When is the self-intersection of a subvariety a fibration?

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Abstract

We provide a necessary and sufficient condition for the derived self-intersection of a smooth subscheme inside a smooth scheme to be a fibration over the subscheme. As a consequence we deduce a generalized HKR isomorphism. We also investigate the relationship of our result to path spaces in homotopy theory, Buchweitz-Flenner formality in algebraic geometry, and draw parallels with similar results in Lie theory and symplectic geometry.

Introduction

0.1. Let Y be a smooth scheme and let X be a smooth subscheme of Y. The main goal of this paper is to give a complete answer to the following question.

> *Under what circumstances is the derived self-intersection* $X \times^R_Y X$ *a fibration over* X*?*

0.2. Denote the closed embedding $X \hookrightarrow Y$ by i and let W be the derived intersection $X \times^R_Y X$. In this context we can think of W as the ordinary scheme X with structure sheaf replaced by a *structure complex* $\mathscr{O}_W$, which is a commutative differential graded algebra with certain properties. Simplifying things slightly for ease of exposition in the introduction we can think of $\mathscr{O}_W$ as the commutative algebra object $i^* i_* \mathscr{O}_X$ of $\mathbf{D}(X)$. (All our functors are implicitly derived.) Here $\mathbf{D}(X)$ denotes the derived category of coherent sheaves on X.

0.3. Any object $E \in \mathbf{D}(X)$ concentrated in strictly positive degrees can be regarded as a linear fibration over X by considering the dg-scheme over X with structure complex $\mathbb{S}(E^\vee)$, the symmetric algebra of the dual of E. Ignoring the difference between dg- and derived, the main question of this paper can be rephrased as

Under what circumstances is the object $i^*i_*\mathscr{O}_X$ *of* $\mathbf{D}(X)$ *of the form* $\mathbb{S}(E^\vee)$ *for some* $E \in \mathbf{D}(X)$*?*

0.4. Our question is motivated by the case where the inclusion $X \hookrightarrow Y$ is the diagonal embedding $\Delta : X \hookrightarrow X \times X$. In this case, using an analogy with homotopy theory, the self-intersection W can be understood as the free loop space $\mathscr{L}X$. Then $\mathscr{L}X$ fibers over X via the map which associates to a loop its base point. Equivalently one has the HKR isomorphism [16], [19]

$$\Delta^*\mathscr{O}_\Delta \cong \mathbb{S}(\Omega^1_X[1])$$

which identifies W with the linear fibration $T_X[-1]$ over X, where T_X is the tangent bundle of X.

0.5. Our criterion for fibering W over X can be expressed in two equivalent ways, one as an extension property for the normal bundle $N = N_{X/Y}$, and another as the vanishing of a cohomological extension class. We shall say the closed embedding $X \hookrightarrow Y$ satisfies condition $(*)$ if it satisfies either of the following two equivalent properties:

$(*)$ The normal bundle N extends to a vector bundle $\overline{N}$ on the first infinitesimal neighborhood X' of X in Y.

The morphism α_N defined below vanishes.

0.6. For any object $V \in \mathbf{D}(X)$ its Atiyah class is a morphism $\mathsf{at}_V : V \to V \otimes \Omega^1_X[1]$ in $\mathbf{D}(X)$. The short exact sequence

$$0 \to N^\vee \to \Omega^1_Y|_X \to \Omega^1_X \to 0$$

gives rise to an extension class $\eta : \Omega^1_X \to N^\vee[1]$. The morphism α_V is defined as the composition

$$\alpha_V : \quad V \xrightarrow{\mathsf{at}_V} V \otimes \Omega_X[1] \xrightarrow{\mathsf{id}_V \otimes \eta[1]} V \otimes N^\vee[2].$$

If V is a vector bundle on X the class α_V vanishes if and only if there exists a vector bundle $\overline{V}$ on X' whose restriction to X is isomorphic to V, see [9, 13]. Thus the two conditions in $(*)$ are equivalent. We review this equivalence in (2.10).

The main result of this paper is the following theorem.

0.7. Theorem. *The derived self-intersection* $W = X \times^{R}_{Y} X$ *fibers over* X *if and only if the closed embedding* $i : X \hookrightarrow Y$ *satisfies condition* $(*)$. *In this case*

$$\mathscr{O}_W = i^* i_* \mathscr{O}_X \cong \mathbb{S}(N^\vee[1]) = \bigoplus_j \wedge^j N^\vee[j],$$

where $N^\vee$ *denotes the conormal bundle of* X *in* Y, *hence* W *is identified with the linear fibration* $N[-1]$.

As an immediate consequence of the above theorem we obtain the following corollary.

0.8. Corollary. *Assume the closed embedding* $i : X \hookrightarrow Y$ *satisfies* $(*)$. *Then the hypercohomology spectral sequence*

$$^2E^{pq} = H^p(X, \wedge^q N) \Rightarrow \mathrm{Hom}^{p+q}_{\mathbf{D}(X)}(i^* i_* \mathscr{O}_X, \mathscr{O}_X) = \mathrm{Ext}^{p+q}_Y(\mathscr{O}_X, \mathscr{O}_X)$$

degenerates at the ^2E*-page, giving rise to a direct sum decomposition*

$$\mathrm{Ext}^n_Y(\mathscr{O}_X, \mathscr{O}_X) \cong \bigoplus_{p+q=n} H^p(X, \wedge^q N).$$

0.9. There are many instances where condition $(*)$ is satisfied. This is the case, for example, when there is a finite group G acting on Y and X is the fixed locus of the action. Then the normal bundle sequence splits and its class η vanishes, so condition $(*)$ is trivially satisfied. In particular this happens for the case of the diagonal embedding $\Delta : X \hookrightarrow X \times X$ and we recover the Hochschild-Kostant-Rosenberg isomorphism. While this already shows that Theorem 0.7 can be regarded as a generalization of the HKR isomorphism, the connection between these two results is deeper, in a sense that we make precise now.

0.10. By analogy with homotopy theory define spaces $\Pi_k(Y, X)$ along with maps $X \to \Pi_k(Y, X)$ by setting $\Pi_0(Y, X) = Y$ and recursively for $k \geq 0$ by

$$\Pi_{k+1}(Y, X) = X \times^{R}_{\Pi_k(Y,X)} X.$$

The map $X \to \Pi_k(Y, X)$ is the original inclusion for $k = 0$, and the diagonal for $k \geq 1$. (We assume that we work in a context where derived fiber products exist, be it simplicial schemes, dg-schemes of Ciocan-Fontanine and Kapranov [5], or derived schemes of Lurie [12] and Toën–Vezzosi [17].)

If instead of algebraic varieties X and Y were topological spaces and the derived fiber product was understood as the homotopy fiber product then the spaces $\Pi_k(Y, X)$ would be obtained through the familiar path construction and would be identified with

$$\Pi_k(Y, X) = \{f : D^k \to Y \mid f(\partial D^k) \subset X\},$$

where D^k is the k-ball. These are the spaces that appear in the definition of relative homotopy groups.

0.11. In the context of algebraic geometry Buchweitz and Flenner [1] proved the strongest possible version of the HKR theorem. If $X \to Y$ is any morphism of schemes and if $\Delta : X \to X \times^R_Y X$ denotes the diagonal morphism then they construct an isomorphism

$$\Delta^* \mathscr{O}_\Delta \cong \mathbb{S}(\mathbb{L}[1]),$$

where $\mathbb{L}$ is the relative cotangent complex of the morphism $X \to Y$. The usual HKR isomorphism for a smooth variety X is recovered by considering the structure morphism $X \to \mathsf{pt}$, whereby $\mathbb{L} = \Omega^1_X[0]$.

By contrast, if the map under consideration is a closed embedding of smooth schemes, its cotangent complex is $\mathbb{L} = N^\vee[1]$. Theorem 0.7 can be rephrased as saying that $i^* i_* \mathscr{O}_X \cong \mathbb{S}(\mathbb{L})$ if and only if $(*)$ holds.

0.12. The two isomorphisms of structure complexes described above can be regarded as isomorphisms of spaces

$$\begin{array}{lcll} \Pi_1(Y, X) & \cong & \mathbb{L}^\vee & \text{if condition } (*) \text{ holds;} \\ \Pi_2(Y, X) & \cong & \mathbb{L}^\vee[-1] & \text{with no assumptions.} \end{array}$$

The fact that in one case we get $\mathbb{L}^\vee$ and its shift in the other is no surprise. In the world of algebraic topology there is a homotopy equivalence

$$\Omega \Pi_{i-1}(Y, X) = \Pi_i(Y, X)$$

which is the natural analogue of the fact that $\mathbb{L}^\vee$ and $\mathbb{L}^\vee[-1]$ are shifts of one another. What is much more surprising is the fact that while we need condition $(*)$ for the space $\Pi_1(Y, X)$ to fiber over X, there is no condition needed for $\Pi_2(Y, X)$.

