colon \mathcal{N} \to \mathcal{N}$, which we call the *double mirror involution*. In the above terms, our proof of Theorem 1.2 boils down to the following statement: there exists a fixed point $s \in \mathcal{N}$ of $\tau$ such that the derivative $\mathrm{d}_s\tau \in \mathrm{GL}(\mathrm{T}_s\mathcal{N})$ is not contained in the image of the homomorphism $\mathrm{Aut}_{\mathcal{N}}(s) \to \mathrm{GL}(\mathrm{T}_s\mathcal{N})$. For this, we use our description of the automorphism groups of $\mathrm{GPK}^3$ threefolds to show the traces of involutions in the image of $\mathrm{Aut}_{\mathcal{N}}(s) \to \mathrm{GL}(\mathrm{T}_s\mathcal{N})$ are contained in an explicit finite list (Proposition 4.1), and then we exhibit a fixed point $s \in \mathcal{N}$ of $\tau$ such that $\mathrm{tr}(\mathrm{d}_s\tau)$ does not occur in this list (Lemma 6.3).

1.4. **Organization of the paper.** In §2, we prove the results on the geometry of a fixed $\mathrm{GPK}^3$ threefold described above. In §3, we construct the moduli stacks $\mathcal{M}$ and $\mathcal{N}$ of $\mathrm{GPK}^3$ threefolds and $\mathrm{GPK}^3$ data, and prove Theorem 1.7. In §4, we prove our results on the action of automorphism groups on tangent spaces (Theorem 1.8 and Proposition 4.1). In §5, we show that the operation of passing to the double mirror preserves smoothness of $\mathrm{GPK}^3$ threefolds, use this to define the double mirror involution $\tau$ of $\mathcal{N}$, and compute the derivative of $\tau$. In §6 we prove Theorem 1.2. In §7 we prove Theorem 1.6. Finally, in Appendix A we gather some Borel–Weil–Bott computations which are used in the main body of the paper.

1.5. **Notation.** We work over an algebraically closed ground field $k$ of characteristic 0. As above, $V$ and $W$ denote fixed $k$-vector spaces of dimensions 5 and 10, $\mathbf{P} = \mathbf{P}(W)$, and $\mathbf{P}^\vee = \mathbf{P}(W^\vee)$. We fix an isomorphism $\phi\colon \wedge^2 V \xrightarrow{\sim} W$, and let $\mathrm{Gr} \subset \mathbf{P}$ denote the corresponding embedded $\mathrm{Gr}(2, V)$. Further, we set $G = \mathrm{PGL}(W)$ and $H = \mathrm{PGL}(V)$, and denote by $\mathfrak{g}$ and $\mathfrak{h}$ their Lie algebras; there are embeddings $H \to G$ and $\mathfrak{h} \to \mathfrak{g}$ by virtue of the isomorphism $\phi$. Given a variety $Z$ with a morphism to $\mathbf{P}$, we write $\mathcal{O}_Z(1)$ for the pullback of $\mathcal{O}_{\mathbf{P}}(1)$.

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## 2. Geometry of $\mathrm{GPK}^3$ threefolds

In this section, we show that a $\mathrm{GPK}^3$ threefold is contained in a unique pair of Grassmannians in $\mathbf{P}$ (Proposition 2.3). The key ingredient for this is the stability of the restrictions of the normal bundles of these Grassmannians (Proposition 2.1). As a consequence, we obtain an explicit description of the automorphism groups of $\mathrm{GPK}^3$ threefolds (Lemma 2.4).

2.1. **The Grassmannians containing a $\mathbf{GPK^3}$ threefold.** Recall that if $X$ is a smooth $n$-dimensional projective variety with an ample divisor $H$, then the slope of a torsion free sheaf $\mathcal{E}$ on $X$ is defined by

$$\mu(\mathcal{E}) = \frac{c_1(\mathcal{E}) \cdot H^{n-1}}{\mathrm{rk}(\mathcal{E})}.$$

Here, $c_1(\mathcal{E})$ is the first Chern class of the line bundle $\det(\mathcal{E}) = ((\wedge^r \mathcal{E})^\vee)^\vee$, where $r = \mathrm{rk}(\mathcal{E})$. The sheaf $\mathcal{E}$ is called *slope stable* if for every subsheaf $\mathcal{F} \subset \mathcal{E}$ such that $0 < \mathrm{rk}(\mathcal{F}) < \mathrm{rk}(\mathcal{E})$, we have

$$\mu(\mathcal{F}) < \mu(\mathcal{E}).$$

If $X \subset \mathbf{P}$ is a GPK$^3$ threefold, we set $H = c_1(\mathcal{O}_X(1))$.

**Proposition 2.1.** *Let $X = \mathrm{Gr}_1 \cap \mathrm{Gr}_2 \subset \mathbf{P}$ be a GPK$^3$ threefold, and let $\mathrm{N}_i = \mathrm{N}_{\mathrm{Gr}_i/\mathbf{P}}$ be the normal bundle of $\mathrm{Gr}_i \subset \mathbf{P}$. Then $\mathrm{N}_i|_X$ is slope stable.*

*Proof.* By Lemma A.4 there is an isomorphism $\mathrm{N}_i \cong \mathcal{Q}_i^\vee(2)$, where $\mathcal{Q}_i$ is the tautological rank 3 quotient bundle. Hence it suffices to prove that the bundle $\mathcal{Q}_i|_X$ is stable. Let $\mathcal{F} \subset \mathcal{Q}_i|_X$ be a subsheaf of rank $r = 1$ or 2. Since $H$ generates $\mathrm{Pic}(X)$ (see [14]), we can write $c_1(\mathcal{F}) = tH$ for some $t \in \mathbf{Z}$. Then taking the $r$-th exterior power of the inclusion $\mathcal{F} \subset \mathcal{Q}_i|_X$ and passing to double duals, we get a nonzero section of $\wedge^r(\mathcal{Q}_i|_X)(-tH)$. Hence $t \leq 0$ by Lemma A.10. We conclude

$$\mu(\mathcal{F}) = tH^3/r < H^3/3 = \mu(\mathcal{Q}_i|_X). \qquad \square$$

The following result shows the representation of a GPK$^3$ threefold as an intersection of two Grassmannians is unique.

**Lemma 2.2.** *Let $X \subset \mathbf{P}$ be a GPK$^3$ threefold. Assume $\phi_i \colon \wedge^2 V \xrightarrow{\sim} W$, $1 \leq i \leq 4$, are isomorphisms whose corresponding Grassmannian embeddings $\mathrm{Gr}_i \subset \mathbf{P}$ satisfy*

$$X = \mathrm{Gr}_1 \cap \mathrm{Gr}_2 = \mathrm{Gr}_3 \cap \mathrm{Gr}_4.$$

*Then either $\mathrm{Gr}_1 = \mathrm{Gr}_3$ and $\mathrm{Gr}_2 = \mathrm{Gr}_4$, or $\mathrm{Gr}_1 = \mathrm{Gr}_4$ and $\mathrm{Gr}_2 = \mathrm{Gr}_3$.*

*Proof.* As above, let $\mathrm{N}_i = \mathrm{N}_{\mathrm{Gr}_i/\mathbf{P}}$. The restrictions $\mathrm{N}_i|_X$ all have the same slope, and are stable by Proposition 2.1. Hence any morphism $\mathrm{N}_i|_X \to \mathrm{N}_j|_X$ is either zero or an isomorphism. Considering the inclusion $\mathrm{N}_i|_X \subset \mathrm{N}_{X/\mathbf{P}}$, $i = 1, 2$, followed by projection onto the summands of the decomposition $\mathrm{N}_{X/\mathbf{P}} \cong \mathrm{N}_3|_X \oplus \mathrm{N}_4|_X$, we conclude that either $\mathrm{N}_1|_X \cong \mathrm{N}_3|_X$ and $\mathrm{N}_2|_X \cong \mathrm{N}_4|_X$, or $\mathrm{N}_1|_X \cong \mathrm{N}_4|_X$ and $\mathrm{N}_2|_X \cong \mathrm{N}_3|_X$. Hence to finish, it suffices to show that the isomorphism class of $\mathrm{N}_i|_X$ determines $\mathrm{Gr}_i \subset \mathbf{P}$.

By Lemma A.4, $\mathrm{N}_i|_X$ determines the restriction $\mathcal{Q}_i|_X$ of the tautological rank 3 quotient bundle via $(\mathrm{N}_i|_X)^\vee(2) \cong \mathcal{Q}_i|_X$. The inclusion $\mathrm{Gr}_i \subset \mathbf{P}$ is determined by $\mathcal{Q}_i$ as follows: $V \cong \mathrm{H}^0(\mathrm{Gr}_i, \mathcal{Q}_i)$, taking the third exterior power induces an isomorphism $\wedge^3 V \cong W^\vee$ (note that $\wedge^3 \mathcal{Q}_i = \mathcal{O}(1)$), and the dual isomorphism $\wedge^3 V^\vee \cong W$ is identified with $\phi_i$ under the isomorphism $\wedge^2 V \cong \wedge^3 V^\vee$. But the restriction maps $V \cong \mathrm{H}^0(\mathrm{Gr}_i, \mathcal{Q}_i) \to \mathrm{H}^0(X, \mathcal{Q}_i|_X)$ and $W^\vee \cong \mathrm{H}^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(1)) \to \mathrm{H}^0(X, \mathcal{O}_X(1))$ are isomorphisms by Lemma A.11. The isomorphism class of $\mathrm{N}_i|_X$ thus determines $\phi_i \colon \wedge^2 V \xrightarrow{\sim} W$ up to an isomorphism of $V$, and hence determines the Grassmannian $\mathrm{Gr}_i$. $\square$

As a slight strengthening of Lemma 2.2, we prove that a GPK$^3$ threefold is contained in a unique pair of Grassmannians. We note, however, that Lemma 2.2 already suffices for our purposes in this paper.

**Proposition 2.3.** *Let $X = \mathrm{Gr}_1 \cap \mathrm{Gr}_2 \subset \mathbf{P}$ be a GPK$^3$ threefold. Let $\mathrm{Gr}_3 \subset \mathbf{P}$ be the image of an embedding $\mathrm{Gr}(2, V) \hookrightarrow \mathbf{P}$ given by an isomorphism $\phi_3 \colon \wedge^2 V \xrightarrow{\sim} W$. If $X \subset \mathrm{Gr}_3$, then either $\mathrm{Gr}_3 = \mathrm{Gr}_1$ or $\mathrm{Gr}_3 = \mathrm{Gr}_2$.*

*Proof.* Let $\mathrm{N}_i = \mathrm{N}_{\mathrm{Gr}_i/\mathbf{P}}$ for $i = 1, 2, 3$. We have an injective morphism

$$(\mathrm{N}_3|_X)^\vee \to (\mathrm{N}_1|_X)^\vee \oplus (\mathrm{N}_2|_X)^\vee.$$

We claim that one of the projections $\alpha_1\colon (\mathrm{N}_3|_X)^\vee \to (\mathrm{N}_1|_X)^\vee$ or $\alpha_2\colon (\mathrm{N}_3|_X)^\vee \to (\mathrm{N}_2|_X)^\vee$ is an isomorphism. Since the $(\mathrm{N}_i|_X)^\vee$ all have the same rank and determinant, $\alpha_k$ is an isomorphism if it is injective. Hence it suffices to show that either $\mathcal{K} = \ker(\alpha_2) \subset (\mathrm{N}_1|_X)^\vee$ or $\mathcal{I} = \mathrm{im}(\alpha_2) \subset (\mathrm{N}_2|_X)^\vee$ vanishes. If $\mu$ is the common slope of the $(\mathrm{N}_i|_X)^\vee$, then either $\mu(\mathcal{K}) \geq \mu$ or $\mu(\mathcal{I}) \geq \mu$. Since $(\mathrm{N}_1|_X)^\vee$ and $(\mathrm{N}_2|_X)^\vee$ are slope stable by Proposition 2.1, it follows that either $\mathcal{K}$ or $\mathcal{I}$ vanishes.

So we may assume $\alpha_1\colon (\mathrm{N}_3|_X)^\vee \to (\mathrm{N}_1|_X)^\vee$ is an isomorphism. Note that this means $\mathrm{Gr}_2$ and $\mathrm{Gr}_3$ intersect transversely along $X$. We will show that $X = \mathrm{Gr}_2 \cap \mathrm{Gr}_3$, which by Lemma 2.2 proves the proposition. The intersection $\mathrm{Gr}_2 \cap \mathrm{Gr}_3$ consists of components of dimension at least 3. If there are no components of dimension at least 4, then $X$ and $\mathrm{Gr}_2 \cap \mathrm{Gr}_3$ have the same degree in $\mathbf{P}$, forcing the inclusion $X \subset \mathrm{Gr}_2 \cap \mathrm{Gr}_3$ to be an equality. Hence it suffices to show there are no components of dimension at least 4. By the transversality of $\mathrm{Gr}_2$ and $\mathrm{Gr}_3$ along $X$, the components of the intersection $\mathrm{Gr}_2 \cap \mathrm{Gr}_3$ which are not equal to $X$ must be disjoint from $X$. But by Lemma A.9 the class of $X$ in the Chow ring of $\mathrm{Gr}_2$ is $5H^3$, which implies $X$ intersects nontrivially any effective cycle in $\mathrm{Gr}_3$ of dimension at least 4. □

2.2. **Automorphism groups.** GPK$^3$ threefolds can alternatively be described as intersections of translates of a fixed Grassmannian. Namely, fix an isomorphism $\phi\colon \wedge^2 V \xrightarrow{\sim} W$. Let $\mathrm{Gr} \subset \mathbf{P}$ denote the corresponding embedded Grassmannian $\mathrm{Gr}(2, V)$. Let $G = \mathrm{PGL}(W)$. Then for any $(g_1, g_2) \in G \times G$ we set

$$X_{g_1,g_2} = g_1\mathrm{Gr} \cap g_2\mathrm{Gr} \subset \mathbf{P}. \tag{2.1}$$

By definition, GPK$^3$ threefolds are precisely the smooth $X_{g_1,g_2}$. Note that setting $H = \mathrm{PGL}(V)$, there is an embedding $H \to G$ induced by the isomorphism $\wedge^2 V \cong W$.

**Lemma 2.4.** *Let $X = X_{g_1,g_2}$ be a GPK$^3$ threefold. The automorphism group scheme* $\mathrm{Aut}(X)$ *is finite and reduced, and can be described explicitly as*

$$\mathrm{Aut}(X) = (g_1Hg_1^{-1} \cap g_2Hg_2^{-1}) \cup (g_2Hg_1^{-1} \cap g_1Hg_2^{-1}) \subset G.$$

*Proof.* Since $X$ is a Calabi–Yau variety of Picard number 1, it follows that $\mathrm{Aut}(X)$ is finite and reduced. Further, $\mathrm{Aut}(X)$ embeds in $G$ as the automorphisms $a \in G$ of $\mathbf{P}$ which fix $X$, i.e. which satisfy

$$ag_1\mathrm{Gr} \cap ag_2\mathrm{Gr} = g_1\mathrm{Gr} \cap g_2\mathrm{Gr}.$$

By Proposition 2.3 this means either $ag_1\mathrm{Gr} = g_1\mathrm{Gr}$ and $ag_2\mathrm{Gr} = g_2\mathrm{Gr}$, or $ag_1\mathrm{Gr} = g_2\mathrm{Gr}$ and $ag_2\mathrm{Gr} = g_1\mathrm{Gr}$. The first case is equivalent to $a \in g_1Hg_1^{-1} \cap g_2Hg_2^{-1}$ and the second to $a \in g_2Hg_1^{-1} \cap g_1Hg_2^{-1}$. □

## 3. Moduli of GPK$^3$ threefolds

The goal of this section is to prove Theorem 1.7. In §3.1 we construct the moduli stack $\mathcal{M}$ of GPK$^3$ threefolds, and show it is a smooth separated Deligne–Mumford stack of finite type over $k$. In §3.2 we construct the moduli stack $\mathcal{N}$ of GPK$^3$ data (as a quotient of an open subspace of $\mathrm{PGL}(W) \times \mathrm{PGL}(W)$) and the PGL-parameterization $f\colon \mathcal{N} \to \mathcal{M}$, and show that $\mathcal{N}$ has the same properties as $\mathcal{M}$. In §3.3 we show that the derivative of $f$ at any point is an isomorphism. Finally, in §3.4 we combine these results to prove Theorem 1.7.

3.1. **The moduli stack of $\mathbf{GPK^3}$ threefolds.** Let $P \in \mathbf{Q}[t]$ be the Hilbert polynomial of a $\mathrm{GPK}^3$ threefold. Let Hilb denote the Hilbert scheme of closed subschemes of $\mathbf{P}$ with Hilbert polynomial $P$, and let $\mathrm{Hilb}^\circ \subset \mathrm{Hilb}$ denote the open subscheme parameterizing Calabi–Yau threefolds which are smooth deformations of a $\mathrm{GPK}^3$ threefold. The natural (left) action of $G = \mathrm{PGL}(W)$ on Hilb preserves $\mathrm{Hilb}^\circ$. Let

$$\mathcal{M} = [G\backslash \mathrm{Hilb}^\circ] \tag{3.1}$$

be the quotient stack and $q_{\mathcal{M}} \colon \mathrm{Hilb}^\circ \to \mathcal{M}$ the quotient morphism. We call $\mathcal{M}$ the *moduli stack of $GPK^3$ threefolds* (although strictly speaking $\mathcal{M}$ parameterizes smooth deformations of $\mathrm{GPK}^3$ threefolds).

**Lemma 3.1.** *The stack $\mathcal{M}$ is a smooth separated Deligne–Mumford stack of finite type over $k$. Moreover, $\mathcal{M}$ admits a coarse moduli space $\pi_{\mathcal{M}} \colon \mathcal{M} \to M$, where $M$ is a separated algebraic space of finite type over $k$.*

*Proof.* A geometric point of $\mathrm{Hilb}^\circ$ corresponds to a Calabi–Yau threefold of Picard number 1, so its stabilizer in $G$ is finite and reduced. Hence $\mathcal{M}$ is Deligne–Mumford. The scheme $\mathrm{Hilb}^\circ$ is of finite type over $k$, and it is smooth by the Bogomolov–Tian–Todorov theorem on unobstrutedness of Calabi–Yau varieties. Hence $\mathcal{M}$ is smooth and of finite type over $k$. Separatedness of $\mathcal{M}$ follows from a result of Matsusaka and Mumford [22]. Finally, the existence of a coarse moduli space with the stated properties then follows from a result of Keel and Mori [16]. □

**Remark 3.2.** Let $X$ be a $\mathrm{GPK}^3$ threefold. By Lemma 2.4, the automorphism group scheme $\mathrm{Aut}(X)$ coincides with the automorphism group scheme $\mathrm{Aut}_{\mathcal{M}}([X])$ of the corresponding point $[X] \in \mathcal{M}$.

3.2. **The** PGL**-parameterization.** In §2.2 we observed any $\mathrm{GPK}^3$ threefold can be written in the form (2.1). We obtain a parameterization of $\mathcal{M}$ by quotienting by the redundancies in this description, as follows. The quotient $G/H$ is a quasi-projective variety, which can be thought of as a parameter space for embeddings of the Grassmannian $\mathrm{Gr}(2,V)$ into $\mathbf{P}$. Namely, for any point $g \in G$ the corresponding Grassmannian is $g\mathrm{Gr}$, which only depends on the coset $gH$. Similarly $X_{g_1,g_2} \subset \mathbf{P}$ only depends on the coset $(g_1H, g_2H)$, and the family of these varieties over $G \times G$ descends to a closed subscheme

$$\mathcal{X} \subset G/H \times G/H \times \mathbf{P}.$$

Let $U \subset G/H \times G/H$ be the open subscheme parameterizing the smooth 3-dimensional fibers of $\mathcal{X}$. Then the restriction $\mathcal{X}_U \to U$ is a family of $\mathrm{GPK}^3$ threefolds, such that every $\mathrm{GPK}^3$ threefold occurs as a fiber. This family induces a morphism $\tilde{f} \colon U \to \mathrm{Hilb}^\circ$.

The group $\mathbf{Z}/2 \times G$ acts on $G/H \times G/H$ (on the left), where $\mathbf{Z}/2$ acts by swapping the two factors and $G$ acts by multiplication. This action preserves $U \subset G/H \times G/H$. Let

$$\mathcal{N} = [(\mathbf{Z}/2 \times G)\backslash U] \tag{3.2}$$

be the quotient stack and $q_{\mathcal{N}} \colon U \to \mathcal{N}$ the quotient morphism. We call $\mathcal{N}$ the *moduli stack of $GPK^3$ data*.

The morphism $\tilde{f} \colon U \to \mathrm{Hilb}^\circ$ is equivariant with respect to the projection $\mathbf{Z}/2 \times G \to G$, and hence descends to a morphism

$$f \colon \mathcal{N} \to \mathcal{M},$$

which we call the PGL-*parameterization of* $\mathcal{M}$.

We note the following consequence of Proposition 2.3 and Lemma 2.4. Given a stack $\mathcal{Y}$ over $k$, we denote by $|\mathcal{Y}(k)|$ the set of isomorphism classes of the $k$-points $\mathcal{Y}(k)$.

**Lemma 3.3.** *We have:*

*(1) $f$ induces an injection $|\mathcal{N}(k)| \to |\mathcal{M}(k)|$.*
*(2) $f$ induces isomorphisms between the automorphism group schemes of points.*

We have the following analog of Lemma 3.1 for $\mathcal{N}$.

**Lemma 3.4.** *The stack $\mathcal{N}$ is a smooth separated Deligne–Mumford stack of finite type over $k$. Moreover, $\mathcal{N}$ admits a coarse moduli space $\pi_{\mathcal{N}} \colon \mathcal{N} \to N$, where $N$ is a separated algebraic space of finite type over $k$.*

*Proof.* Since the scheme $U$ is smooth and of finite type over $k$, so is the quotient stack $\mathcal{N}$. Since $\mathcal{M}$ is Deligne–Mumford, so is $\mathcal{N}$ by Lemma 3.3. It remains to show that $\mathcal{N}$ is separated; then the existence of a coarse moduli space with the stated properties follows from a result of Keel and Mori [16]. By the valuative criterion, this amounts to the following. Let $R$ be a valuation ring with field of fractions $K$, let $x, x' \colon \operatorname{Spec}(R) \to \mathcal{N}$ be two $\operatorname{Spec}(R)$-points of $\mathcal{N}$, and let $\gamma \colon x|_K \xrightarrow{\sim} x'|_K$ be an isomorphism of the restrictions to $\operatorname{Spec}(K)$. Then we must show there exists an isomorphism $\tilde{\gamma} \colon x \xrightarrow{\sim} x'$ restricting to $\gamma$. The points $x$ and $x'$ correspond to the data of embeddings $\mathrm{Gr}_{1,R}, \mathrm{Gr}_{2,R} \subset \mathbf{P}_R$ and $\mathrm{Gr}'_{1,R}, \mathrm{Gr}'_{2,R} \subset \mathbf{P}_R$ of $\mathrm{Gr}_R(2,V)$ such that

$$X_R = \mathrm{Gr}_{1,R} \cap \mathrm{Gr}_{2,R} \quad \text{and} \quad X'_R = \mathrm{Gr}'_{1,R} \cap \mathrm{Gr}'_{2,R}$$

are families of GPK$^3$ threefolds over $\operatorname{Spec}(R)$. Using the presentation $\mathbf{Z}/2 = \{\pm 1\}$, the isomorphism $\gamma$ corresponds to a point $(\epsilon, a) \in \mathbf{Z}/2 \times G(K)$ such that

$$\begin{aligned} a\mathrm{Gr}_{1,K} &= \mathrm{Gr}'_{1,K},\ a\mathrm{Gr}_{2,K} = \mathrm{Gr}'_{2,K} && \text{if } \epsilon = +1, \\ a\mathrm{Gr}_{1,K} &= \mathrm{Gr}'_{2,K},\ a\mathrm{Gr}_{2,K} = \mathrm{Gr}'_{1,K} && \text{if } \epsilon = -1. \end{aligned}$$

In particular, $a$ is an automorphism of $\mathbf{P}_K$ such that $aX_K = X'_K$. By separatedness of the moduli stack $\mathcal{M}$ (Lemma 3.1), we can find $\tilde{a} \in G(R)$ restricting to $a$ such that $\tilde{a}X_R = X'_R$. We claim

$$\begin{aligned} \tilde{a}\mathrm{Gr}_{1,R} &= \mathrm{Gr}'_{1,R},\ a\mathrm{Gr}_{2,R} = \mathrm{Gr}'_{2,R} && \text{if } \epsilon = +1, \\ \tilde{a}\mathrm{Gr}_{1,R} &= \mathrm{Gr}'_{2,R},\ a\mathrm{Gr}_{2,R} = \mathrm{Gr}'_{1,R} && \text{if } \epsilon = -1. \end{aligned}$$

Indeed, by the separatedness of the parameter space $G/H$ for embeddings of $\mathrm{Gr}(2,V)$ into $\mathbf{P}$, if two embeddings of $\mathrm{Gr}_R(2,V)$ into $\mathbf{P}_R$ coincide after restriction to $K$, then they coincide. Hence $(\epsilon, \tilde{a}) \in \mathbf{Z}/2 \times G(R)$ gives the desired extension of $\gamma$ to an isomorphism $\tilde{\gamma} \colon x \xrightarrow{\sim} x'$. □

3.3. **Derivative of the** PGL**-parameterization.** Let $s \in U$ be a point, let $t = \tilde{f}(s) \in \mathrm{Hilb}^\circ$, and denote by $[s] \in \mathcal{N}$ and $[t] \in \mathcal{M}$ their images. Our goal is to prove that the derivative $\mathrm{d}_{[s]}f \colon \mathrm{T}_{[s]}\mathcal{N} \to \mathrm{T}_{[t]}\mathcal{M}$ is an isomorphism. In fact, we will prove a slightly more precise result, which gives an explicit description of these tangent spaces.

To formulate this, let $\mathrm{act}_s \colon G \to U$ be the action morphism at $s$ given by $\mathrm{act}_s(g) = g \cdot s$, and similarly let $\mathrm{act}_t \colon G \to \mathrm{Hilb}^\circ$ be the action morphism at $t$ given by $\mathrm{act}_t(g) = g \cdot t$. Then there is a commutative diagram

$$\begin{array}{ccccc} G & \xrightarrow{\mathrm{act}_s} & U & \xrightarrow{q_{\mathcal{N}}} & \mathcal{N} \\ \| & & \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f} \\ G & \xrightarrow{\mathrm{act}_t} & \mathrm{Hilb}^\circ & \xrightarrow{q_{\mathcal{M}}} & \mathcal{M} \end{array} \tag{3.3}$$

Let $\mathfrak{g}$ denote the Lie algebra of $G$.

**Proposition 3.5.** *Taking derivatives in* (3.3) *gives a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{g} & \xrightarrow{\mathrm{d}_1\mathrm{act}_s} & \mathrm{T}_sU & \xrightarrow{\mathrm{d}_s q_{\mathcal{N}}} & \mathrm{T}_{[s]}\mathcal{N} & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \mathrm{d}_s\tilde{f}}\,\wr & & \downarrow{\scriptstyle \mathrm{d}_{[s]}f}\,\wr & & \\ 0 & \longrightarrow & \mathfrak{g} & \xrightarrow{\mathrm{d}_1\mathrm{act}_t} & \mathrm{T}_t\mathrm{Hilb}^\circ & \xrightarrow{\mathrm{d}_t q_{\mathcal{M}}} & \mathrm{T}_{[t]}\mathcal{M} & \longrightarrow & 0 \end{array} \tag{3.4}$$

*with exact rows and vertical maps isomorphisms.*

From the presentations of $\mathcal{N}$ and $\mathcal{M}$ as quotient stacks it follows that the rows of (3.4) are right exact, but since $\mathcal{N}$ and $\mathcal{M}$ are Deligne–Mumford (Lemmas 3.4 and 3.1), they are in fact exact. Hence to prove Proposition 3.5, it suffices to show $\mathrm{d}_s\tilde{f}\colon \mathrm{T}_sU \to \mathrm{T}_t\mathrm{Hilb}^\circ$ is an isomorphism.