0.13. It is interesting to relate Theorem 0.7 to other similar results. It was noted by M. Mustaţă that condition $(*)$ is very similar to the condition that appears in Deligne-Illusie's theorem [7] on the degeneration of the Hodge-to-de Rham spectral sequence. In both cases the condition needed is the existence of a lifting to the first infinitesimal neighborhood. It would be very interesting to explore this connection further.

Secondly, in symplectic geometry it was noted that the spectral sequence that relates the singular and Floer cohomologies of a Lagrangian submanifold which is the fixed locus of an anti-symplectic involution is expected to degenerate. This appears to be mirror to the fact that condition $(*)$ holds when X is the fixed locus of an involution on Y and then the spectral sequence of Corollary 0.8 degenerates.

0.14. Finally there is a strong analogy between the world of varieties and that of Lie algebras, initially discovered by Kontsevich and Kapranov. In this theory we associate to the variety X the Lie algebra object $\mathsf{T}_\mathsf{X}[-1]$ of $\mathbf{D}(\mathsf{X})$, where T_X is the tangent bundle of X. The analogue of Theorem 0.7 is the following result, which will appear in [2].

0.15. Theorem (Calaque-Căldăraru-Tu). *Let $\mathfrak{h} \hookrightarrow \mathfrak{g}$ be an inclusion of Lie algebras and let $\mathfrak{n}$ denote the quotient $\mathfrak{g}/\mathfrak{h}$ as an $\mathfrak{h}$-module. Then there is a Poincaré-Birkhoff-Witt-type isomorphism of Lie modules*

$$\mathfrak{U}\mathfrak{g}/\mathfrak{h}\mathfrak{U}\mathfrak{g} \cong \mathfrak{S}\mathfrak{g}/\mathfrak{h}\mathfrak{S}\mathfrak{g} \cong \mathfrak{S}\mathfrak{n}$$

if and only if a certain condition $(**)$ is satisfied. This condition is the exact analogue of condition $(*)$, expressed either as an extension property or as the vanishing of a natural cohomology class.

0.16. The body of the paper consists of two parts. In the first part we prove the "if" part of Theorem 0.7 and we discuss briefly the dependence of the resulting isomorphism on the choice of lifting $\overline{\mathsf{N}}$ of the normal bundle. The second part is devoted to proving the reverse implication and to giving an explicit example of a closed embedding $\mathsf{X} \hookrightarrow \mathsf{Y}$ where condition $(*)$ is not fulfilled. We also discuss the relationship of our class α_N with a natural L_∞-coalgebra structure arising on the cotangent complex.

0.17. Throughout the paper we work in slightly greater generality than that stated in this introduction. The subscheme X is only assumed to be a local complete intersection, and is not required to be smooth. All schemes are assumed to be over a field $\mathbf{k}$ of characteristic zero or greater than $\operatorname{codim}\mathsf{X}$.

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1. The first implication

In this section we prove the "if" part of Theorem 0.7. We also discuss how the resulting isomorphism depends on the choice of lifting $\overline{\mathsf{N}}$ of N.

1.1. Let $i : X \hookrightarrow Y$ be an lci closed embedding over a field $\mathbf{k}$ which is assumed to be of characteristic zero or greater than the codimension of i. We are interested in understanding the object $i^*i_*\mathcal{O}_X$ of $\mathbf{D}(X)$.

The first observation is that $i^*i_*\mathcal{O}_X$ is a commutative algebra object in $\mathbf{D}(X)$. Indeed the map from the derived tensor product to the underived one gives a morphism in $\mathbf{D}(Y)$

$$i_*\mathcal{O}_X \otimes i_*\mathcal{O}_X \to i_*\mathcal{O}_X$$

which, together with the canonical map $\mathcal{O}_Y \to i_*\mathcal{O}_X$, makes $i_*\mathcal{O}_X$ into a commutative algebra object of $\mathbf{D}(Y)$. (More abstractly one can argue that $i_*\mathcal{O}_X = i_*i^*\mathcal{O}_Y$ is an algebra object by noting that i_*i^* is a monad on $\mathbf{D}(Y)$, see [18, 8.6.1], and using the projection formula.) Since i^* is monoidal, the object $i^*i_*\mathcal{O}_X$ of $\mathbf{D}(X)$ inherits a commutative algebra structure from the same structure on $i_*\mathcal{O}_X$.

1.2. We pause for a moment to discuss the distinction between working in the derived category or at dg-level. While we phrase all our results below at the derived level, the actual multiplication maps in all our algebra objects arise as explicit associative maps on complexes. Thus our algebra objects can be regarded as dg-algebras rather than algebras in the derived category. We stick to using the derived category notation for ease of exposition.

1.3. The lci assumption on i implies that the conormal bundle $N^\vee_{X/Y}$ of X in Y is a vector bundle on X. Throughout this section we shall denote this vector bundle by E for simplicity.

A local computation analogous to the one in [4, Appendix A] shows that the cohomology sheaves of $i^*i_*\mathcal{O}_X$ are the same as those of the symmetric algebra $\mathbb{S}(E[1])$,

$$\mathbb{S}(E[1]) = \bigoplus_k \wedge^k E[k].$$

1.4. Theorem. *Let X' denote the first infinitesimal neighborhood of X in Y. Assume that there exists a vector bundle F on X' whose restriction to X is isomorphic to the conormal bundle E of X in Y. Then there exists an isomorphism I of commutative algebra objects in $\mathbf{D}(X)$*

$$\mathrm{I} : i^*i_*\mathcal{O}_X \cong \mathbb{S}(E[1]).$$

The isomorphism I may depend on the choice of the bundle F.

1.5. The proof of Theorem 1.4 is divided into two parts. In the first part, using the existence of the vector bundle F we construct a global morphism of algebra objects

$$I : i^* i_* \mathscr{O}_X \to \mathbb{S}(E[1]).$$

Determining whether a morphism in $\mathbf{D}(X)$ is an isomorphism is a local question (Lemma 1.12 below). In the second part of the proof we describe the restriction of the morphism I to a sufficiently small open set, and we argue that this restriction is an isomorphism.

1.6. The crux of the proof lies in a computation which is familiar in the category of vector spaces, but which makes sense equally well in $\mathbf{D}(X)$. For the sake of clarity of exposition we shall present it in the category of vector spaces and we leave it to the reader to fill in the details for $\mathbf{D}(X)$.

1.7. Let V be a finite dimensional vector space. The graded vector space $\oplus_{k\geq 0} V^{\otimes k}$ can be endowed with two distinct algebra structures, the free algebra $\mathbb{T}(V)$ and the shuffle product structure on the free coalgebra $\mathbb{T}^c(V)$. The product in $\mathbb{T}(V)$ is the usual, non-commutative product,

$$(v_1|\cdots|v_p)\cdot(v_{p+1}|\cdots|v_{p+q}) = v_1|\cdots|v_{p+q},$$

while multiplication in $\mathbb{T}^c(V)$ is commutative, given by the shuffle product [18, 6.5.11]

$$(v_1|\cdots|v_p)\cdot(v_{p+1}|\cdots|v_{p+q}) = \sum_{\sigma \text{ is a } p-q-\text{shuffle}} v_{\sigma_1}|\cdots|v_{\sigma_{p+q}}.$$

(We have used a vertical bar to denote tensor products between elements in V.)

The symmetric algebra $\mathbb{S}(V)$ is naturally a subalgebra of $\mathbb{T}^c(V)$ and a quotient algebra of $\mathbb{T}(V)$. The inclusion and quotient maps are given by

$$\mathbb{S}(V) \to \mathbb{T}^c(V) \quad v_1 v_2 \cdots v_k \mapsto \sum_{\sigma\in\Sigma_k} v_{\sigma_1}|v_{\sigma_2}|\cdots|v_{\sigma_k},$$

and

$$\mathbb{T}(V) \to \mathbb{S}(V) \quad v_1|v_2|\cdots|v_k \mapsto v_1 v_2 \cdots v_k.$$

1.8. Define the exponential map to be the isomorphism $\mathsf{exp} : \mathbb{T}^c(V) \xrightarrow{\sim} \mathbb{T}(V)$ which multiplies by $1/k!$ on $V^{\otimes k}$. (It is here that we need to assume that $\mathsf{char}(\mathbf{k}) = 0$.) The heart of the proof of Theorem 1.4 is the observation, which can be confirmed by a quick calculation, that the composition

$$\mathbb{T}^c(V) \xrightarrow{\mathsf{exp}} \mathbb{T}(V) \to \mathbb{S}(V)$$

is a commutative ring morphism splitting the natural map $\mathbb{S}(V) \to \mathbb{T}^c(V)$, i.e., the composition

$$\mathbb{S}(V) \to \mathbb{T}^c(V) \xrightarrow{\mathsf{exp}} \mathbb{T}(V) \to \mathbb{S}(V)$$

is the identity.