To this end, we factor $\tilde{f}\colon U \to \mathrm{Hilb}^\circ$ as follows. Let $Q \in \mathbf{Q}[t]$ be the Hilbert polynomial of $\mathrm{Gr} \subset \mathbf{P}$, and let $\mathrm{Hilb}_Q$ be the Hilbert scheme of closed subschemes of $\mathbf{P}$ with Hilbert polynomial $Q$. Let $\mathcal{Y} \subset \mathrm{Hilb}_Q \times \mathbf{P}$ denote the universal family, and let $\mathcal{Y}_i \to \mathrm{Hilb}_Q \times \mathrm{Hilb}_Q$, $i = 1, 2$, denote its pullback along each of the projections. Define $U' \subset \mathrm{Hilb}_Q \times \mathrm{Hilb}_Q$ to be the open subscheme over which the fibers of the morphism

$$\mathcal{Y}_1 \times_{\mathbf{P}} \mathcal{Y}_2 \to \mathrm{Hilb}_Q \times \mathrm{Hilb}_Q$$

are smooth deformations of a $\mathrm{GPK}^3$ threefold, and let

$$\tilde{f}'\colon U' \to \mathrm{Hilb}^\circ$$

be the induced morphism. The morphism $\tilde{f}$ factors through $\tilde{f}'$. Indeed, consider the closed subscheme $\mathcal{Z} \subset G/H \times \mathbf{P}$ whose fiber over $[g] \in G/H$ is $g\mathrm{Gr} \subset \mathbf{P}$. This induces a morphism

$$\gamma\colon G/H \to \mathrm{Hilb}_Q$$

such that $\gamma \times \gamma\colon G/H \times G/H \to \mathrm{Hilb}_Q \times \mathrm{Hilb}_Q$ takes $U$ into $U'$, and hence induces a morphism

$$j\colon U \to U'.$$

It follows from the definitions that $\tilde{f} = \tilde{f}' \circ j$. Thus $\mathrm{d}_s\tilde{f} = \mathrm{d}_{j(s)}\tilde{f}' \circ \mathrm{d}_s j$, so to prove Proposition 3.5 it suffices to show $\mathrm{d}_s j$ and $\mathrm{d}_{j(s)}\tilde{f}'$ are isomorphisms. This is the content of the next two lemmas.

**Lemma 3.6.** *The map* $\mathrm{d}_s j\colon \mathrm{T}_sU \to \mathrm{T}_{j(s)}U'$ *is an isomorphism.*

*Proof.* By the definition of $j$, it suffices to show that for any $g \in G$ the map

$$\mathrm{d}_{gH}\gamma\colon \mathrm{T}_{gH}(G/H) \to \mathrm{T}_{\gamma(gH)}\mathrm{Hilb}_Q$$

is an isomorphism. The point $\gamma(gH) \in \mathrm{Hilb}_Q$ corresponds to the subscheme $Z = g\mathrm{Gr} \subset \mathbf{P}$, and there is a canonical isomorphism $\mathrm{T}_{\gamma(gH)}\mathrm{Hilb}_Q \cong \mathrm{H}^0(Z, \mathrm{N}_{Z/\mathbf{P}})$. The exact sequence

$$0 \to \mathrm{T}_Z \to \mathrm{T}_{\mathbf{P}}|_Z \to \mathrm{N}_{Z/\mathbf{P}} \to 0$$

induces a long exact sequence

$$0 \to \mathrm{H}^0(Z, \mathrm{T}_Z) \to \mathrm{H}^0(Z, \mathrm{T}_{\mathbf{P}}|_Z) \to \mathrm{H}^0(Z, \mathrm{N}_{Z/\mathbf{P}}) \to \mathrm{H}^1(Z, \mathrm{T}_Z) \to \dots$$

We have $\mathrm{H}^1(Z, \mathrm{T}_Z) = 0$, so the first three terms form a short exact sequence. Moreover, there is a canonical isomorphism $\mathfrak{h} \cong \mathrm{H}^0(Z, \mathrm{T}_Z)$, since $\mathrm{H}^0(Z, \mathrm{T}_Z)$ is identified with the tangent space to $H \cong \mathrm{Aut}(Z)$. By the same reason $\mathfrak{g} \cong \mathrm{H}^0(\mathbf{P}, \mathrm{T}_{\mathbf{P}})$. By Lemma A.3 the restriction map $\mathrm{H}^0(\mathbf{P}, \mathrm{T}_{\mathbf{P}}) \to \mathrm{H}^0(Z, \mathrm{T}_{\mathbf{P}}|_Z)$ is an isomorphism, so we get an isomorphism $\mathfrak{g} \cong \mathrm{H}^0(\mathbf{P}, \mathrm{T}_{\mathbf{P}}|_Z)$. The isomorphisms $\mathfrak{h} \cong \mathrm{H}^0(Z, \mathrm{T}_Z)$ and $\mathfrak{g} \cong \mathrm{H}^0(Z, \mathrm{T}_{\mathbf{P}}|_Z)$ are compatible with the canonical isomorphism $\mathrm{T}_{gH}(G/H) \cong \mathfrak{g}/\mathfrak{h}$, i.e. they fit into a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{h} & \longrightarrow & \mathfrak{g} & \longrightarrow & \mathrm{T}_{gH}(G/H) & \longrightarrow & 0 \\ & & \downarrow \wr & & \downarrow \wr & & \downarrow \mathrm{d}_{gH}\gamma & & \\ 0 & \longrightarrow & \mathrm{H}^0(Z, \mathrm{T}_Z) & \longrightarrow & \mathrm{H}^0(Z, \mathrm{T}_{\mathbf{P}}|_Z) & \longrightarrow & \mathrm{H}^0(Z, \mathrm{N}_{Z/\mathbf{P}}) & \longrightarrow & 0 \end{array}$$

So $\mathrm{d}_{gH}\gamma$ is an isomorphism. □

**Lemma 3.7.** *The map* $\mathrm{d}_{j(s)}\tilde{f}' \colon \mathrm{T}_{j(s)}U' \to \mathrm{T}_t\mathrm{Hilb}^\circ$ *is an isomorphism.*

*Proof.* Write $s = (g_1H, g_2H)$ and let $\mathrm{Gr}_i = g_i\mathrm{Gr}$, so that $X = \mathrm{Gr}_1 \cap \mathrm{Gr}_2$ is the GPK$^3$ threefold corresponding to $s$. Let $\mathrm{N}_i = \mathrm{N}_{\mathrm{Gr}_i/\mathbf{P}}$. Then since $U' \subset \mathrm{Hilb}_Q \times \mathrm{Hilb}_Q$ is an open subscheme, there is a canonical isomorphism

$$\mathrm{T}_{j(s)}U' \cong \mathrm{H}^0(\mathrm{Gr}_1, \mathrm{N}_1) \oplus \mathrm{H}^0(\mathrm{Gr}_2, \mathrm{N}_2).$$

Similarly, since $\mathrm{N}_{X/\mathbf{P}} \cong \mathrm{N}_1|_X \oplus \mathrm{N}_2|_X$, there is an isomorphism

$$\mathrm{T}_t\mathrm{Hilb}^\circ \cong \mathrm{H}^0(X, \mathrm{N}_1|_X) \oplus \mathrm{H}^0(X, \mathrm{N}_2|_X).$$

Under the above isomorphisms, the map $\mathrm{d}_{j(s)}\tilde{f}' \colon \mathrm{T}_{j(s)}U' \to \mathrm{T}_t\mathrm{Hilb}^\circ$ is identified with the direct sum of the restriction maps $\mathrm{H}^0(\mathrm{Gr}_i, \mathrm{N}_i) \to \mathrm{H}^0(X, \mathrm{N}_i|_X)$. Now use Lemma A.12. □

3.4. **Proof of Theorem 1.7.** We have already shown $\mathcal{N}$ and $\mathcal{M}$ are smooth separated Deligne–Mumford stacks of finite type over $k$ (Lemmas 3.4 and 3.1), so we just need to show $f$ is an open immersion. The separatedness of $\mathcal{N}$ guarantees that $f$ is separated. By Proposition 3.5 and the smoothness of $\mathcal{N}$ and $\mathcal{M}$, the morphism $f$ is étale. Now the result follows by replacing $\mathcal{M}$ with the image of $f$ and applying [27, Tag 0DUD], whose hypotheses hold by the above observations and Lemma 3.3.

## 4. The infinitesimal structure of $\mathcal{N}$

In this section, we study the moduli stack $\mathcal{N}$ infinitesimally. Our goal is to prove Theorem 1.8 from the introduction, as well as the following result.

**Proposition 4.1.** *Let* $s \in \mathcal{N}$ *be a point. Let* $\gamma \in \mathrm{Aut}_{\mathcal{N}}(s)$ *be an element such that the induced map* $\gamma_* \in \mathrm{GL}(\mathrm{T}_s\mathcal{N})$ *is an involution. Then*

$$\mathrm{tr}(\gamma_*) \in \{51, 3, 1, -3, -5, -13, -15, -35\}\,.$$

We start in §4.1 by spelling out an explicit presentation of the tangent space to any point $s \in \mathcal{N}$. In §4.2, we combine this with our description of $\mathrm{Aut}_{\mathcal{N}}(s)$ from Lemma 2.4 to prove some preliminary results, by analyzing the eigenvalues of the induced maps on $\mathrm{T}_s\mathcal{N}$. As an easy consequence of our analysis, we prove Theorem 1.8 and Proposition 4.1.

4.1. **An explicit presentation of the tangent space to $\mathcal{N}$.** Proposition 3.5 gives a presentation of the tangent spaces to $\mathcal{N}$. To make this explicit, we start with some preliminary remarks.

For any $g \in G$ there is an isomorphism

$$\mathfrak{g}/\mathfrak{h} \cong \mathrm{T}_{gH}(G/H). \tag{4.1}$$

If we regard $\mathrm{T}_{gH}(G/H)$ as the set of $k[\varepsilon]/(\varepsilon^2)$-points of $G/H$ based at $gH$, then this identification is induced by the map

$$\begin{aligned} \eta_g \colon \mathfrak{g} &\to \mathrm{T}_{gH}(G/H) \\ R &\mapsto g(1+\varepsilon R)H. \end{aligned}$$

As the notation indicates, the identification (4.1) depends on the choice of representative for the coset $gH \in G/H$. Namely, suppose $gH = g'H$. Then there is a commutative diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mathrm{Ad}_{(g')^{-1}g}} & \mathfrak{g} \\ {\scriptstyle \eta_g}\downarrow & & \downarrow{\scriptstyle \eta_{g'}} \\ \mathrm{T}_{gH}(G/H) & = & \mathrm{T}_{g'H}(G/H) \end{array}$$

Here, for any $a \in G$ we use the notation $\mathrm{Ad}_a \colon \mathfrak{g} \to \mathfrak{g}$ for the action of $a$ under the adjoint representation, i.e. $\mathrm{Ad}_a(R) = aRa^{-1}$. This follows from the computation

$$g(1+\varepsilon R)H = (1+\varepsilon gRg^{-1})gH = (1+\varepsilon gRg^{-1})g'H = g'(1+\varepsilon(g')^{-1}gRg^{-1}g')H.$$

Next we note that the group $G$ acts on $G/H$ on the left. For $gH \in G/H$ the derivative at $1 \in G$ of the action morphism $\mathrm{act}_{gH} \colon G \to G/H$, $\mathrm{act}_{gH}(a) = agH$, gives a map

$$\mathfrak{g} \to \mathrm{T}_{gH}(G/H).$$

Under the identification (4.1), this map takes the form

$$\begin{aligned} \mathfrak{g} &\to \mathfrak{g}/\mathfrak{h} \\ S &\mapsto g^{-1}Sg. \end{aligned}$$

Indeed, this follows from the observation

$$(1+\varepsilon S)gH = g(1+\varepsilon g^{-1}Sg)H.$$

Combined with Proposition 3.5, the above discussion gives the following.

**Lemma 4.2.** *Let $(g_1, g_2) \in G \times G$ be such that $[(g_1H, g_2H)] \in \mathcal{N}$.*

*(1) There is a short exact sequence*

$$0 \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h} \to \mathrm{T}_{[(g_1H, g_2H)]}\mathcal{N} \to 0 \tag{4.2}$$

*where the first map is given by*

$$\begin{aligned} \mathfrak{g} &\to \mathfrak{g}/\mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h} \\ S &\mapsto (g_1^{-1}Sg_1, g_2^{-1}Sg_2). \end{aligned} \tag{4.3}$$

(2) *The sequence* (4.2) *depends on the choice of representatives* $g_1, g_2$ *for the cosets* $g_1H, g_2H$*, but is canonical in the following sense. If* $g_1H = g_1'H$ *and* $g_2H = g_2'H$*, then there is a commutative diagram of exact sequences*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{g} & \longrightarrow & \mathfrak{g}/\mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h} & \longrightarrow & \mathrm{T}_{[(g_1H,g_2H)]}\mathcal{N} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow {\scriptstyle (\mathrm{Ad}_{(g_1')^{-1}g_1},\, \mathrm{Ad}_{(g_2')^{-1}g_2})} & & \| & & \\
0 & \longrightarrow & \mathfrak{g} & \longrightarrow & \mathfrak{g}/\mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h} & \longrightarrow & \mathrm{T}_{[(g_1'H,g_2'H)]}\mathcal{N} & \longrightarrow & 0
\end{array}$$

In particular, suppose $[(g_1H, g_2H)], [(g_1'H, g_2'H)] \in \mathcal{N}$. Then to specify a map

$$\alpha\colon \mathrm{T}_{[(g_1H,g_2H)]}\mathcal{N} \to \mathrm{T}_{[(g_1'H,g_2'H)]}\mathcal{N}$$

it suffices to specify a map

$$\beta\colon \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}$$

which preserves the subspace $\mathfrak{h} \oplus \mathfrak{h} \subset \mathfrak{g} \oplus \mathfrak{g}$ and the image of the map $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}$ in (4.3). To indicate this situation, we will simply say $\alpha$ *is induced by the map* $\beta$. We emphasize once again that this notion depends on the choice of representatives $g_1, g_2, g_2', g_2'$, which we regard as being made implicitly in the notation for the domain and target of $\alpha$.

4.2. **Eigenvalue analysis.** Recall that $\mathcal{N}$ is defined as the quotient of an open subscheme $U \subset G/H \times G/H$ by the group $\mathbf{Z}/2 \times G$. Hence, letting $\sigma \in \mathbf{Z}/2$ denote the generator, for any point $[(g_1H, g_2H)] \in \mathcal{N}$ and $a \in G$ there are corresponding isomorphisms

$$\gamma_{(1,a)}\colon [(g_1H, g_2H)] \xrightarrow{\sim} [(ag_1H, ag_2H)],$$
$$\gamma_{(\sigma,a)}\colon [(g_1H, g_2H)] \xrightarrow{\sim} [(ag_2H, ag_1H)].$$

These isomorphisms of points of $\mathcal{N}$ induce isomorphisms of tangent spaces, which we denote respectively by

$$a_*\colon \mathrm{T}_{[(g_1H,g_2H)]}\mathcal{N} \to \mathrm{T}_{[(ag_1H,ag_2H)]}\mathcal{N},$$
$$(\sigma, a)_*\colon \mathrm{T}_{[(g_1H,g_2H)]}\mathcal{N} \to \mathrm{T}_{[(ag_2H,ag_1H)]}\mathcal{N}.$$

We also simply write $\sigma_*$ for $(\sigma, 1)_*$. The next lemma follows immediately by unwinding the definitions.

**Lemma 4.3.** *Let* $(g_1, g_2) \in G \times G$ *be such that* $[(g_1H, g_2H)] \in \mathcal{N}$.

(1) *The map*

$$a_*\colon \mathrm{T}_{[(g_1H,g_2H)]}\mathcal{N} \to \mathrm{T}_{[(ag_1H,ag_2H)]}\mathcal{N}$$

*is induced by the identity map*

$$\mathrm{id}\colon \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}.$$

(2) *The map*

$$\sigma_*\colon \mathrm{T}_{[(g_1H,g_2H)]}\mathcal{N} \to \mathrm{T}_{[(g_2H,g_1H)]}\mathcal{N}$$

*is induced by the transposition map*

$$\mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}$$
$$(R_1, R_2) \mapsto (R_2, R_1).$$

Below we will be concerned with automorphisms of a point $[(g_1H, g_2H)] \in \mathcal{N}$. As observed in Lemma 3.3, the automorphism group of $[(g_1H, g_2H)] \in \mathcal{N}$ coincides with that of $[X_{g_1,g_2}] \in \mathcal{M}$. This latter group consists of elements of two types, according to Lemma 2.4. We turn this into a definition:

**Definition 4.4.** Let $(g_1, g_2) \in G \times G$ be such that $s = [(g_1H, g_2H)] \in \mathcal{N}$, i.e. such that $X_{g_1,g_2}$ is a GPK$^3$ threefold. We say an automorphism $a \in G$ of $X_{g_1,g_2}$ is:

(1) of *type I* if $a \in g_1Hg_1^{-1} \cap g_2Hg_2^{-1}$;
(2) of *type II* if $a \in g_2Hg_1^{-1} \cap g_1Hg_2^{-1}$.

We say $\gamma \in \mathrm{Aut}_{\mathcal{N}}(s)$ is of *type I* or *type II* according to the type of the corresponding element of $\mathrm{Aut}_{\mathcal{M}}([X_{g_1,g_2}])$ under the isomorphism $\mathrm{Aut}_{\mathcal{N}}(s) \cong \mathrm{Aut}_{\mathcal{M}}([X_{g_1,g_2}])$.

In the situation of Definition 4.4, the automorphism of $[(g_1H, g_2H)]$ corresponding to $a$ is $\gamma_{(1,a)}$ from above if $a$ is of type I, and is $\gamma_{(\sigma,a)}$ if $a$ is of type II. Via the isomorphism $\mathrm{T}_{[(g_1H,g_2H)]}\mathcal{N} \cong \mathrm{T}_{[X_{g_1,g_2}]}\mathcal{M}$ from Proposition 3.5, the induced map

$$a_* \colon \mathrm{T}_{[X_{g_1,g_2}]}\mathcal{M} \to \mathrm{T}_{[X_{g_1,g_2}]}\mathcal{M}$$

is identified with

$$\begin{aligned} a_* \colon \mathrm{T}_{[(g_1H,g_2H)]}\mathcal{N} &\to \mathrm{T}_{[(g_1H,g_2H)]}\mathcal{N} \quad \text{if } a \text{ is of type I,} \\ (\sigma, a)_* \colon \mathrm{T}_{[(g_1H,g_2H)]}\mathcal{N} &\to \mathrm{T}_{[(g_1H,g_2H)]}\mathcal{N} \quad \text{if } a \text{ is of type II.} \end{aligned}$$

In the following lemma, we describe these maps explicitly.

**Lemma 4.5.** *Let $(g_1, g_2) \in G \times G$ be such that $[(g_1H, g_2H)] \in \mathcal{N}$. Let $a \in G$ be an automorphism of $X_{g_1,g_2}$.*

*(1) If $a$ is of type I, then the map*

$$a_* \colon \mathrm{T}_{[(g_1H,g_2H)]}\mathcal{N} \to \mathrm{T}_{[(g_1H,g_2H)]}\mathcal{N}$$

*is induced by the map*

$$\begin{aligned} \mathfrak{g} \oplus \mathfrak{g} &\to \mathfrak{g} \oplus \mathfrak{g} \\ (R_1, R_2) &\mapsto ((g_1^{-1}ag_1)R_1(g_1^{-1}ag_1)^{-1}, (g_2^{-1}ag_2)R_2(g_2^{-1}ag_2)^{-1}). \end{aligned}$$

*(2) If $a$ is of type II, then the map*

$$(\sigma, a)_* \colon \mathrm{T}_{[(g_1H,g_2H)]}\mathcal{N} \to \mathrm{T}_{[(g_1H,g_2H)]}\mathcal{N}$$

*is induced by the map*

$$\begin{aligned} \mathfrak{g} \oplus \mathfrak{g} &\to \mathfrak{g} \oplus \mathfrak{g} \\ (R_1, R_2) &\mapsto ((g_1^{-1}ag_2)R_2(g_1^{-1}ag_2)^{-1}, (g_2^{-1}ag_1)R_1(g_2^{-1}ag_1)^{-1}). \end{aligned}$$

*Proof.* By Lemma 4.3(1) the map

$$a_* \colon \mathrm{T}_{[(g_1H,g_2H)]}\mathcal{N} \to \mathrm{T}_{[(ag_1H,ag_2H)]}\mathcal{N}$$

is induced by the identity $\mathrm{id} \colon \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}$. If $a$ is of type I, then we have $ag_1H = g_1H$ and $ag_2H = g_2H$. Hence (1) follows from the comparison between the presentations for $\mathrm{T}_{[(ag_1H,ag_2H)]}$ and $\mathrm{T}_{[(g_1H,g_2H)]}$ given by Lemma 4.2(2). Part (2) follows similarly. □

Set $\tilde{H} = \mathrm{GL}(V)$, $\tilde{G} = \mathrm{GL}(W)$, $\tilde{\mathfrak{h}} = \mathfrak{gl}(V)$, and $\tilde{\mathfrak{g}} = \mathfrak{gl}(W)$, so that we have commutative diagrams

$$\begin{array}{ccc} \tilde{H} & \longrightarrow & \tilde{G} \\ \downarrow & & \downarrow \\ H & \longrightarrow & G \end{array} \qquad\qquad \begin{array}{ccc} \tilde{\mathfrak{h}} & \longrightarrow & \tilde{\mathfrak{g}} \\ \downarrow & & \downarrow \\ \mathfrak{h} & \longrightarrow & \mathfrak{g} \end{array} \tag{4.4}$$

where the rows are embeddings and the columns surjections with 1-dimensional kernels. In the following proofs, at certain points it will be convenient to work with the spaces in the top row. For readability, we commit the following abuse of notation: given $g \in G$ we use the same symbol to denote a fixed lift $g \in \tilde{G}$; similarly for $H$ and $\tilde{H}$; and we choose our lifts compatibly, i.e. if $g \in G$ is the image of $g_0 \in H$ under $H \to G$, we choose a lift of $g_0$ that maps to $g$.

**Lemma 4.6.** *Let $(g_1, g_2) \in G \times G$ be such that $[(g_1H, g_2H)] \in \mathcal{N}$. Let $a \in G$ be an automorphism of $X_{g_1,g_2}$.*

*(1) If $a$ is of type I and the map*

$$a_* \colon \mathrm{T}_{[(g_1H,g_2H)]}\mathcal{N} \to \mathrm{T}_{[(g_1H,g_2H)]}\mathcal{N}$$

*is the identity, then $a = 1$.*

*(2) If $a$ is of type II and the map*

$$(\sigma, a)_* \colon \mathrm{T}_{[(g_1H,g_2H)]}\mathcal{N} \to \mathrm{T}_{[(g_1H,g_2H)]}\mathcal{N}$$

*is an involution, then $a^2 = 1$.*

*Proof.* Note that there is an isomorphism $[(g_1H, g_2H)] \cong [(H, g_1^{-1}g_2H)] \in \mathcal{N}$. Hence we may assume $g_1 = 1$ and $g_2 = g$. Further, note that $a \in G$ must have finite order by Lemma 2.4, so in particular $a$ is diagonalizable. Since the square of an automorphism of type II is of type I, (2) follows from (1).

So assume $a$ is of type I, i.e. $a \in H \cap gHg^{-1} \subset G$. Then by Lemma 4.5(1) the map $a_* \colon \mathrm{T}_{[(H,gH)]}\mathcal{N} \to \mathrm{T}_{[(H,gH)]}\mathcal{N}$ is induced by the map

$$\begin{aligned} \mathfrak{g} \oplus \mathfrak{g} &\to \mathfrak{g} \oplus \mathfrak{g} \\ (R_1, R_2) &\mapsto (aR_1a^{-1}, (g^{-1}ag)R_2(g^{-1}ag)^{-1}). \end{aligned} \tag{4.5}$$

Recall that this means that in terms of the presentation

$$\begin{array}{ccccccccc} 0 & \to & \mathfrak{g} & \to & \mathfrak{g}/\mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h} & \to & \mathrm{T}_{[(H,gH)]}\mathcal{N} & \to & 0 \\ & & S & \mapsto & (S, g^{-1}Sg) & & & & \end{array} \tag{4.6}$$

given by (4.2), $a_*$ is induced by (4.5).

Let $a_0 \in H$ be the element whose image under the embedding $H \to G$ is $a$. Let $\lambda_i, 1 \le i \le 5$, be the eigenvalues of $a_0$. Then $\lambda_i\lambda_j, 1 \le i < j \le 5$, are the eigenvalues of $a$; we must show they are all equal if $a_*$ is the identity. We start by computing the eigenvalues of $a_*$ in terms of the $\lambda_i$. We do so by computing the eigenvalues on each summand in $\mathfrak{g}/\mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}$ and on the subspace $\mathfrak{g} \subset \mathfrak{g}/\mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}$ separately:

*The first $\mathfrak{g}/\mathfrak{h}$ summand.* Consider the map $\tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}$ given by $R \mapsto aRa^{-1}$. It has eigenvalues

$$\frac{\lambda_i\lambda_j}{\lambda_k\lambda_\ell}, \ 1 \le i < j \le 5, \, 1 \le k < \ell \le 5,$$

and similarly the induced map $\tilde{\mathfrak{h}} \to \tilde{\mathfrak{h}}$ has eigenvalues

$$\frac{\lambda_i}{\lambda_k},\ 1 \le i, k \le 5.$$

Hence the induced map $\tilde{\mathfrak{g}}/\tilde{\mathfrak{h}} \to \tilde{\mathfrak{g}}/\tilde{\mathfrak{h}}$, which coincides with the induced map $\mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h}$, has eigenvalues given by the multiset difference

$$\left\{ \frac{\lambda_i\lambda_j}{\lambda_k\lambda_\ell} \,\middle|\, 1 \le i < j \le 5,\ 1 \le k < \ell \le 5 \right\} - \left\{ \frac{\lambda_i}{\lambda_k} \,\middle|\, 1 \le i, k \le 5 \right\}. \tag{4.7}$$

*The second $\mathfrak{g}/\mathfrak{h}$ summand.* Let $b = g^{-1}ag \in G$. By assumption $b$ is in the image of the embedding $H \to G$; let $b_0$ be its preimage. Let $\mu_i, 1 \le i \le 5$, be the eigenvalues of $b_0$, so that $\mu_i\mu_j, 1 \le i < j \le 5$, are the eigenvalues of $b$. Note that we have an equality of multisets

$$\{\mu_i\mu_j \,|\, 1 \le i < j \le 5\} = \{\lambda_i\lambda_j \,|\, 1 \le i < j \le 5\}, \tag{4.8}$$

but the multisets $\{\mu_i \,|\, 1 \le i \le 5\}$ and $\{\lambda_i \,|\, 1 \le i \le 5\}$ need not coincide. The above argument shows that the map $\mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h}$ induced by $R \mapsto bRb^{-1}$ has eigenvalues given by the multiset

$$\left\{ \frac{\mu_i\mu_j}{\mu_k\mu_\ell} \,\middle|\, 1 \le i < j \le 5,\ 1 \le k < \ell \le 5 \right\} - \left\{ \frac{\mu_i}{\mu_k} \,\middle|\, 1 \le i, k \le 5 \right\}. \tag{4.9}$$

*The subspace $\mathfrak{g} \subset \mathfrak{g}/\mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}$.* The map (4.5) induces the map $S \mapsto aSa^{-1}$ on the copy of $\mathfrak{g} \subset \mathfrak{g}/\mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}$ embedded as in (4.6). The above argument shows this map has eigenvalues given by the multiset

$$\left\{ \frac{\lambda_i\lambda_j}{\lambda_k\lambda_\ell} \,\middle|\, 1 \le i < j \le 5,\ 1 \le k < \ell \le 5 \right\} - \{1\}, \tag{4.10}$$

where we have removed a single 1 eigenvalue corresponding to the kernel of $\tilde{\mathfrak{g}} \to \mathfrak{g}$.