1.9. We now return to the problem of understanding $i^*i_*\mathscr{O}_X$. Begin by fixing notation. Let $\mathscr{I}$ denote the ideal sheaf of X in Y, and let X' be the first infinitesimal neighborhood of X in Y, i.e., the subscheme of Y cut out by $\mathscr{I}^2$. The map i then factors into the closed embeddings

$$X \xrightarrow{f} X' \xrightarrow{g} Y.$$

The coherent sheaf $\mathscr{I}/\mathscr{I}^2$, regarded as a sheaf on X, is the conormal bundle $E := N^\vee_{X/Y}$. Fix a vector bundle F on X' together with an isomorphism $F|_X \cong E$. The exterior powers of E will be denoted by $E^k := \wedge^k E$, and similarly for F.

The following proposition shows that the free coalgebra arises naturally in the context of studying $i^*i_*\mathscr{O}_X$.

1.10. Proposition. *A choice of bundle F and isomorphism $F|_X \cong E$ determines an isomorphism of commutative algebras*

$$f^*f_*\mathscr{O}_X \xrightarrow{\sim} \mathbb{T}^c(E[1]),$$

where $\mathbb{T}^c(E[1])$ denotes the free coalgebra on $E[1]$, i.e., the object

$$\mathbb{T}^c(E[1]) = \bigoplus_{k\geq 0} E^{\otimes k}[k]$$

of $\mathbf{D}(X)$, endowed with the shuffle product.

Proof. In order to compute $f^*f_*\mathscr{O}_X$ we need to construct an explicit resolution $K^{\otimes}_{\bullet}$ of $f_*\mathscr{O}_X$ on X'. Regarding the short exact sequence of sheaves on Y

$$0 \to \mathscr{I}/\mathscr{I}^2 \to \mathscr{O}_Y/\mathscr{I}^2 \to \mathscr{O}_Y/\mathscr{I} \to 0$$

as a sequence of sheaves on X' we get

$$0 \to f_* E \to \mathcal{O}_{X'} \to f_* \mathcal{O}_X \to 0.$$

Consider the composite morphism of sheaves on X'

$$F \to f_*(F|_X) \cong f_* E \to \mathcal{O}_{X'},$$

whose cokernel is $f_* \mathcal{O}_X$. Its dual is a section s of $F^\vee$ which vanishes precisely along X, and the morphism above can be regarded as the operation $\lrcorner s$ of contracting s.

The complex $K_\bullet^\otimes$ of sheaves on X' can be written as follows

$$K_\bullet^\otimes : \quad \cdots \xrightarrow{\lrcorner s} F^{\otimes k} \xrightarrow{\lrcorner s} F^{\otimes (k-1)} \xrightarrow{\lrcorner s} \cdots \xrightarrow{\lrcorner s} \mathcal{O}_{X'} \to 0,$$

where the maps $\lrcorner s$ are given by

$$\lrcorner s : F^{\otimes k} \to F^{\otimes (k-1)} \qquad (\lrcorner s)(f_1 | \cdots | f_k) = s(f_1) \cdot f_2 | \cdots | f_k.$$

Since the kernel of the original map $\lrcorner s : F \to \mathcal{O}_{X'}$ is $f_* E$, the kernel of the differential at the k-th step in the above complex is immediately seen to be

$$f_* E \otimes F^{\otimes (k-1)} \cong f_* E^{\otimes k}.$$

It is a straightforward exercise to verify that $K_\bullet^\otimes$ is exact everywhere except at the last step, where the cokernel is $f_* \mathcal{O}_X$. Thus $K_\bullet^\otimes$ is a resolution of $f_* \mathcal{O}_X$ on X'.

Having a resolution of $f_* \mathcal{O}_X$ on X' allows us to compute $f^* f_* \mathcal{O}_X$, thus yielding an isomorphism

$$I' : f^* f_* \mathcal{O}_X \cong \bigoplus_k E^{\otimes k}[k] = \mathbb{T}^c(E[1]),$$

as objects of $\mathbf{D}(X)$. Indeed, since s vanishes along X, all the differentials in the restriction of $K_\bullet^\otimes$ to X vanish. Since $F^{\otimes k}|_X \cong E^{\otimes k}$, we get the above isomorphism. To finish the proof of Proposition 1.10 we only need to check that I' respects the algebra operations on the two sides.

For $k \geq 0$ consider the map of vector bundles

$$\star_k : \bigoplus_{p+q=k} F^{\otimes p} \otimes F^{\otimes q} \to F^{\otimes k}$$

given by the formula of the shuffle product defined earlier. It is straightforward to check that the collection of maps $\star_\bullet$ is a chain map of complexes

$K^{\otimes}_{\bullet} \otimes K^{\otimes}_{\bullet} \to K^{\otimes}_{\bullet}$ which represents the natural map $f_*\mathscr{O}_X \otimes f_*\mathscr{O}_X \to f_*\mathscr{O}_X$ in $\mathbf{D}(X)$. Indeed, we only need to check the equality of the two compositions in the diagram below:

$$\begin{array}{ccc}
(x_1|\cdots|x_p) \otimes (x_{p+1}|\cdots|x_{p+q}) & \overset{d}{\longmapsto} & s(x_1)\cdot(x_2|\cdots|x_p) \otimes (x_{p+1}|\cdots|x_{p+q}) \\
 & & +(x_1|\cdots|x_p) \otimes s(x_{p+1})\cdot(x_{p+2}|\cdots|x_{p+q}) \\
\Big\downarrow \star & & \Big\downarrow \star \\
\sum_{\sigma \text{ is a } p-q-\text{shuffle}} x_{\sigma_1}|\cdots|x_{\sigma_{p+q}} & \overset{d}{\longmapsto} & \sum_{\sigma \text{ is a } p-q-\text{shuffle}} s(x_{\sigma_1})\cdot x_{\sigma_2}|\cdots|x_{\sigma_{p+q}}.
\end{array}$$

(We have written the maps without signs for simplicity, but in fact all the morphisms have signs in them, obtained by the rule that says that the x_i's behave like odd elements in a graded vector space.)

Having represented the multiplication map $f_*\mathscr{O}_X \otimes f_*\mathscr{O}_X \to f_*\mathscr{O}_X$ by the chain map $\star_\bullet$, the product map of the algebra $f^*f_*\mathscr{O}_X$ is obtained by applying the functor f^* to $\star_\bullet$. It is obvious from the definition of $\star_\bullet$ that the induced map under the isomorphism

$$f^*f_*\mathscr{O}_X \cong \bigoplus_k E^{\otimes k}$$

is given precisely by the shuffle product, thus I' is an algebra map. $\square$

1.11. Our candidate map for the isomorphism

$$I : i^*i_*\mathscr{O}_X \to \mathbb{S}(E[1])$$

is the composite map

$$i^*i_*\mathscr{O}_X = f^*g^*g_*f_*\mathscr{O}_X \to f^*f_*\mathscr{O}_X \xrightarrow{\sim} \mathbb{T}^c(E[1]) \xrightarrow{\exp} \mathbb{T}(E[1]) \to \mathbb{S}(E[1]),$$

where the first morphism is the counit of the adjunction $g^* \dashv g_*$, the middle one is multiplying by $1/k!$ on $E^{\otimes k}[k]$, and the last one is the natural projection map. It is obvious that I is an algebra map: the counit map is an algebra map by standard facts about monads, and the composition $\mathbb{T}^c(E[1]) \xrightarrow{\exp} \mathbb{T}(E[1]) \to \mathbb{S}(E[1])$ is a homomorphism by a calculation entirely analogous to the one for vector spaces in (1.7). Thus in order to complete the proof of Theorem 1.4 we only need to argue that I is an isomorphism in $\mathbf{D}(X)$. This can be checked locally, as the following lemma shows.

1.12. Lemma. *Let* X *be a scheme, and let* $f : A \to B$ *be a morphism of objects in* $\mathbf{D}(X)$*. Then* f *is an isomorphism if and only if* $f|_U : A|_U \to B|_U$ *is an isomorphism in* $\mathbf{D}(U)$ *for all open sets* U *in a covering of* X*.*

Proof. The map f is an isomorphism if and only if the induced maps $H^\bullet(f)$ on cohomology sheaves $H^\bullet(A) \to H^\bullet(B)$ are isomorphisms. Since checking that a map of sheaves is an isomorphism is a local question, and the maps on cohomology commute with restriction, the result follows. □

1.13. In order to complete the proof of Theorem 1.4 we shall compare two resolutions of $\mathscr{O}_X$, one on Y and one on X'. If X were cut globally out of Y by a section $\bar{s}$ of a vector bundle $\overline{F}$, then the Koszul resolution of the section $\bar{s}$ would be a resolution of $\mathscr{O}_X$ on Y. This may not be the case globally, but the following lemma shows that such a vector bundle and section always exist locally.