*The eigenvalues of $a_*$.* Combining all of the above, we conclude that the eigenvalues of $a_*$ are given by the multiset sum of (4.7) and (4.9) minus (4.10), i.e. by

$$\{1\} + \left\{ \frac{\mu_i\mu_j}{\mu_k\mu_\ell} \,\middle|\, 1 \le i < j \le 5,\ 1 \le k < \ell \le 5 \right\} - \left\{ \frac{\lambda_i}{\lambda_k} \,\middle|\, 1 \le i, k \le 5 \right\} - \left\{ \frac{\mu_i}{\mu_k} \,\middle|\, 1 \le i, k \le 5 \right\}. \tag{4.11}$$

Recall that to finish we need to show that if the support of (4.11) is $\{1\}$, then the $\lambda_i$ coincide. To see this, first note that every $\lambda_i/\lambda_k$ appears at least three times in

$$\left\{ \frac{\lambda_i\lambda_j}{\lambda_k\lambda_\ell} \,\middle|\, 1 \le i < j \le 5,\ 1 \le k < \ell \le 5 \right\}, \tag{4.12}$$

hence the difference (4.7) has the same support as (4.12). Similarly, the multiset (4.9) has the same support as

$$\left\{ \frac{\mu_i\mu_j}{\mu_k\mu_\ell} \,\middle|\, 1 \le i < j \le 5,\ 1 \le k < \ell \le 5 \right\}. \tag{4.13}$$

But using (4.8) we see that twice the multiset (4.11) coincides with the sum of $\{1, 1\}$, (4.7), and (4.9). It follows that if the support of (4.11) is $\{1\}$, then so are the supports of (4.12) and (4.13), and hence all $\lambda_i$ coincide. □

In the next proof, we use the following convenient notation. Given an endomorphism $\psi$ of a $k$-vector space and $\lambda \in k$, we write $\mathrm{mult}_\lambda(\psi)$ for the multiplicity of the eigenvalue $\lambda$ for $\psi$. If $\psi$ is an involution, its eigenvalues are $\pm 1$, and we say: $\psi$ is of type $(p,q)$ if $\mathrm{mult}_1(\psi) = p$ and $\mathrm{mult}_{-1}(\psi) = q$; $\psi$ is of type $\{p,q\}$ if it is either of type $(p,q)$ or $(q,p)$. Keep in mind below our abuse of notation by which given $a$ in $G$ or $H$, we fix a lift to $\tilde{G}$ or $\tilde{H}$ denoted by the same symbol; if $a$ is an involution, we choose our lift to be an involution as well.

**Lemma 4.7.** *Let $(g_1, g_2) \in G \times G$ be such that $[(g_1H, g_2H)] \in \mathcal{N}$. Let $1 \neq a \in G$ be an automorphism of $X_{g_1,g_2}$ which satisfies $a^2 = 1$.*

(1) *If $a$ is of type I, then the trace of the map*
$$a_* \colon \mathrm{T}_{[(g_1H,g_2H)]}\mathcal{N} \to \mathrm{T}_{[(g_1H,g_2H)]}\mathcal{N}$$
*is one of the following:* $3, -5, -13$.

(2) *If $a$ is of type II, then the trace of the map*
$$(\sigma, a)_* \colon \mathrm{T}_{[(g_1H,g_2H)]}\mathcal{N} \to \mathrm{T}_{[(g_1H,g_2H)]}\mathcal{N}$$
*is one of the following:* $1, -3, -15, -35$.

*Proof.* As in the proof of Lemma 4.6, we may assume $g_1 = 1$ and $g_2 = g$.

Assume $a$ is of type I. To compute the trace of $a_*$, it suffices to compute $\mathrm{mult}_1(a_*)$. For this, we follow the proof of Lemma 4.6. As there, let $a_0 \in H$ be a preimage of $a$, and let $b_0 \in H$ be a preimage of $b = g^{-1}ag$. Then the formula (4.11) for the multiset of eigenvalues of $a_*$ shows
$$\mathrm{mult}_1(a_*) = 1 + \mathrm{mult}_1(\mathrm{Ad}_b \colon \tilde{\mathfrak{g}}/\tilde{\mathfrak{h}} \to \tilde{\mathfrak{g}}/\tilde{\mathfrak{h}}) - \mathrm{mult}_1(\mathrm{Ad}_{a_0} \colon \tilde{\mathfrak{h}} \to \tilde{\mathfrak{h}}) - \mathrm{mult}_1(\mathrm{Ad}_{b_0} \colon \tilde{\mathfrak{h}} \to \tilde{\mathfrak{h}}).$$

To compute the above quantity, we use the following remark. Let $\psi \colon L \to L$ be an involution of a $k$-vector space $L$ of type $\{p,q\}$. Then the $+1$ eigenspace of the map
$$\begin{aligned} \mathrm{Ad}_\psi \colon \mathfrak{gl}(L) &\to \mathfrak{gl}(L) \\ R &\mapsto \psi R \psi^{-1} \end{aligned}$$
consists of $R \in \mathfrak{gl}(L)$ that commute with $\psi$, and hence $\mathrm{mult}_1(\mathrm{Ad}_\psi) = p^2 + q^2$. Since $a_0 \neq 1$ by assumption, the involution $a_0$ is either of type $\{4,1\}$ or $\{3,2\}$, and $a$ is of type $\{4,6\}$. Similarly, $b_0$ is either of type $\{4,1\}$ or $\{3,2\}$, and $b$ is of type $\{4,6\}$. Thus, using the above formula we find
$$\mathrm{mult}_1(a_*) = \begin{cases} 19 & \text{if } a_0 \text{ and } b_0 \text{ are of type } \{4,1\}, \\ 23 & \text{if } a_0 \text{ and } b_0 \text{ are of opposite } \{p,q\} \text{ types}, \\ 27 & \text{if } a_0 \text{ and } b_0 \text{ are of type } \{3,1\}. \end{cases}$$
Since $\dim \mathrm{T}_{[(H,gH)]}\mathcal{N} = 51$, this gives for the trace
$$\mathrm{tr}(a_*) = \begin{cases} -13 & \text{if } a_0 \text{ and } b_0 \text{ are of type } \{4,1\}, \\ -5 & \text{if } a_0 \text{ and } b_0 \text{ are of opposite } \{p,q\} \text{ types}, \\ 3 & \text{if } a_0 \text{ and } b_0 \text{ are of type } \{3,1\}, \end{cases}$$
and hence proves (1) of the lemma.

Now assume $a$ is of type II, i.e. $a \in gH \cap Hg^{-1} \subset G$. Then by Lemma 4.5(2) the map $(\sigma, a)_* \colon \mathrm{T}_{[(H,gH)]}\mathcal{N} \to \mathrm{T}_{[(H,gH)]}\mathcal{N}$ is induced by the map
$$\begin{aligned} \mathfrak{g} \oplus \mathfrak{g} &\to \mathfrak{g} \oplus \mathfrak{g} \\ (R_1, R_2) &\mapsto ((ag)R_2(ag)^{-1}, (g^{-1}a)R_1(g^{-1}a)^{-1}). \end{aligned} \tag{4.14}$$

That is, in terms of the presentation

$$\begin{array}{ccccccc} 0 & \to & \mathfrak{g} & \to & \mathfrak{g}/\mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h} & \to & \mathrm{T}_{[(H,gH)]}\mathcal{N} & \to & 0 \\ & & S & \mapsto & (S, g^{-1}Sg) & & & & \end{array} \tag{4.15}$$

given by (4.2), $(\sigma, a)_*$ is induced by (4.14).

We compute $\mathrm{mult}_1((\sigma, a)_*)$ by computing the multiplicity of the eigenvalue 1 on the terms $\mathfrak{g}/\mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}$ and $\mathfrak{g}$ in (4.15) separately.

*The* $\mathfrak{g}/\mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}$ *term.* Let $b = ag \in G$, which by assumption is in the image of the embedding $H \to G$. Note that $b^{-1} = g^{-1}a^{-1} = g^{-1}a$, hence (4.14) can be written

$$(R_1, R_2) \mapsto (bR_2b^{-1}, b^{-1}R_1b)$$

The $+1$ eigenspace consists of $(R_1, R_2)$ such that $R_2 = b^{-1}R_1b$. Hence the induced map $\mathfrak{g}/\mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}$ has eigenvalue 1 with multiplicity $75 = \dim \mathfrak{g}/\mathfrak{h}$.

*The* $\mathfrak{g}$ *term.* For $S \in \mathfrak{g}$ the map (4.14) sends $(S, g^{-1}Sg) \mapsto (aSa^{-1}, g^{-1}(aSa^{-1})g)$. Hence the induced action on the term $\mathfrak{g}$ in (4.15) is $S \mapsto aSa^{-1}$. By assumption $1 \neq a \in G$ is an involution, and so has type $\{p, q\}$ for some $p + q = 10$ and $p, q \geq 1$. By the observation from above, the map $\tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}$ given by $S \mapsto aSa^{-1}$ has eigenvalue 1 with multiplicity $p^2 + q^2$; thus the corresponding map $\mathfrak{g} \to \mathfrak{g}$ has eigenvalue 1 with multiplicity $p^2 + q^2 - 1$.

Combining the above, we conclude $\mathrm{mult}_1((\sigma, a)_*) = 76 - p^2 - q^2$ where the type $\{p, q\}$ of $a$ is one of the following: $\{9, 1\}, \{8, 2\}, \{7, 3\}, \{6, 4\}, \{5, 5\}$. Note that for $\{p, q\} = \{9, 1\}$ this gives $\mathrm{mult}_1((\sigma, a)_*) = -6$, which is nonsense; so this case does not occur. Since $\dim \mathrm{T}_{[(H,gH)]}\mathcal{N} = 51$, we find $\mathrm{tr}((\sigma, a)_*) = 101 - 2(p^2 + q^2)$. Plugging in $\{p, q\} = \{8, 2\}, \{7, 3\}, \{6, 4\}, \{5, 5\}$ gives the values in (2) of the lemma. □

4.3. **Proof of Theorem 1.8.** Suppose $\gamma \in \mathrm{Aut}_{\mathcal{N}}(s)$ acts trivially on $\mathrm{T}_s\mathcal{N}$. If $\gamma$ is of type I, then $\gamma = 1$ by Lemma 4.6(1). If $\gamma$ is of type II, then $\gamma^2$ is of type I, and hence $\gamma^2 = 1$ by the previous sentence. But then by Lemma 4.7(2), either $\gamma = 1$ or $\mathrm{tr}(\gamma_*) \in \{1, -3, -15, -35\}$. By assumption $\gamma_* = \mathrm{id}$ and hence $\mathrm{tr}(\gamma_*) = 51$, so we conclude $\gamma = 1$. □

4.4. **Proof of Proposition 4.1.** This follows by combining Theorem 1.8 and Lemma 4.7. (The case $\mathrm{tr}(\gamma_*) = 51$ corresponds to $\gamma = 1$.) □

## 5. The double mirror involution

We begin this section by showing that the operation of passing to the double mirror preserves smoothness of $\mathrm{GPK}^3$ threefolds. Using this, in §5.2 we define the double mirror involution $\tau$ of the moduli stack $\mathcal{N}$ of $\mathrm{GPK}^3$ data. In §5.3 we compute the derivative of $\tau$.

5.1. **Simultaneous smoothness.** Let

$$X = \mathrm{Gr}_1 \cap \mathrm{Gr}_2 \subset \mathbf{P},$$
$$Y = \mathrm{Gr}_1^\vee \cap \mathrm{Gr}_2^\vee \subset \mathbf{P}^\vee,$$

be $\mathrm{GPK}^3$ threefolds corresponding to isomorphisms $\phi_i \colon \wedge^2 V \xrightarrow{\sim} W$, $i = 1, 2$, as in §1. We aim to show the following result, which is analogous to [5, Corollary 2.3].

**Proposition 5.1.** *The variety $X$ is a smooth threefold if and only if the same is true of $Y$.*

**Remark 5.2.** If $X$ and $Y$ are of expected dimension, Proposition 5.1 follows from Theorem 1.1, but we give a more direct proof below.

We start by recalling a basic fact about projective duality of $\mathrm{Gr}(2, V)$.