1.14. Lemma. *Let* $(R, \mathfrak{m})$ *be a regular local ring and let* J *be an ideal in* R *of height* $k \geq 1$*. Let* $f_1, \ldots, f_k$ *be elements of* R *whose reduction modulo* J^2 *generate the* R/J^2*-module* J/J^2*. Then* $f_1, \ldots, f_k$ *are a regular sequence for the* R*-module* J*.*

Proof. The elements $f_1 \mod J^2, \ldots, f_k \mod J^2$ generate J/J^2, hence their reductions mod $\mathfrak{m}$ generate the $R/\mathfrak{m}$ vector space $(J/J^2)/\mathfrak{m} = J/\mathfrak{m}$. By Nakayama's lemma, $f_1, \ldots f_k$ generate J. Since J was assumed to be of height k, the result follows by Krull's Hauptidealsatz. □

1.15. We want to show that the restrictions of the map I to small enough open sets in X are quasi-isomorphisms. The previous lemma shows that by replacing Y with a sufficiently small neighborhood around any of its points we can assume that there exists a vector bundle $\overline{F}$ on Y and a section $\bar{s}$ of $\overline{F}$ such that the restriction of $\overline{F}$ to X' is isomorphic to F, and the section $\bar{s}$ maps to s under this isomorphism. Consider the Koszul resolution

$$K_\bullet^\wedge : \quad 0 \to \overline{F}^k \xrightarrow{\lrcorner\bar{s}} \overline{F}^{k-1} \xrightarrow{\lrcorner\bar{s}} \cdots \xrightarrow{\lrcorner\bar{s}} \overline{F} \xrightarrow{\lrcorner\bar{s}} \mathscr{O}_Y \to 0,$$

where we have denoted by $\overline{F}^k$ the vector bundle $\wedge^k \overline{F}$ on Y. The differentials in $K_\bullet^\wedge$ can be written down explicitly, in a similar way we did for $K_\bullet^\otimes$:

$$\lrcorner\bar{s} : \wedge^k \overline{F} \to \wedge^{k-1}\overline{F} \qquad (\lrcorner\bar{s})(f_1 \wedge \cdots \wedge f_k) = \sum_{i=1}^{k} (-1)^i \bar{s}(v_i) v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_k.$$

Restricting $K_\bullet^\wedge$ to X' yields a non-exact complex, representing $g^* g_* f_* \mathscr{O}_X$. The adjunction $g^* \dashv g_*$ yields the counit map

$$g^* g_* f_* \mathscr{O}_X \to f_* \mathscr{O}_X.$$

Since $K^{\otimes}_{\bullet}$ is a resolution of $f_*\mathscr{O}_X$ on X' it is reasonable to search for a map of complexes

$$K^{\wedge}_{\bullet}|_{X'} \to K^{\otimes}_{\bullet}.$$

A straightforward computation shows that the following diagram of bundles on X' commutes for any $k \geq 0$

$$\begin{array}{ccc} F^k & \xrightarrow{\lrcorner \bar{s}} & F^{k-1} \\ \downarrow \epsilon & & \downarrow \epsilon \\ F^{\otimes k} & \xrightarrow{\lrcorner s} & F^{\otimes (k-1)}, \end{array}$$

where the top map is the restriction of the map $\lrcorner \bar{s}$ to X', and the vertical maps are the symmetrizations maps defined below:

$$\epsilon(v_1 \wedge v_2 \wedge \cdots \wedge v_k) = \sum_{\sigma \in \Sigma_k} (-1)^{\epsilon(\sigma)} v_{\sigma_1} | v_{\sigma_2} | \cdots | v_{\sigma_k}.$$

(This map is the analogue of the map $\mathbb{S}(V) \to \mathbb{T}^c(V)$ discussed earlier, with signs included to take into account the fact that $F[1]$ in the symmetric monoidal category $\mathbf{D}(X)$ behaves like an odd vector space in the category of $\mathbf{Z}_2$-graded vector spaces.) Thus we get a map of complexes $\epsilon_\bullet$ from the restriction of the Koszul resolution to X' to the second resolution, and it is immediate to check that it represents the counit map above.

Restricting further the map $\epsilon_\bullet : K^{\wedge}_{\bullet} \to K^{\otimes}_{\bullet}$ to X by pulling-back via f, all the differentials in both complexes vanish. Therefore we conclude that the natural map

$$i^* i_* \mathscr{O}_X \to f^* f_* \mathscr{O}_X$$

arising by contracting $g^* g_*$ is given by the map of complexes $\epsilon : \mathbb{S}(E[1]) \to \mathbb{T}^c(E[1])$. Thus composing the map $I : i^* i_* \mathscr{O}_X \to \mathbb{S}(E[1])$ with the (local) isomorphism $\mathbb{S}(E[1]) \xrightarrow{\sim} i^* i_* \mathscr{O}_X$ obtained from the restriction of the Koszul resolution gives the composition

$$\mathbb{S}(E[1]) \to \mathbb{T}^c(E[1]) \xrightarrow{\exp} \mathbb{T}(E[1]) \to \mathbb{S}(E[1])$$

which we know is an isomorphism. Therefore the maps induced by I are locally isomorphisms on cohomology sheaves, and thus I is globally a quasi-isomorphism. This concludes the proof of Theorem 1.4. □

1.16. There are two clear situations in which we can guarantee the existence of a vector bundle F on the first infinitesimal neighborhood X' extending the conormal bundle $\mathsf{E} = \mathsf{N}^{\vee}_{\mathsf{X}/\mathsf{Y}}$. One is when X is a global complete intersection in Y, that is there exists a vector bundle V on Y of rank $\mathsf{codim}(\mathsf{X}/\mathsf{Y})$ and a section of V whose vanishing locus is X scheme-theoretically. In this case $\mathsf{V}^{\vee}|_{\mathsf{X}} = \mathsf{N}^{\vee}$, and hence $\mathsf{F} = \mathsf{V}^{\vee}|_{\mathsf{X}'}$ extends E.

1.17. A more interesting situation is when the sequence

$$0 \to \mathsf{N}^{\vee} \to \Omega^1_{\mathsf{Y}}|_{\mathsf{X}} \to \Omega_{\mathsf{X}} \to 0$$

is split. Then the class η which appears in the definition of the obstruction α_{V} in (0.6) vanishes, and hence any object of $\mathbf{D}(\mathsf{X})$ extends to X'. This can be understood geometrically by noting that if η vanishes then there is a map $\pi : \mathsf{X}' \to \mathsf{X}$ splitting the natural inclusion $\mathsf{X} \hookrightarrow \mathsf{X}'$, see [8, 20.5.12 (iv)]. Then any $\mathsf{V} \in \mathbf{D}(\mathsf{X})$ extends to the object $\pi^*\mathsf{V} \in \mathbf{D}(\mathsf{X}')$ whose restriction to X is V.

1.18. The above sequence is split in two situations of interest. One is when $i : \mathsf{X} \hookrightarrow \mathsf{Y}$ is split by a morphism $\pi : \mathsf{Y} \to \mathsf{X}$. Such is the case for the diagonal embedding $\mathsf{X} \to \mathsf{X} \times \mathsf{X}$ which is split by either projection on the factors, or for the zero section of a bundle map $\mathsf{E} \to \mathsf{X}$. Another situation where the tangent sequence splits is when there is a finite group G of order not divisible by $\mathsf{char}\,\mathbf{k}$ which acts on Y, and X is the fixed locus Y^{G}. Then the dual map

$$\mathsf{T}_{\mathsf{X}} \to \mathsf{T}_{\mathsf{Y}}|_{\mathsf{X}}$$

is split by the map that takes a tangent vector ν to Y at a point of X to

$$\frac{1}{\mathsf{ord}(\mathsf{G})} \sum_{g \in \mathsf{G}} g \cdot \nu.$$

Since the diagonal is the fixed locus of the involution which interchanges the two factors, this gives rise to another HKR-type isomorphism through the corresponding splitting of the tangent sequence.

1.19. An interesting question that arises in the study of the diagonal embedding is whether the two HKR isomorphisms

$$\mathsf{HKR}_1, \mathsf{HKR}_2 : \Delta^*\Delta_*\mathscr{O}_{\mathsf{X}} \cong \mathbb{S}(\Omega^1_{\mathsf{X}}[1])$$

induced by the splittings π_1, π_2 of Δ are different. While we are unable to answer this question at the moment, we give strong evidence that these splittings induce the *same* HKR morphism.