**Lemma 5.3.** *Let $x \in \mathrm{Gr}(2, V) \subset \mathbf{P}(\wedge^2 V)$ be a point corresponding to a 2-plane $A \subset V$. Let $y \in \mathbf{P}(\wedge^2 V^\vee)$ be a point corresponding to a hyperplane $H \subset \mathbf{P}(\wedge^2 V)$. Then $H$ is tangent to $\mathrm{Gr}(2, V)$ at $x$ if and only if the following equivalent conditions hold:*

*(1) if $K \subset V$ denotes the kernel of the 2-form on $V$ corresponding to $y$, then $A \subset K$.*
*(2) $y \in \mathrm{Gr}(2, V^\vee) \subset \mathbf{P}(\wedge^2 V^\vee)$ and if $B \subset V^\vee$ is the corresponding 2-plane with orthogonal $B^\perp = \ker(V \to B^\vee)$, then $A \subset B^\perp$.*

*Moreover, in this case $K = B^\perp$.*

**Remark 5.4.** The equivalence of $H$ being tangent to $\mathrm{Gr}(2, V)$ at $x$ with (1) holds for $V$ of any dimension, while the equivalence with (2) is special to the case $\dim V = 5$. Note that (2) says in particular that the projective dual of $\mathrm{Gr}_i \subset \mathbf{P}$ is $\mathrm{Gr}_i^\vee \subset \mathbf{P}^\vee$, as the notation indicates.

We will deduce Proposition 5.1 from an auxiliary result, which describes the loci in $X$ and $Y$ where the defining Grassmannians do not intersect transversally. Given $x \in X = \mathrm{Gr}_1 \cap \mathrm{Gr}_2$, we let $x_i \in \mathrm{Gr}_i$ be the two corresponding points, and we write $A_{x_i} \subset V$ for the corresponding 2-planes. Similarly, for $y \in Y = \mathrm{Gr}_1^\vee \cap \mathrm{Gr}_2^\vee$ we let $y_i \in \mathrm{Gr}_i^\vee$ be the corresponding points, and write $B_{y_i} \subset V^\vee$ for the corresponding 2-planes. Define $Z \subset X \times Y$ to be the locus of pairs $(x, y)$ such that $A_{x_i} \subset B_{y_i}^\perp$ for $i = 1, 2$, and let $\mathrm{pr}_X \colon Z \to X$ and $\mathrm{pr}_Y \colon Z \to Y$ be the two projections.

**Lemma 5.5.** *The following hold:*

*(1) A point $x \in X$ has $\dim \mathrm{T}_{X,x} > 3$ if and only if it is in the image $\mathrm{pr}_X(Z)$.*
*(2) A point $y \in Y$ has $\dim \mathrm{T}_{Y,y} > 3$ if and only if it is in the image $\mathrm{pr}_Y(Z)$.*

*Proof.* The condition $\dim \mathrm{T}_{X,x} > 3$ is equivalent to $\mathrm{T}_{\mathrm{Gr}_1,x}$ and $\mathrm{T}_{\mathrm{Gr}_2,x}$ intersecting non-transversely in $\mathrm{T}_{\mathbf{P},x}$, i.e. to the existence of a hyperplane in $\mathrm{T}_{\mathbf{P},x}$ containing both $\mathrm{T}_{\mathrm{Gr}_i,x}$, or equivalently to the existence of a projective hyperplane $H \subset \mathbf{P}$ tangent to both $\mathrm{Gr}_i$ at $x$. But by Lemma 5.3, the existence of such an $H$ is equivalent to the existence of a point $y \in Y$ such that $(x, y) \in Z$. This proves part (1) of the lemma. Part (2) follows by symmetry (note that $Z$ can also be described as the locus of $(x, y)$ such that $B_{y_i} \subset A_{x_i}^\perp$, $i = 1, 2$). □

*Proof of Proposition 5.1.* If $X$ is a smooth threefold, then by Lemma 5.5(1) the correspondence $Z \subset X \times Y$ is empty. Hence $Y$, which a priori has dimension at least 3, is in fact a smooth threefold by Lemma 5.5(2). By symmetry, we conclude conversely that if $Y$ is a smooth threefold, then so is $X$. □

5.2. **The double mirror involution.** For any $(g_1, g_2) \in G \times G$, we have defined a Grassmannian intersection $X_{g_1,g_2}$ by (2.1). Let

$$Y_{g_1,g_2} = (g_1\mathrm{Gr})^\vee \cap (g_2\mathrm{Gr})^\vee \subset \mathbf{P}^\vee \tag{5.1}$$

be the corresponding double mirror.

We can identify $Y_{g_1,g_2}$ with an explicit Grassmannian intersection in $\mathbf{P}$, as follows. Fix from now on an isomorphism $V \cong V^\vee$ (or more explicitly, a basis for $V$). This induces an isomorphism $\wedge^2 V \cong \wedge^2 V^\vee$, and hence an isomorphism

$$\theta \colon W \xrightarrow{\phi^{-1}} \wedge^2 V \cong \wedge^2 V^\vee \xrightarrow{(\phi^{-1})^*} W^\vee,$$

which identifies $\mathrm{Gr}$ with $\mathrm{Gr}^\vee$. Given $g \in G$, its transpose is by definition the automorphism of $W$ given by $g^T = \theta^{-1} \circ g^* \circ \theta$, and its inverse transpose is $g^{-T} = (g^{-1})^T$. Here and below, we slightly abuse notation by not distinguishing between $g \in G = \mathrm{PGL}(W)$ and a lift of $g$ to $\mathrm{GL}(W)$.

**Lemma 5.6.** *The isomorphism* $\theta^{-1}\colon \mathbf{P}^\vee \xrightarrow{\sim} \mathbf{P}$ *induces an isomorphism* $Y_{g_1,g_2} \cong X_{g_1^{-T}, g_2^{-T}}$.

*Proof.* For any $g \in G$, it follows from the definitions that

$$(g\mathrm{Gr})^\vee = (g^{-1})^* \mathrm{Gr}^\vee \subset \mathbf{P}^\vee.$$

The result follows. □

The involution

$$\begin{aligned} G \times G &\to G \times G \\ (g_1, g_2) &\mapsto (g_1^{-T}, g_2^{-T}) \end{aligned}$$

induces an involution of $\tilde{\tau}$ of $G/H \times G/H$. By Proposition 5.1 combined with Lemma 5.6, the involution $\tilde{\tau}$ preserves the open subscheme $U \subset G \times G$ appearing in the definition (3.2) of the stack $\mathcal{N}$, and corresponds to passing to the double mirror GPK$^3$ threefold on this locus. We denote by

$$\tau\colon \mathcal{N} \to \mathcal{N}$$

the induced involution, which we call the *double mirror involution* of $\mathcal{N}$.

5.3. **Derivative of the double mirror involution.** In the following result, we use the terminology introduced directly after Lemma 4.2.

**Lemma 5.7.** *Let* $(g_1, g_2) \in G \times G$ *be such that* $[(g_1 H, g_2 H)] \in \mathcal{N}$. *Then the derivative*

$$\mathrm{d}_{[(g_1 H, g_2 H)]}\tau\colon \mathrm{T}_{[(g_1 H, g_2 H)]}\mathcal{N} \to \mathrm{T}_{[(g_1^{-T} H, g_2^{-T} H)]}\mathcal{N}$$

*is induced by the map*

$$\begin{aligned} \mathfrak{g} \oplus \mathfrak{g} &\to \mathfrak{g} \oplus \mathfrak{g} \\ (R_1, R_2) &\mapsto (-R_1^T, -R_2^T). \end{aligned}$$

*Proof.* This follows easily from the definitions and the observation

$$(1 + \varepsilon R)^{-T} = 1 - \varepsilon R^T$$

for $R \in \mathfrak{g}$. □

## 6. Proof of Theorem 1.2

The moduli spaces and morphisms constructed in §3 and §5 can be summarized by the diagram

$$\begin{array}{ccc} \tau \circlearrowright \mathcal{N} & \xrightarrow{f} & \mathcal{M} \\ \downarrow{\scriptstyle \pi_{\mathcal{N}}} & & \downarrow{\scriptstyle \pi_{\mathcal{M}}} \\ N & \longrightarrow & M \end{array}$$

where $f$ is the PGL-parameterization of the moduli stack $\mathcal{M}$ of GPK$^3$ threefolds by the moduli stack $\mathcal{N}$ of GPK$^3$ data, $\tau$ is the double mirror involution, and $\pi_{\mathcal{N}}$ and $\pi_{\mathcal{M}}$ are coarse moduli spaces.

Two GPK$^3$ threefolds are birational if and only if they are isomorphic, since they are Calabi–Yau of Picard number 1. Recall that $f\colon \mathcal{N} \to \mathcal{M}$ induces an injection $|\mathcal{N}(k)| \to |\mathcal{M}(k)|$

by Lemma 3.3. (In fact $f$ is an open immersion by Theorem 1.7, but we only need the weaker statement about points for the following argument.) Moreover, the locus $\mathcal{Z} \subset \mathcal{N}$ where the morphisms $\pi_{\mathcal{N}} \circ \tau$ and $\pi_{\mathcal{N}}$ agree is closed, because $N$ is separated by Lemma 3.4. Hence Theorem 1.2 is equivalent to the assertion that $\mathcal{Z}$ does not coincide with $\mathcal{N}$, i.e. that the morphisms $\pi_{\mathcal{N}} \circ \tau$ and $\pi_{\mathcal{N}}$ do not coincide. We will prove this after a sequence of lemmas.

**Lemma 6.1.** *Let $s \in \mathcal{N}$ be a point such that $\tau(s) = s$. If $\pi_{\mathcal{N}} \circ \tau = \pi_{\mathcal{N}}$, then the derivative*

$$\mathrm{d}_s\tau \colon \mathrm{T}_s\mathcal{N} \to \mathrm{T}_s\mathcal{N}$$

*is contained in the image of the homomorphism* $\mathrm{Aut}_{\mathcal{N}}(s) \to \mathrm{GL}(\mathrm{T}_s\mathcal{N})$.

*Proof.* Let $\Gamma = \mathrm{Aut}_{\mathcal{N}}(s)$. By Luna's étale slice theorem (see [2, Theorem 2.1]), we may find an integral affine scheme $Y = \mathrm{Spec}(R)$ with a $\Gamma$-action, a point $y \in Y$, and an involution $\tau_Y \colon Y \to Y$, such that:

(1) $\tau_Y(y) = y$.
(2) There is a $\Gamma$-equivariant isomorphism $\mathrm{T}_yY \cong \mathrm{T}_s\mathcal{N}$ under which the maps $\mathrm{d}_y\tau_Y$ and $\mathrm{d}_s\tau$ are identified.
(3) If $\pi_Y \colon Y \to Y/\!\!/\Gamma = \mathrm{Spec}(R^\Gamma)$ denotes the GIT quotient, then $\pi_Y \circ \tau_Y = \pi_Y$.

Consider the ring map $R^\Gamma \to R$ corresponding to $\pi_Y$. Passing the fraction fields, we obtain a Galois field extension $K(R)^\Gamma \to K(R)$. Since $\tau_Y$ restricts to an automorphism of $K(R)$ over $K(R)^\Gamma$ by (3), we conclude that $\tau_Y$ coincides with the action of an element $\gamma \in \Gamma$ over the generic point of $Y$, and hence $\tau_Y$ coincides with the action of $\gamma$ on all of $Y$. Now the result follows from (2) by taking the derivative of $\tau_Y$ at $y$. □

**Lemma 6.2.** *There exists $g \in G$ with $g = g^{-T}$ such that $[(H, gH)] \in \mathcal{N}$, i.e. such that $X_{1,g}$ is smooth.*

*Proof.* We verified this by an easy *Macaulay2* computation with a random orthogonal matrix in GL(10). □

Note that $[(H, gH)] \in \mathcal{N}$ as in Lemma 6.2 satisfies $\tau([(H, gH)]) = [(H, g^{-T}H)] = [(H, gH)]$.

**Lemma 6.3.** *Let $g \in G$ be as in Lemma 6.2. Then* $\mathrm{tr}(\mathrm{d}_{[(H,gH)]}\tau) = -1$.

*Proof.* By Lemma 5.7 the map $\mathrm{d}_{[(H,gH)]}\tau \colon \mathrm{T}_{[(H,gH)]}\mathcal{N} \to \mathrm{T}_{[(H,gH)]}\mathcal{N}$ is induced by the map

$$\begin{gathered} \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g} \\ (R_1, R_2) \mapsto (-R_1^T, -R_2^T). \end{gathered} \tag{6.1}$$

That is, in terms of the presentation

$$\begin{array}{ccccccccc} 0 & \to & \mathfrak{g} & \to & \mathfrak{g}/\mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h} & \to & \mathrm{T}_{[(H,gH)]}\mathcal{N} & \to & 0 \\ & & S & \mapsto & (S, g^{-1}Sg) & & & & \end{array} \tag{6.2}$$

given by (4.2), $\mathrm{d}_{[(H,gH)]}\tau$ is induced by (6.1). On each copy of $\mathfrak{g}$ and $\mathfrak{h}$ appearing in (6.2), the map induced by (6.1) is $R \mapsto -R^T$. In general, given a vector space $L$, the trace of the map $\mathfrak{pgl}(L) \to \mathfrak{pgl}(L)$ given by $R \mapsto -R^T$ is $-\dim(L) + 1$. Using this and additivity of traces, the result follows. □

Now we can complete the proof of Theorem 1.2. Let $g \in G$ be as in Lemma 6.2 and let $s = [(H, gH)] \in \mathcal{N}$. Then the involution $\mathrm{d}_s\tau$ is not in the image of $\mathrm{Aut}_{\mathcal{N}}(s) \to \mathrm{GL}(\mathrm{T}_s\mathcal{N})$ by Proposition 4.1 and Lemma 6.3. Hence the morphisms $\pi_{\mathcal{N}} \circ \tau$ and $\pi_{\mathcal{N}}$ do not coincide by Lemma 6.1. □

## 7. Proof of Theorem 1.6

Let $X$ and $Y$ be GPK$^3$ double mirrors,

$$X = \mathrm{Gr}_1 \cap \mathrm{Gr}_2 \subset \mathbf{P},$$
$$Y = \mathrm{Gr}_1^\vee \cap \mathrm{Gr}_2^\vee \subset \mathbf{P}^\vee,$$

corresponding to isomorphisms $\phi_i \colon \wedge^2 V \xrightarrow{\sim} W$, $i = 1, 2$.