1.20. By adjunction the two isomorphisms HKR_1, HKR_2 can be regarded as morphisms

$$\mathscr{O}_\Delta \to \Delta_* \mathbb{S}(\Omega^1_X[1]),$$

which in turn can be regarded as natural transformations f and g between the identity functor of $\mathbf{D}(X)$ and the functor $- \otimes \mathbb{S}(\Omega^1_X[1])$. Instead of arguing that the maps on the level of kernels agree, we'll prove the weaker statement that the induced natural transformations are the same.

1.21. The natural transformations f and g give, for $E \in \mathbf{D}(X)$, maps

$$E \to E \otimes \mathbb{S}(\Omega^1_X[1]) = \bigoplus_i E \otimes \Omega^i_X[i],$$

which, for simplicity, we denote by f and g as well.

The components of these maps can easily be understood (see also [3] for more details). Explicitly, if f is obtained using the second projection to split Δ, then the component $f_i : E \to E \otimes \Omega^i_X[i]$ of f is

$$f_i = \epsilon \circ (\mathsf{id}_{\Omega_X^{\otimes(i-1)}[i-1]} \otimes \mathsf{at}_E) \circ (\mathsf{id}_{\Omega_X^{\otimes(i-2)}[i-2]} \otimes \mathsf{at}_E) \circ \cdots \circ \mathsf{at}_E,$$

where ϵ is the antisymmetrization map $\Omega_X^{\otimes i} \to \Omega^i_X$. Similarly, if g corresponds to the first projection, then g_i is given by

$$g_i = \epsilon \circ \mathsf{at}_{E\otimes\Omega_X^{\otimes(i-1)}[i-1]} \circ \mathsf{at}_{E\otimes\Omega_X^{\otimes(i-2)}[i-2]} \circ \cdots \circ \mathsf{at}_E.$$

The fact that these two maps are the same follows easily from the fact that the Atiyah class gives rise to a Lie coalgebra structure on $\Omega^1_X[1]$, which makes E into a Lie comodule.

1.22. It is obvious that $f_i = g_i$ for $i = 0, 1$. We exemplify the calculation that $f_2 = g_2$, and leave the details that $f_i = g_i$ for $i \geq 3$ to the reader. Denoting by $\mathfrak{g} = T_X[-1]$, the dual of $\Omega^1_X[1]$, and using $E^\vee$ instead of E, the duals of the maps f_2 and g_2 are the maps

$$\mathfrak{g} \otimes \mathfrak{g} \otimes E \longrightarrow \mathfrak{g} \otimes E \longrightarrow E$$

given by, respectively,

$$x \otimes y \otimes e \mapsto x \otimes (y \cdot e) \mapsto x \cdot (y \cdot e)$$

and

$$\begin{aligned} x \otimes y \otimes e \mapsto x \cdot (y \otimes e) &= [x, y] \otimes e + y \otimes (x \cdot e) \\ &\mapsto [x, y] \cdot e + y \cdot (x \cdot e). \end{aligned}$$

(We have written the above maps in component notation for clarity, but these maps make sense in an arbitrary symmetric monoidal category, like $\mathbf{D}(X)$.) Note the the equality

$$x \cdot (y \otimes e) = [x, y] \otimes e + y \otimes (x \cdot e)$$

is nothing but the way $\mathfrak{g}$ acts on the tensor product of representations $\mathfrak{g} \otimes E$. In other words the above map $\mathfrak{g} \otimes \mathfrak{g} \otimes E \to \mathfrak{g} \otimes E$ is the dual of $\mathsf{at}_{\Omega^1_X[1] \otimes E}$.

The equality $f_2 = g_2$ follows now from the fact that E is a representation of $\mathfrak{g}$ in $\mathbf{D}(X)$. See [15] for more details.

2. The reverse implication

In this section we argue that if $i^* i_* \mathscr{O}_X$ is formal (isomorphic to $\mathbb{S}(N^*[1])$), then the obstruction class α_N from (0.6) vanishes. We then give an explicit example of a closed embedding of smooth varieties for which α_N does not vanish. In a final part of this section we discuss the relationship of the class α_N to an L_∞-coalgebra structure on the relative cotangent complex.

2.1. As before let $i : X \hookrightarrow Y$ be a closed lci embedding with conormal bundle $E = N^\vee_{X/Y}$ and let V be a vector bundle on X. Consider the truncation $\tau^{\geq -1}(i^* i_* V)$. It has only two non-trivial cohomology sheaves, H^0 and H^{-1}, which are naturally isomorphic to V and $V \otimes E$, respectively. Therefore $\tau^{\geq -1}(i^* i_* V)$ fits into a triangle

$$V \otimes E[1] \to \tau^{\geq -1} i^* i_* V \to V \to V \otimes E[2].$$

2.2. Definition. *The rightmost map* $V \to V \otimes E[2]$ *in the above triangle will be called the* HKR class *of* V, *denoted by* $\alpha_V \in \mathsf{Ext}^2(V, V \otimes E)$.

2.3. *A priori* the notation in the above definition seems ambiguous, as we have already defined α_V using a different formula in (0.6). However, the bulk of this section is devoted to proving the following two facts:

a) the HKR class α_V equals $\mathsf{at}_V \circ \eta$, the previously defined α_V;

b) α_V is the obstruction to lifting V to X', the first infinitesimal neighborhood of X in Y.

2.4. It is easy to see that the HKR class depends only on the embedding $f: X \to X'$ of X into its first infinitesimal neighborhood. Indeed, the truncation of the natural map $i^*i_*V \to f^*f_*V$ is an isomorphism

$$\tau^{\geq -1} i^* i_* V \xrightarrow{\sim} \tau^{\geq -1} f^* f_* V.$$

The claim that the HKR class α_V is the obstruction to extending V to a vector bundle $\overline{V}$ on X' is naturally formulated in the language of gerbes: locally on X an extension of V always exists, and the local extensions form a gerbe over X for the sheaf $\underline{\mathsf{Hom}}(V, V \otimes E)$. The claim is that α_V is the class of this gerbe.

2.5. Let us be more precise about these definitions. For an open subset $U \subset X$ let $U' \subset X'$ be its first infinitesimal neighborhood. Denote by $\mathscr{E}\mathscr{G}(U)$ the category of extensions of $V|_U$ to a vector bundle on U'. Thus objects of $\mathscr{E}\mathscr{G}(U)$ are vector bundles $\overline{V}_U$ on U' equipped with isomorphisms $(\overline{V}_U)|_U \cong V|_U$. (To simplify notation we will usually omit this isomorphism.) Clearly $\mathscr{E}\mathscr{G}(U)$ is a groupoid. As U varies the groupoids $\mathscr{E}\mathscr{G}(U)$ form a sheaf of groupoids $\mathscr{E}\mathscr{G}$.

If U is small enough an extension $\overline{V}_U$ exists; moreover, any two extensions are locally isomorphic. Finally the automorphism group of an extension $\overline{V}_U$ equals $\mathsf{Hom}(V|_U, (V \otimes E)|_U)$. In other words $\mathscr{E}\mathscr{G}$ is a gerbe over $\underline{\mathsf{Hom}}(V, V \otimes E)$.

2.6. Proposition. *The class*

$$[\mathscr{E}\mathscr{G}] \in H^2(X, \underline{\mathsf{Hom}}(V, V \otimes E)) = \mathsf{Ext}^2(V, V \otimes E)$$

equals α_V as defined in (2.2).

Proof. The statement is probably well known; since we were unable to locate a reference, we include a detailed proof.

Let $C^\bullet = C^{-1} \xrightarrow{d} C^0$ be a two-term complex of $\mathscr{O}_X$-modules with cohomology sheaves H^0, H^{-1}, such that H^0 is locally free. Then $C^\bullet$ gives rise to a gerbe $\mathscr{G}$ over $\underline{\mathsf{Hom}}(H^0, H^{-1})$ as follows. For every open set U there is an associated groupoid $\mathscr{G}(U)$. Its objects are elements φ of $\mathsf{Hom}(H^0|_U, C^0|_U)$ that lift the identity automorphism of $H^0|_U$; equivalently φ is a section of $C^0 \to H^0$ over U. Morphisms between between $\varphi, \varphi' \in \mathsf{Hom}(H^0|_U, C^0|_U)$ are elements $\psi \in \mathsf{Hom}(H^0|_U, C^{-1}|_U)$ such that $d\psi = \varphi' - \varphi$. As U varies the groupoids $\mathscr{G}(U)$ naturally form a gerbe over $\underline{\mathsf{Hom}}(H^0, H^{-1})$ whose class

$$[\mathscr{G}] \in \mathsf{Hom}(H^0, H^{-1}[2])$$

is precisely the rightmost map in the triangle

$$H^{-1}[1] \to C^\bullet \to H^0 \to H^{-1}[2].$$

Essentially $\mathscr{G}$ is the gerbe of splittings of the complex $C^\bullet$.