The proof of [4, Theorem 2.12] applies directly to show that if $X$ is not birational to $Y$, then $[X] \neq [Y]$ in $\mathrm{K}_0(\mathrm{Var}/k)$. Hence by Theorem 1.2, the second part of Theorem 1.6 holds.

To prove the first part of Theorem 1.6, we consider an incidence correspondence between $\mathrm{Gr}_1$ and $\mathrm{Gr}_2^\vee$. Namely, we consider the intersection

$$\mathrm{Q}(\mathrm{Gr}_1, \mathrm{Gr}_2^\vee) = \mathrm{Q} \times_{\mathbf{P} \times \mathbf{P}^\vee} (\mathrm{Gr}_1 \times \mathrm{Gr}_2^\vee)$$

of the canonical $(1,1)$ divisor $\mathrm{Q} \subset \mathbf{P} \times \mathbf{P}^\vee$ with the product $\mathrm{Gr}_1 \times \mathrm{Gr}_2^\vee \subset \mathbf{P} \times \mathbf{P}^\vee$. We will calculate the class of $\mathrm{Q}(\mathrm{Gr}_1, \mathrm{Gr}_2^\vee)$ in $\mathrm{K}_0(\mathrm{Var}/k)$ in two ways, using the two projections

$$\begin{array}{ccc} & \mathrm{Q}(\mathrm{Gr}_1, \mathrm{Gr}_2^\vee) & \\ {\scriptstyle p_1} \swarrow & & \searrow {\scriptstyle p_2} \\ \mathrm{Gr}_1 & & \mathrm{Gr}_2^\vee. \end{array}$$

Given $x \in \mathbf{P}$, let $x_i = \phi_i^{-1}(x) \in \mathbf{P}(\wedge^2 V)$ be the corresponding point for $i = 1, 2$. Similarly, given $y \in \mathbf{P}^\vee$ let $y_i = \phi_i^*(y) \in \mathbf{P}(\wedge^2 V^\vee)$ for $i = 1, 2$. Further, for a point $\omega$ in $\mathbf{P}(\wedge^2 V)$ or $\mathbf{P}(\wedge^2 V^\vee)$, we write $\mathrm{rk}(\omega)$ for the rank of $\omega$ considered as a skew form (defined up to scalars); note that either $\mathrm{rk}(\omega) = 2$ or $\mathrm{rk}(\omega) = 4$. By definition we have

$$X = \{x \in \mathbf{P} \mid \mathrm{rk}(x_1) = \mathrm{rk}(x_2) = 2\} \ \subset \mathrm{Gr}_1 = \{x \in \mathbf{P} \mid \mathrm{rk}(x_1) = 2\},$$
$$Y = \left\{y \in \mathbf{P}^\vee \;\middle|\; \mathrm{rk}(y_1) = \mathrm{rk}(y_2) = 2\right\} \subset \mathrm{Gr}_2^\vee = \left\{y \in \mathbf{P}^\vee \;\middle|\; \mathrm{rk}(y_2) = 2\right\},$$

and hence also

$$\mathrm{Gr}_1 \setminus X = \{x \in \mathbf{P} \mid \mathrm{rk}(x_1) = 2,\ \mathrm{rk}(x_2) = 4\},$$
$$\mathrm{Gr}_2^\vee \setminus Y = \left\{y \in \mathbf{P}^\vee \;\middle|\; \mathrm{rk}(y_1) = 4,\ \mathrm{rk}(y_2) = 2\right\}.$$

For $\omega$ in $\mathbf{P}(\wedge^2 V)$ or $\mathbf{P}(\wedge^2 V^\vee)$, we let $H_\omega$ denote the corresponding hyperplane in the dual projective space. Then for $x \in \mathrm{Gr}_1$ and $y \in \mathrm{Gr}_2$ we have

$$p_1^{-1}(x) \cong H_{x_2} \cap \mathrm{Gr}(2, V^\vee) \subset \mathbf{P}(\wedge^2 V^\vee),$$
$$p_2^{-1}(y) \cong H_{y_1} \cap \mathrm{Gr}(2, V) \subset \mathbf{P}(\wedge^2 V).$$

Recall that a morphism of varieties $g \colon Z \to S$ is called a *piecewise trivial fibration with fiber* $F$ if there is a finite partition $S = S_1 \sqcup S_2 \sqcup \cdots \sqcup S_n$, with each $S_i \subset Y$ a locally closed subset such that $g^{-1}(S_i) \cong S_i \times F$ as $S_i$-schemes.

**Lemma 7.1.** *The following hold:*

*(1) The morphisms*

$$p_1^{-1}(X) \to X \quad \textit{and} \quad p_2^{-1}(Y) \to Y$$

*are piecewise trivial fibrations, with fiber a hyperplane section of* $\mathrm{Gr}(2, V)$ *defined by a rank* 2 *skew form.*

*(2) The morphisms*

$$p_1^{-1}(\mathrm{Gr}_1 \setminus X) \to \mathrm{Gr}_1 \setminus X \quad \textit{and} \quad p_2^{-1}(\mathrm{Gr}_2 \setminus Y) \to \mathrm{Gr}_2 \setminus Y$$

*are piecewise trivial fibrations, with fiber a hyperplane section of* $\mathrm{Gr}(2, V)$ *defined by a rank* 4 *skew form.*

*Proof.* By the above discussion, this follows as in [21, Lemma 3.3] from the fact that skew forms over $k$ can be put into one of the standard forms according to their rank. □

Next we calculate the class of the fibers appearing in Lemma 7.1.

**Lemma 7.2.** *If* $\omega \in \mathbf{P}(\wedge^2 V^\vee)$*, then*

$$[H_\omega \cap \mathrm{Gr}(2, V)] = \begin{cases} (\mathbf{L}^2 + \mathbf{L} + 1)(\mathbf{L}^3 + \mathbf{L}^2 + 1) & \textit{if } \mathrm{rk}(\omega) = 2, \\ (\mathbf{L}^2 + 1)(\mathbf{L}^3 + \mathbf{L}^2 + \mathbf{L} + 1) & \textit{if } \mathrm{rk}(\omega) = 4. \end{cases}$$

*Proof.* First assume $\mathrm{rk}(\omega) = 2$. Then the kernel $K \subset V$ of $\omega$ regarded as a skew form has $\dim K = 3$, and

$$H_\omega \cap \mathrm{Gr}(2, V) = \{A \in \mathrm{Gr}(2, V) \mid A \cap K \neq 0\}.$$

Consider the closed subset

$$Z = \{A \in \mathrm{Gr}(2, V) \mid A \subset K\} \subset H_\omega \cap \mathrm{Gr}(2, V),$$

with open complement

$$U = \{A \in \mathrm{Gr}(2, V) \mid \dim(A \cap K) = 1\} \subset H_\omega \cap \mathrm{Gr}(2, V).$$

Note that $Z \cong \mathbf{P}^2$. Further, the natural morphism $U \to \mathbf{P}(K) \cong \mathbf{P}^2$ is a Zariski locally trivial fibration, whose fiber over $[v] \in \mathbf{P}(K)$ is the complement in $\mathbf{P}(V/\langle v \rangle) \cong \mathbf{P}^3$ of $\mathbf{P}(K/\langle v \rangle) \cong \mathbf{P}^1$. Hence we have

$$[H_\omega \cap \mathrm{Gr}(2, V)] = [\mathbf{P}^2] + [\mathbf{P}^2]([\mathbf{P}^3] - [\mathbf{P}^1]) = (\mathbf{L}^2 + \mathbf{L} + 1)(\mathbf{L}^3 + \mathbf{L}^2 + 1).$$

Now assume $\mathrm{rk}(\omega) = 4$. Let $V_4 \subset V$ be a 4-dimensional subspace such that the restriction of the form $\omega$ to $V_4$ has full rank. Consider the closed subset

$$Z = \{A \in H_\omega \cap \mathrm{Gr}(2, V) \mid A \subset V_4\} \subset H_\omega \cap \mathrm{Gr}(2, V),$$

with open complement

$$U = \{A \in H_\omega \cap \mathrm{Gr}(2, V) \mid \dim(A \cap V_4) = 1\} \subset H_\omega \cap \mathrm{Gr}(2, V).$$

Note that $Z$ is isomorphic to a smooth quadric hypersurface in $\mathbf{P}^4$, whose class is well-known to be $[Z] = [\mathbf{P}^3]$. Further, the natural morphism $U \to \mathbf{P}(V_4) \cong \mathbf{P}^3$ is a Zariski locally trivial fibration, whose fiber over $[v] \in \mathbf{P}(V_4)$ consists of $[v'] \in \mathbf{P}(V/\langle v \rangle)$ such that $\omega(v, v') = 0$ and $v' \notin V_4/\langle v \rangle$. That is, the fiber is isomorphic to the complement in $\mathbf{P}(\langle v \rangle^{\perp_\omega}/\langle v \rangle) \cong \mathbf{P}^2$ of $\mathbf{P}((\langle v \rangle^{\perp_\omega} \cap V_4)/\langle v \rangle) \cong \mathbf{P}^1$, where $\langle v \rangle^{\perp_\omega} \subset V$ denotes the orthogonal of $\langle v \rangle \subset V$ with respect to the skew form $\omega$. Hence we have

$$[H_\omega \cap \mathrm{Gr}(2, V)] = [\mathbf{P}^3] + [\mathbf{P}^3]([\mathbf{P}^2] - [\mathbf{P}^1]) = (\mathbf{L}^2 + 1)(\mathbf{L}^3 + \mathbf{L}^2 + \mathbf{L} + 1),$$

as claimed. □

Now we can prove Theorem 1.6. Using the first projection $p_1$, we have

$$[\mathrm{Q}(\mathrm{Gr}_1, \mathrm{Gr}_2^\vee)] = [p_1^{-1}(X)] + [p_1^{-1}(\mathrm{Gr}_1 \setminus X)].$$

But if $g\colon Z \to S$ is a piecewise trivial fibration with fiber $F$, then $[Z] = [S][F]$. Hence using Lemmas 7.1 and 7.2, we find

$$\begin{aligned}[\mathrm{Q}(\mathrm{Gr}_1, \mathrm{Gr}_2^\vee)] &= [X](\mathbf{L}^2+\mathbf{L}+1)(\mathbf{L}^3+\mathbf{L}^2+1) + ([\mathrm{Gr}(2,V)] - [X])(\mathbf{L}^2+1)(\mathbf{L}^3+\mathbf{L}^2+\mathbf{L}+1),\\ &= [X]\mathbf{L}^4 + [\mathrm{Gr}(2,V)](\mathbf{L}^2+1)(\mathbf{L}^3+\mathbf{L}^2+\mathbf{L}+1).\end{aligned}$$

The same argument applied to the second projection $p_2$ shows

$$[\mathrm{Q}(\mathrm{Gr}_1, \mathrm{Gr}_2^\vee)] = [Y]\mathbf{L}^4 + [\mathrm{Gr}(2,V)](\mathbf{L}^2+1)(\mathbf{L}^3+\mathbf{L}^2+\mathbf{L}+1).$$

We conclude

$$([X]-[Y])\mathbf{L}^4 = 0. \qquad \square$$

## Appendix A. Borel–Weil–Bott computations

The purpose of this appendix is to collect some coherent cohomology computations on Grassmannians and GPK$^3$ threefolds, which are invoked in the main text. The key tool is Borel–Weil–Bott, which we review in §A.1.

A.1. **Borel–Weil–Bott.** For this subsection, we let $V$ denote an $n$-dimensional vector space over $k$ (in the rest of the paper $n = 5$). The Borel–Weil–Bott Theorem for $\mathrm{GL}(V)$ allows us to compute the coherent cohomology of $\mathrm{GL}(V)$-equivariant bundles on a Grassmannian $\mathrm{Gr}(r, V)$ (in the rest of the paper we only need the case $r = 2$). To state the result, we need some notation. Our exposition follows [17, §2.6].

The weight lattice of $\mathrm{GL}(V)$ is isomorphic to $\mathbf{Z}^n$ via the map taking the $d$-th fundamental weight, i.e. the highest weight of $\wedge^d V$, to the sum of the first $d$ basis vectors of $\mathbf{Z}^n$. Under this isomorphism, the dominant integral weights of $\mathrm{GL}(V)$ correspond to nonincreasing sequences of integers $\lambda = (\lambda_1, \ldots, \lambda_n)$. For such a $\lambda$, we denote by $\Sigma^\lambda V$ the corresponding irreducible representation of $\mathrm{GL}(V)$ of highest weight $\lambda$. The only facts we shall need about these representations are the following:

(1) If $\lambda = (1, \ldots, 1, 0, \ldots, 0)$ with the first $d$ entries equal to 1, then $\Sigma^\lambda V = \wedge^d V$.
(2) If $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\lambda_1 + m, \ldots, \lambda_n + m)$ for some $m \in \mathbf{Z}$, then there is an isomorphism of $\mathrm{GL}(V)$-representations $\Sigma^\mu V \cong \Sigma^\lambda V \otimes \det(V)^{\otimes m}$.
(3) Given $\lambda = (\lambda_1, \ldots, \lambda_n)$, set $\lambda^\vee = (-\lambda_n, -\lambda_{n-1}, \ldots, -\lambda_1)$. Then there is an isomorphism of $\mathrm{GL}(V)$-representations $\Sigma^{\lambda^\vee} V \cong (\Sigma^\lambda V)^\vee$.

The construction $V \mapsto \Sigma^\lambda V$ for a dominant integral weight $\lambda$ globalizes to vector bundles over a scheme, and the above identities continue to hold. We are interested in the case where the base scheme is the Grassmannian $\mathrm{Gr}(r, V)$. Denote by $\mathcal{U}$ the tautological rank $r$ bundle on $\mathrm{Gr}(r, V)$, and by $\mathcal{Q}$ the rank $n - r$ quotient of $V \otimes \mathcal{O}_{\mathrm{Gr}(r,V)}$ by $\mathcal{U}$, so that there is an exact sequence

$$0 \to \mathcal{U} \to V \otimes \mathcal{O} \to \mathcal{Q} \to 0.$$

Then every $\mathrm{GL}(V)$-equivariant bundle on $\mathrm{Gr}(r, V)$ is of the form $\Sigma^\alpha \mathcal{U}^\vee \otimes \Sigma^\beta \mathcal{Q}^\vee$ for some nonincreasing sequences of integers $\alpha \in \mathbf{Z}^r$ and $\beta \in \mathbf{Z}^{n-r}$.

The symmetric group $\mathrm{S}_n$ acts on the weight lattice $\mathbf{Z}^n$ by permuting the factors. Denote by $\ell : \mathrm{S}_n \to \mathbf{Z}$ the standard length function. We say $\lambda \in \mathbf{Z}^n$ is *regular* if all of its components are distinct; in this case, there is a unique $\sigma \in \mathrm{S}_n$ such that $\sigma(\lambda)$ is a strictly decreasing sequence. Finally, let

$$\rho = (n, n-1, \dots, 2, 1) \in \mathbf{Z}^n$$

be the sum of the fundamental weights.

The following result can be deduced from the usual statement of Borel–Weil–Bott by pushing forward equivariant line bundles on the flag variety to the Grassmannian. For a vector space $L$ and an integer $p$, we write $L[p]$ for the single-term complex of vector spaces with $L$ in degree $-p$.

**Proposition A.1.** *Let the notation be as above. Let $\alpha \in \mathbf{Z}^r$ and $\beta \in \mathbf{Z}^{n-r}$ be nonincreasing sequences of integers, and let $\lambda = (\alpha, \beta) \in \mathbf{Z}^n$ be their concatenation. If $\lambda + \rho$ is not regular, then*

$$\mathrm{R}\Gamma(\mathrm{Gr}(r, V), \Sigma^\alpha \mathcal{U}^\vee \otimes \Sigma^\beta \mathcal{Q}^\vee) \cong 0$$

*If $\lambda + \rho$ is regular and $\sigma \in \mathrm{S}_n$ is the unique element such that $\sigma(\lambda + \rho)$ is a strictly decreasing sequence, then*

$$\mathrm{R}\Gamma(\mathrm{Gr}(r, V), \Sigma^\alpha \mathcal{U}^\vee \otimes \Sigma^\beta \mathcal{Q}^\vee) \cong \Sigma^{\sigma(\lambda+\rho)-\rho} V^\vee[-\ell(\sigma)].$$

A.2. **Computations on** $\mathrm{Gr}$**.** From now on, we assume $\dim V = 5$, fix an identification $\wedge^2 V \cong W$, and let $\mathrm{Gr} \subset \mathbf{P}$ denote the corresponding embedded Grassmannian $\mathrm{Gr}(2, V)$.

**Lemma A.2.** *The ideal sheaf $\mathcal{I}_{\mathrm{Gr}/\mathbf{P}}$ of $\mathrm{Gr} \subset \mathbf{P}$ admits a resolution of the form*

$$0 \to \mathcal{O}(-5) \to V^\vee \otimes \mathcal{O}(-3) \to V \otimes \mathcal{O}(-2) \to \mathcal{I}_{\mathrm{Gr}/\mathbf{P}} \to 0. \tag{A.1}$$

*Proof.* This follows from [13, Theorem 2.2] by regarding $\mathrm{Gr} \subset \mathbf{P}$ as a Pfaffian variety. □

**Lemma A.3.** *The restriction map $\mathrm{H}^0(\mathbf{P}, \mathrm{T}_\mathbf{P}) \to \mathrm{H}^0(\mathrm{Gr}, \mathrm{T}_\mathbf{P}|_{\mathrm{Gr}})$ is an isomorphism.*

*Proof.* Taking cohomology of the exact sequence

$$0 \to \mathcal{I}_{\mathrm{Gr}/\mathbf{P}} \otimes \mathrm{T}_\mathbf{P} \to \mathrm{T}_\mathbf{P} \to \mathrm{T}_\mathbf{P}|_{\mathrm{Gr}} \to 0,$$

we see it is enough to show $\mathrm{H}^k(\mathbf{P}, \mathcal{I}_{\mathrm{Gr}/\mathbf{P}} \otimes \mathrm{T}_\mathbf{P}) = 0$ for $k = 0, 1$. In fact, we claim the sheaf $\mathcal{I}_{\mathrm{Gr}/\mathbf{P}} \otimes \mathrm{T}_\mathbf{P}$ has no cohomology. Indeed, $\mathrm{R}\Gamma(\mathbf{P}, \mathrm{T}_\mathbf{P}(-t)) \cong 0$ for $2 \leq t \leq 9$, as can be seen from the exact sequence

$$0 \to \mathcal{O} \to W \otimes \mathcal{O}(1) \to \mathrm{T}_\mathbf{P} \to 0,$$

so the claim follows by tensoring the resolution (A.1) with $\mathrm{T}_\mathbf{P}$ and taking cohomology. □

**Lemma A.4.** *The normal bundle $\mathrm{N}_{\mathrm{Gr}/\mathbf{P}}$ of $\mathrm{Gr} \subset \mathbf{P}$ satisfies*

$$\mathrm{N}_{\mathrm{Gr}/\mathbf{P}} \cong \wedge^2 \mathcal{Q}(1) \cong \mathcal{Q}^\vee(2).$$

**Remark A.5.** The proof below goes through to show that if $V$ has arbitrary dimension $n$, then there are isomorphisms $\mathrm{N}_{\mathrm{Gr}/\mathbf{P}} \cong \wedge^2 \mathcal{Q}(1) \cong (\wedge^{n-4} \mathcal{Q}^\vee)(2)$, recovering the statement of the lemma when $n = 5$.

*Proof.* The normal bundle fits into a commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{U}^\vee \otimes \mathcal{U} & \longrightarrow & \mathcal{O} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{U}^\vee \otimes V & \longrightarrow & \wedge^2 V \otimes \mathcal{O}(1) & \longrightarrow & \mathcal{E} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{T}_{\mathrm{Gr}} & \longrightarrow & \mathrm{T}_{\mathbf{P}}|_{\mathrm{Gr}} & \longrightarrow & \mathrm{N}_{\mathrm{Gr}/\mathbf{P}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

with exact rows and columns. Here, the map $\mathcal{U}^\vee \otimes \mathcal{U} \to \mathcal{O}$ is given by evaluation. The map $\mathcal{U}^\vee \otimes V \to \wedge^2 V \otimes \mathcal{O}(1)$ can be described as follows. Since $\det(\mathcal{U}^\vee) \cong \mathcal{O}(1)$ there is a natural isomorphism $\mathcal{U}^\vee \cong \mathcal{U}(1)$, and the map in question is the composition

$$\mathcal{U}^\vee \otimes V \cong \mathcal{U} \otimes V(1) \hookrightarrow V \otimes V \otimes \mathcal{O}(1) \to \wedge^2 V \otimes \mathcal{O}(1).$$

The sheaf $\mathcal{E}$ is by definition the cokernel of this map. Due to the exact sequence

$$0 \to \mathcal{U} \to V \otimes \mathcal{O} \to \mathcal{Q} \to 0$$

we therefore have an isomorphism $\mathcal{E} \cong (\wedge^2 \mathcal{Q})(1)$. Hence also $\mathcal{E} \cong \mathcal{Q}^\vee(2)$ as $\det(\mathcal{Q}) \cong \mathcal{O}(1)$. It remains to note that $\mathcal{E} \cong \mathrm{N}_{\mathrm{Gr}/\mathbf{P}}$ by the snake lemma. □

**Lemma A.6.** *We have*

$$\mathrm{R}\Gamma(\mathrm{Gr}, \mathcal{Q}(-t)) \cong \begin{cases} V[0] & t = 0, \\ 0 & 1 \le t \le 5, \\ \Sigma^{(t-2,t-2,t-3,3,3)} V^\vee[-6] & t \ge 6, \end{cases}$$

$$\mathrm{R}\Gamma(\mathrm{Gr}, \wedge^2 \mathcal{Q}(-t)) \cong \begin{cases} \wedge^2 V[0] & t = 0, \\ 0 & 1 \le t \le 5, \\ \Sigma^{(t-2,t-3,t-3,3,3)} V^\vee[-6] & t \ge 6. \end{cases}$$

*Proof.* Note that $\mathcal{Q}(-t) \cong \Sigma^{(t,t,t-1)}\mathcal{Q}^\vee$ and $\wedge^2 \mathcal{Q}(-t) \cong \Sigma^{(t,t-1,t-1)}\mathcal{Q}^\vee$. Now the result follows from Proposition A.1. □

**Lemma A.7.** *For $2 \le t \le 6$, we have* $\mathrm{R}\Gamma(\mathrm{Gr}, \mathrm{N}_{\mathrm{Gr}/\mathbf{P}}(-t)) \cong 0$.

*Proof.* Combine Lemmas A.4 and A.6. □

A.3. **Computations on a GPK$^3$ threefold.** Let $X = \mathrm{Gr}_1 \cap \mathrm{Gr}_2$ be a GPK$^3$ threefold. We write $\mathcal{Q}_i$ for the tautological rank 3 quotient bundle on $\mathrm{Gr}_i$, and $\mathrm{N}_i = \mathrm{N}_{\mathrm{Gr}_i/\mathbf{P}}$ for the normal bundle of $\mathrm{Gr}_i \subset \mathbf{P}$.

**Lemma A.8.** *For $i = 1, 2$, the ideal sheaf $\mathcal{I}_{X/\mathrm{Gr}_i}$ of $X \subset \mathrm{Gr}_i$ admits a resolution of the form*

$$0 \to \mathcal{O}(-5) \to V^\vee \otimes \mathcal{O}(-3) \to V \otimes \mathcal{O}(-2) \to \mathcal{I}_{X/\mathrm{Gr}_i} \to 0 \qquad \text{(A.2)}$$

*Proof.* Analogously to Lemma A.2, this follows by regarding $X \subset \mathrm{Gr}_i$ as a Pfaffian variety. □

**Lemma A.9.** *The class of $X$ in the Chow ring of $\mathrm{Gr}_i$ is $5H^3$, where $H$ denotes the Plücker hyperplane class.*

*Proof.* By (A.2) there is a resolution of $\mathcal{O}_X$ on $\mathrm{Gr}_i$ of the form

$$0 \to \mathcal{O}(-5) \to V^\vee \otimes \mathcal{O}(-3) \to V \otimes \mathcal{O}(-2) \to \mathcal{O} \to \mathcal{O}_X \to 0. \tag{A.3}$$

The result follows by taking $\mathrm{ch}_3$. □

**Lemma A.10.** *For $t \geq 1$ we have*

$$\mathrm{H}^0(X, \mathcal{Q}_i|_X(-tH)) = \mathrm{H}^0(X, \wedge^2(\mathcal{Q}_i|_X)(-tH)) = 0.$$

*Proof.* From (A.3) we get a resolution

$$0 \to \mathcal{Q}_i(-(t+5)) \to V^\vee \otimes \mathcal{Q}_i(-(t+3)) \to V \otimes \mathcal{Q}_i(-(t+2)) \to \mathcal{Q}_i(-tH) \to \mathcal{Q}_i|_X(-tH) \to 0.$$

Let $\mathcal{R}_i^\bullet$ be the complex concentrated in degrees $[-3, 0]$ given by the first four terms, so that there is a quasi-isomorphism

$$\mathcal{R}_i^\bullet \simeq \mathcal{Q}_i|_X(-tH).$$

Then the resulting spectral sequence

$$E_1^{p,q} = \mathrm{H}^q(X, \mathcal{R}_i^p) \implies \mathrm{H}^{p+q}(X, \mathcal{Q}_i|_X(-tH))$$

combined with Lemma A.6 shows $\mathrm{H}^0(X, \mathcal{Q}_i|_X(-tH)) = 0$ for $t \geq 1$. The same argument also proves $\mathrm{H}^0(X, \wedge^2(\mathcal{Q}_i|_X)(-tH)) = 0$ for $t \geq 1$. □

**Lemma A.11.** *The restriction maps*

$$V \cong \mathrm{H}^0(\mathrm{Gr}_i, \mathcal{Q}_i) \to \mathrm{H}^0(X, \mathcal{Q}_i|_X), \ i = 1, 2,$$
$$W^\vee \cong \mathrm{H}^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(1)) \to \mathrm{H}^0(X, \mathcal{O}_X(1)),$$

*are isomorphisms.*

*Proof.* Taking cohomology of the exact sequence

$$0 \to \mathcal{I}_{X/\mathrm{Gr}_i} \otimes \mathcal{Q}_i \to \mathcal{Q}_i \to \mathcal{Q}_i|_X \to 0,$$

the first claim follows from the vanishing $\mathrm{R}\Gamma(\mathrm{Gr}_i, \mathcal{I}_{X/\mathrm{Gr}_i} \otimes \mathcal{Q}_i) = 0$, which is a consequence of the resolution (A.2) combined with Lemma A.6. The second claim is proved similarly. □

**Lemma A.12.** *The restriction maps $\mathrm{H}^0(\mathrm{Gr}_i, \mathrm{N}_i) \to \mathrm{H}^0(X, \mathrm{N}_i|_X)$, $i = 1, 2$, are isomorphisms.*

*Proof.* Taking cohomology of the exact sequence

$$0 \to \mathcal{I}_{X/\mathrm{Gr}_i} \otimes \mathrm{N}_i \to \mathrm{N}_i \to \mathrm{N}_i|_X \to 0,$$

we see it is enough to show $\mathrm{H}^k(\mathrm{Gr}_i, \mathcal{I}_{X/\mathrm{Gr}_i} \otimes \mathrm{N}_i) = 0$ for $k = 0, 1$. In fact, we claim the sheaf $\mathcal{I}_{X/\mathrm{Gr}_i} \otimes \mathrm{N}_i$ has no cohomology. This follows by tensoring the resolution (A.2) with $\mathrm{N}_i$ and applying Lemma A.7. □

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