Now we will pick a specific two-term complex $C^\bullet$ which will represent $\tau^{\geq -1}(i^* i_* V)$. Choose a truncated resolution of $i_* V$ of the form

$$0 \to F^{-1} \to F^0 \to i_* V \to 0,$$

where F^i are $\mathscr{O}_Y$-modules (not necessarily quasi-coherent) and F^0 is flat. Then the restriction

$$F^\bullet|_X = (F^{-1}|_X \to F^0|_X)$$

represents the object $\tau^{\geq -1}(i^* i_* V)$. Note that F^{-1} is not flat, and $F^{-1}|_X$ refers to the naive (non-derived) restriction. Let $\mathscr{G}$ be the gerbe of splittings of the complex $F^\bullet|_X$.

It remains to construct a morphism $W : \mathscr{G} \to \mathscr{E}\mathscr{G}$ of $\underline{\mathrm{Hom}}(V, V \otimes E)$-gerbes. To simplify the notation, we will only construct it for global sections. On objects, a morphism $\varphi : V \to F^0$ defines an extension $W_\varphi \in \mathscr{E}\mathscr{G}(X)$ by

$$W_\varphi = \{s \in F^0 : s|_X \in \varphi(V)\}/\mathscr{I}_X dF^{-1},$$

where $\mathscr{I}_X \subset \mathscr{O}_Y$ is the ideal sheaf of X. Given two morphisms $\varphi, \varphi' : V \to F^0$ and $\psi : V \to F^{-1}$ such that $d\psi = \varphi' - \varphi$, the corresponding map $W_\varphi \to W_{\varphi'}$ sends $s \in W_\varphi$ to $s + d\psi([s])$ where $[s]$ is the image of $s \in F^0$ in V. It is easy to see that we obtain a morphism of gerbes in this way. Thus $\alpha_V = [\mathscr{G}] = [\mathscr{E}\mathscr{G}]$. □

2.7. We will also be interested in the following modification of the above extension problem. Let $\tilde{V}$ be a vector bundle (or a quasi-coherent sheaf) on X, equipped with a morphism $m : V \otimes E \to \tilde{V}$. We can then consider the problem of constructing exact sequences

$$0 \to i_* \tilde{V} \to \overline{V} \to i_* V \to 0$$

on Y such that the sheaf of ideals $\mathscr{I}_X \subset \mathscr{O}_Y$ acts on $\overline{V}$ as

$$\mathscr{I}_X \otimes \overline{V} \to \mathscr{I}_X \otimes i_* V = i_*(V \otimes E) \xrightarrow{i_* m} i_* \tilde{V} \to \overline{V}.$$

It is easy to see that in this case the obstruction is a class

$$\alpha_{V,m} \in \mathrm{Ext}^2(V, \tilde{V})$$

that is functorial in the pair $(\tilde{V}, m)$. Therefore $\alpha_{V,m} = m \circ \alpha_V$.

2.8. Corollary. *An extension*

$$0 \to i_* \tilde{V} \to \overline{V} \to i_* V \to 0$$

as above exists if and only if $m \circ \alpha_V$ *vanishes.*

2.9. We would now like to relate the class α_V to $\eta \circ \mathsf{at}_V$, which was our original definition of α_V. Let us assume that X and Y are both smooth. This assumption persists through 2.12. It allows us to use the classical Atiyah class instead of its derived counterpart constructed using the cotangent complex. See [9] for a more general statement without this assumption.

As before V is a vector bundle on X. Let us describe the class $\eta \circ \mathsf{at}_V$ explicitly. The Atiyah sequence of V is the exact sequence of coherent sheaves

$$0 \to V \otimes \Omega^1_X \to J^1(V) \to V \to 0,$$

where $J^1(V)$ is the first jet bundle of V. The corresponding extension class $\mathsf{at}_V \in \mathsf{Ext}^1(V, V \otimes \Omega^1_X)$ is called the Atiyah class of V. Composing with the class $\eta \in \mathsf{Ext}^1(\Omega^1_X, E)$ of the short exact sequence

$$0 \to E \to \Omega^1_Y|_X \to \Omega^1_X \to 0$$

we get a morphism

$$\eta \circ \mathsf{at}_V \in \mathsf{Hom}(V, V \otimes E[2]) = \mathsf{Ext}^2(V, V \otimes E).$$

This composite morphism can be interpreted as the obstruction to fitting the two exact sequences above into a diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V \otimes E & \longrightarrow & V \otimes i^*\Omega^1_Y & \longrightarrow & V \otimes \Omega^1_X & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V \otimes E & \longrightarrow & J' & \longrightarrow & J^1(V) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & V & = & V & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

In other words J' is a vector bundle on X equipped with a filtration

$$0 = J'_0 \subset J'_1 \subset J'_2 \subset J'_3 = J'$$

such that

$$J'_1 = V \otimes E \qquad J'_2 = V \otimes i^*\Omega^1_Y$$
$$J'_2/J'_1 = V \otimes \Omega_X \qquad J'_3/J'_1 = J^1(V)$$
$$J'_3/J'_2 = V.$$

2.10. Proposition. $\alpha_V = \eta \circ \mathsf{at}_V$.

Proof. This is Remark II.6.10(b) in [13] (see also Theorem II.6.6), and a particular case of Corollary 3.4 in [9]. Let us sketch the argument.

Let us interpret α_V as the obstruction to extending V to a vector bundle V' on X'. Such an extension V' can be viewed as a sheaf g_*V' on Y. Considering its Atiyah extension

$$0 \to g_*V' \otimes \Omega^1_Y \to J^1(g_*V') \to g^*V' \to 0$$

it is easy to see that the restriction $J' = i^*(J^1(g_*V'))$ fits into the diagram of (2.9). In other words, a solution to the problem obstructed by α_V yields a solution to the problem obstructed by $\eta \circ \mathsf{at}_V$.

Looking at local solutions, we obtain a map between the corresponding two gerbes (the gerbes of local solutions to the two problems). Since the gerbes are over the same sheaf $\underline{\mathsf{Hom}}(V, V \otimes E)$, the map is a 1-isomorphism. Therefore, the classes of the two gerbes coincide. □

2.11. The class $\eta \circ \mathsf{at}_V$ can also be interpreted as follows. Consider the composition

$$i^*\Omega_Y \otimes V \to \Omega_X \otimes V \to J^1(V),$$

and let $J^1_{X/Y}(V)$ be the length two complex

$$i^*\Omega_Y \otimes V \to J^1(V),$$

with $J^1(V)$ located in cohomological degree zero and differential the map above. The cohomology sheaves of $J^1_{X/Y}(V)$ are V and $V \otimes E$ in degrees zero and minus one, respectively. We therefore obtain a triangle

$$V \otimes E[1] \to J^1_{X/Y}(V) \to V \to V \otimes E[2],$$

in which the rightmost map is $\eta \circ \mathsf{at}_V$.

Propositions 2.6 and 2.10 can be interpreted as saying that there is an isomorphism $J^1_{X/Y}(V) \simeq \tau^{\geq -1}(i^*i_*V)$ that relates the triangle above to the triangle of (2.1).

2.12. Remark. The triangle in (2.11) can be viewed as a version of the Atiyah sequence if one views X as a scheme over Y. Here the analogue of the cotangent bundle $\omega^1_X = \mathbb{L}_{X/k}$ in the definition of the usual Atiyah class is taken by the relative cotangent complex $E[1] = N^\vee[1] = \mathbb{L}_{X/Y}$. We thus see that the HKR class α_V is essentially the relative Atiyah class of V viewed as a sheaf on X/Y.

2.13. We now want to concentrate on a special HKR class, namely that of the conormal bundle itself. We return to the original assumptions that Y is smooth, $i : X \hookrightarrow Y$ is a locally complete intersection closed embedding with conormal bundle E. We are interested in the HKR class associated to E,

$$\alpha = \alpha_E \in H^2(X, E^{\otimes 2} \otimes E^\vee).$$

2.14. Proposition. *The class α is skew-symmetric,*

$$\alpha \in H^2(X, \wedge^2 E \otimes E^\vee) \subset H^2(X, E^{\otimes 2} \otimes E^\vee).$$

Proof. We need to check that the image of α in $H^2(X, \mathbb{S}^2(E) \otimes E^\vee)$ vanishes. This image equals $\mathsf{Sym}(\alpha)$, where Sym is the symmetrization map $\mathsf{Sym} : E^{\otimes 2} \to \mathbb{S}^2(E)$. By Corollary 2.8, we need to construct an exact sequence

$$0 \to i_*\mathbb{S}^2 E \to E' \to i_* E \to 0$$

of sheaves on Y such that the action of I_X on E' is given by Sym. Now take $E' = I_X/I_X^3$. □

2.15. We introduce the following notation. For an object $C \in \mathbf{D}(X)$, consider the filtration of C by objects $H^{-k}(C)[k]$. The differential $H^{-k}(C) \to H^{-k-1}(C)[2]$ in the corresponding spectral sequence is denoted by $\delta_k = \delta_k(C)$. Explicitly, δ_k is the shift of the rightmost map in the exact triangle

$$H^{-k-1}(C)[k+1] \to \tau^{\leq -k}\tau^{\geq -k-1} C \to H^{-k}(C)[k] \to H^{-k-1}(C)[k+2].$$

Clearly, $\delta_k(C)$ is functorial in C.

As an example of using this notation the HKR class of a vector bundle V on X is nothing but

$$\alpha_V = \delta_0(i_* i^* V) = \delta_0(f_* f^* V).$$

2.16. It is obvious that if the object C is formal (isomorphic to the direct sum of its cohomology sheaves), then $\delta_k(\mathsf{C}) = 0$ for all k. We will show that the HKR class $\alpha = \alpha_E$ of E is precisely $\delta_1(i^*i_*\mathscr{O}_X)$, thus proving the second implication of Theorem 0.7: if $i^*i_*\mathscr{O}_X$ is formal, then

$$\alpha = \delta_1(i^*i_*\mathscr{O}_X) = 0.$$

Note that both α and $\delta_1(i^*i_*\mathscr{O}_X)$ are maps from $E = H^{-1}(i^*i_*\mathscr{O}_X)$ to the shift by two of $\wedge^2 E = H^{-2}(i^*i_*\mathscr{O}_X)$.

As a side note it is easy to see that the map giving the unit of the algebra structure on $i^*i_*\mathscr{O}_X$ splits the natural projection $i^*i_*\mathscr{O}_X$, so that $i^*i_*\mathscr{O}_X = \mathscr{O}_X \oplus \tau^{<0} i^*i_*\mathscr{O}_X$. Therefore, $\delta_0 = 0$ and δ_1 is the first map that has a chance of not being zero.

2.17. As before we will understand the class $\delta_1(i^*i_*\mathscr{O}_X)$ in two steps, by first studying the behavior of this class with respect to the embedding f of X into X', the first infinitesimal neighborhood of X in Y. Recall that $H^{-k}(f^*f_*\mathscr{O}_X) = E^{\otimes k}$ and we can consider the map

$$\delta_1(f_*f^*\mathscr{O}_X) : E \to E^{\otimes 2}[2].$$

2.18. Lemma. *We have*

$$\delta_1(f_*f^*\mathscr{O}_X) = \alpha;$$

here α is regarded as an element of $H^2(X, \wedge^2 E \otimes E^\vee) \subset H^2(X, E^{\otimes 2} \otimes E^\vee)$.

Proof. Consider the natural exact sequence

$$0 \to f_*E \to \mathscr{O}_{X'} \to f_*\mathscr{O}_X \to 0.$$

Applying f^*, we obtain an exact triangle

$$f^*f_*E \to \mathscr{O}_X \to f^*f_*\mathscr{O}_X \to f^*f_*E[1].$$

The composition

$$\tau^{<0} f^*f_*\mathscr{O}_X \to f^*f_*\mathscr{O}_X \to f^*f_*E[1]$$

is an isomorphism, and therefore

$$\delta_1(f^*f_*\mathscr{O}_X) = \delta_1(\tau^{<0} f^*f_*\mathscr{O}_X) = \delta_1(f^*f_*E[1]) = \delta_0(f^*f_*E) = \alpha_E. \qquad \square$$

2.19. Let $\epsilon : E^{\otimes 2} \to \wedge^2(E)$ be the skew-symmetrization map. By Proposition 2.14, α is already skew-symmetric, so that

$$\alpha = \epsilon(\alpha) \in H^2(X, \wedge^2(E) \otimes E^\vee).$$

We are now ready to prove the main result of this section, which finishes the proof of the second implication of Theorem 0.7.

2.20. Proposition. *For the object* $i^* i_* \mathscr{O}_X \in \mathbf{D}(X)$ *we have*

$$\delta_1(i^* i_* \mathscr{O}_X) = \epsilon(\alpha) = \alpha.$$

Proof. Consider the composition

$$\mathsf{Sym}^2(E)[2] \to E^{\otimes 2}[2] \to H^{-2}(f^* f_* \mathscr{O}_X) \to \tau^{\geq -2}(f^* f_* \mathscr{O}_X),$$

and include it into a triangle

$$\mathsf{Sym}^2(E)[2] \to \tau^{\geq -2}(f^* f_* \mathscr{O}_X) \to C \to \mathsf{Sym}^2(E)[3].$$

By construction,

$$H^{-k}(C) = \begin{cases} \bigwedge^k E & k = 0, 1, 2 \\ 0 & \text{otherwise,} \end{cases}$$

Moreover, the map $H^{-2}(f^* f_* \mathscr{O}_X) \to H^{-2}(C)$ equals ϵ, and so

$$\delta_1(C) = \epsilon(\delta_1(f^* f_* \mathscr{O}_X)) = \epsilon(\alpha).$$

On the other hand, the composition

$$i^* i_* \mathscr{O}_X \to f^* f_* \mathscr{O}_X \to \tau^{\geq -2}(f^* f_* \mathscr{O}_X) \to C$$

induces an isomorphism on cohomology objects in degrees 0, -1, and -2. Therefore, $\delta_1(i^* i_* \mathscr{O}_X) = \delta_1(C) = \alpha$. □

2.21. There is an alternative approach to the HKR class α that we have studied above arising from a canonical Lie coalgebra structure on the shifted cotangent complex $\mathbb{L}_{X/Y}[1]$. This approach is natural if one views α as a version of the Atiyah class (Remark 2.12): in the absolute case, Kapranov [11] showed that the Atiyah class of a smooth manifold X gives a Lie algebra structure on the shifted tangent bundle $T_X[-1]$.

2.22. The coalgebra structure on the shifted cotangent complex is defined for arbitrary morphisms of ringed spaces (or topoi). Let us sketch the construction in this generality.

Recall the definition of the cotangent complex $\mathbb{L}_{X/Y}$ for a morphism $\varphi : X \to Y$ of ringed spaces. We follow the original definition of Illusie [10], which also appears in the works of Ciocan-Fontanine and Kapranov [5], [6].

Consider $\mathscr{O}_X$ as a sheaf of algebras over the sheaf of rings $\varphi^{-1}\mathscr{O}_Y$. There exists a resolution of $\mathscr{O}_X$ of the form

$$(\mathbb{S}(M^\bullet), d) \to \mathscr{O}_X,$$

where $M^\bullet$ is a graded flat $\varphi^{-1}\mathscr{O}_Y$-module concentrated in non-positive degrees, and d is a (degree one) differential on $\mathbb{S}(M^\bullet)$, viewed as a graded $\varphi^{-1}\mathscr{O}_Y$-algebra. Such a resolution can be constructed by the usual iterative procedure ('attaching cells to kill homotopy groups') as in [14, p. 256]. This is the approach used in [5, Theorem 2.6.1] with somewhat different assumptions. (In [5] the authors work with coherent sheaves on quasi-projective schemes, which admit enough locally free sheaves; in our setting there are enough flat sheaves.) Alternatively we can follow [10, I.1.5.5.6] and use the natural 'maximal' resolution of $\mathscr{O}_X$.

Taking the tensor product we obtain a commutative dg-algebra

$$(\mathbb{S}(M^\bullet) \otimes_{\varphi^{-1}\mathscr{O}_Y} \mathscr{O}_X, d)$$

over $\mathscr{O}_X$. In the derived category of $\mathscr{O}_X$-modules this algebra represents the derived tensor product $\mathscr{O}_X \otimes_{\varphi^{-1}\mathscr{O}_Y} \mathscr{O}_X$.

Set $M^\bullet_X = M^\bullet \otimes_{\varphi^{-1}\mathscr{O}_Y} \mathscr{O}_X$. Let us embed $M^\bullet_X$ into $\mathbb{S}(M^\bullet) \otimes_{\varphi^{-1}\mathscr{O}_Y} \mathscr{O}_X$ using the morphism

$$M^\bullet_X \to M^\bullet_X \oplus \mathscr{O}_X = (M^\bullet \otimes_{\varphi^{-1}\mathscr{O}_Y} \mathscr{O}_X) \oplus \mathscr{O}_X \hookrightarrow \mathbb{S}(M^\bullet) \otimes_{\varphi^{-1}\mathscr{O}_Y} \mathscr{O}_X.$$

Here the leftmost arrow equals $(\mathrm{id}, -\varepsilon)$, and $\varepsilon : M^\bullet_X \to \mathscr{O}_X$ is the augmentation map. The embedding induces an identification $\mathbb{S}(M^\bullet_X) = \mathbb{S}(M^\bullet) \otimes_{\varphi^{-1}\mathscr{O}_Y} \mathscr{O}_X$. It is easy to see that the ideal

$$\mathbb{S}^{\geq k}(M^\bullet_X) = \bigoplus_{j \geq k} \mathbb{S}^j(M^\bullet_X)$$

is preserved by the differential d. In particular d induces a differential on

$$\mathbb{S}^{\geq 1}(M^\bullet_X)/\mathbb{S}^{\geq 2}(M^\bullet_X) = M^\bullet_X.$$

The *cotangent complex* $\mathbb{L} = \mathbb{L}_{X/Y}$ is by definition $M_X^\bullet$ equipped with this differential, viewed as an object of the derived category of $\mathscr{O}_X$-modules.

Clearly, $M_X^\bullet$ depends on the choice of resolution of $\mathscr{O}_X$ (unless we use the natural resolution of [10]). However, one can check that it is unique up to quasi-isomorphism, cf. [10, Corollary II.1.2.6.3], [5, Proposition 2.7.7], and [6, Proposition 2.1.2].

2.23. Recall that a square-zero differential d on a free commutative algebra $\mathbb{S}(V)$ is equivalent to a (possibly curved) L_∞-coalgebra structure on V. (Here V is a graded vector space, or any similar object in an appropriate graded symmetric monoidal category.) Thus the specific complex $M_X^\bullet[1]$ constructed above inherits tautologically the structure of an L_∞-coalgebra in the dg-category of complexes of flat $\mathscr{O}_X$-modules. Its structure maps $d_k : M_X^\bullet[1] \to \wedge^k(M_X^\bullet[1])[2-k]$, $k \geq 1$, are defined by the equality

$$d|_{M_X^\bullet} = (d_1, d_2, \dots) : M_X^\bullet \to \mathbb{S}^{\geq 1}(M_X^\bullet) \subset \mathbb{S}(M_X^\bullet).$$

(One can see that the curvature term $d_0 : M_X^\bullet \to \mathscr{O}_X$ of this coalgebra is trivial.)

2.24. Example. By definition, $\mathbb{L}_{X/Y}$ is $M_X^\bullet$ equipped with the differential d_1. Since $d_2 : M_X^\bullet \to \mathbb{S}^2(M_X^\bullet)[1]$ is a chain map of complexes we get a natural map in the derived category

$$\mathbb{L}_{X/Y} \to \mathbb{S}^2(\mathbb{L}_{X/Y})[1],$$

which makes $\mathbb{L}_{X/Y}[1]$ into a Lie coalgebra object in the derived category of $\mathscr{O}_X$-modules.

2.25. Let us now return to the case of a lci embedding of schemes $i : X \hookrightarrow Y$. In this case, $\mathbb{L}_{X/Y} = N_{X/Y}^\vee[1] = E[1]$.

2.26. Proposition. *The bracket*

$$N_{X/Y}^\vee \to \wedge^2 N_{X/Y}^\vee[2]$$

of Example 2.24 is given by the HKR class α.

Proof. As we saw above the complex $(\mathbb{S}(M_X^\bullet), d)$ is filtered by complexes $(\mathbb{S}^{\geq k}(M_X^\bullet), d)$. As a graded sheaf the quotient $\mathbb{S}(M_X^\bullet)^{\geq 1}/\mathbb{S}(M_X^\bullet)^{\geq 3}$ is equal to $M_X^\bullet \bigoplus \mathbb{S}^2(M_X^\bullet)$, however, the differential on the quotient includes both the d_1 and the d_2 components. This provides an identification of the quotient $\mathbb{S}(M_X^\bullet)^{\geq 1}/\mathbb{S}(M_X^\bullet)^{\geq 3}$ with the cone of the morphism

$$d_2 : (M_X^\bullet, d_1) \to (\mathbb{S}^2(M_X^\bullet), d_1).$$

In our case, $(\mathbb{S}^k(\mathsf{M}^\bullet_\mathsf{X}), d_1)$ has cohomology only in degree $-k$. Therefore in the derived category $\mathbf{D}(X)$ we have

$$(\mathbb{S}^{\geq k}(\mathsf{M}^\bullet_\mathsf{X}), d) \simeq \tau^{\leq -k}((\mathbb{S}^k(\mathsf{M}^\bullet_\mathsf{X}), d_1) \simeq \tau^{\leq -k}(\mathscr{O}_X \otimes_{i^{-1}\mathscr{O}_Y} \mathscr{O}_X) = \tau^{\leq -k}(i^* i_* \mathscr{O}_X).$$

Now the statement follows from Proposition 2.20. □

2.27. Remark. There is a similar description of the HKR class α_V for any vector bundle V on X (cf. the definition of the Atiyah classes in [1, Section 4.2]. Namely we can resolve V by a $(\mathbb{S}(\mathsf{M}^\bullet), d)$-module of the form

$$(\mathbb{S}(\mathsf{M}^\bullet) \otimes_{\varphi^{-1}\mathscr{O}_Y} \mathsf{C}^\bullet, d).$$

Here $\mathsf{C}^\bullet$ is a graded sheaf of flat $\varphi^{-1}\mathscr{O}_Y$-modules. The tensor product with $\mathscr{O}_X$ gives a graded $\mathscr{O}_X$-module

$$(\mathbb{S}(\mathsf{M}^\bullet) \otimes_{\varphi^{-1}\mathscr{O}_Y} \mathsf{C}^\bullet) \otimes_{\varphi^{-1}\mathscr{O}_Y} \mathscr{O}_X = \mathbb{S}(\mathsf{M}^\bullet_\mathsf{X}) \otimes_{\varphi^{-1}\mathscr{O}_Y} \mathsf{C}^\bullet = \mathbb{S}(\mathsf{M}^\bullet_\mathsf{X}) \otimes_{\mathscr{O}_X} \mathsf{C}^\bullet_\mathsf{X},$$

where $\mathsf{C}^\bullet_\mathsf{X} = \mathsf{C}^\bullet \otimes_{\varphi^{-1}\mathscr{O}_Y} \mathscr{O}_X$. This tensor product carries a differential d compatible with the action of $(\mathbb{S}(\mathsf{M}^\bullet_\mathsf{X}), d)$.

This provides a coaction $d_k : \mathsf{C}^\bullet_\mathsf{X} \to \bigwedge^k(\mathsf{M}^\bullet_\mathsf{X}[1]) \otimes \mathsf{C}^\bullet_\mathsf{X}[1-k]$ of $\mathsf{M}^\bullet_\mathsf{X}[1]$ on $\mathsf{C}^\bullet_\mathsf{X}$. In particular $d_0 : \mathsf{C}^\bullet_\mathsf{X} \to \mathsf{C}^\bullet_\mathsf{X}[1]$ is a differential, and it is easy to see that it turns $\mathsf{C}^\bullet_\mathsf{X}$ into a resolution of V. Therefore d_1 yields a coaction of $\mathbb{L}_{X/Y}[1]$ on V in $\mathbf{D}(X)$; one can check that the coaction is given by the HKR class α_V. This is a (dual) analogue of the usual statement that $\mathfrak{g} = T_X[-1]$ is a Lie algebra object in $\mathbf{D}(X)$ and every object $F \in \mathbf{D}(X)$ is a representation of $\mathfrak{g}$.

2.28. Remark. Suppose that $i : X \hookrightarrow Y$ is an embedding of quasi-projective schemes. In this case, $\mathsf{M}^\bullet$ can be chosen to be locally free over $i^{-1}\mathscr{O}_Y$, so that $(\mathsf{M}^\bullet_\mathsf{X}, d_1)$ is a complex of locally free $\mathscr{O}_X$-modules. It is perhaps more natural to resolve the $\mathscr{O}_Y$-algebra $i_*\mathscr{O}_X$ instead of the $i^{-1}\mathscr{O}_Y$-algebra $\mathscr{O}_X$: this leads to a resolution of X by a dg-scheme smooth over Y in the sense of [5].

2.29. An example of a non-vanishing HKR class. We shall now argue that the HKR class α is non-zero for the embedding of $X = \mathbb{P}^1 \times \mathbb{P}^1$ into $Y = \mathbb{P}^5$ using the very ample line bundle $\mathscr{O}(1) \boxtimes \mathscr{O}(2)$ on X, thus showing that our theory is non-trivial.

2.30. Lemma. *Consider the cotangent bundle short exact sequence*

$$0 \to E = N^\vee \to \Omega^1_Y|_X \to \Omega^1_X \to 0,$$

and the associated long exact sequence

$$\cdots \to H^1(X, i^*\Omega^1_Y) \to H^1(X, \Omega^1_X) \xrightarrow{H^1(\eta)} H^2(X, E) \to \ldots .$$

Then we have

$$H^1(\eta) \neq 0.$$

Proof. Clearly,

$$H^1(X, \Omega^1_X) = H^1(X, (\Omega_{\mathbb{P}^1} \boxtimes \mathscr{O}_{\mathbb{P}^1}) \oplus (\mathscr{O}_{\mathbb{P}^1} \boxtimes \Omega_{\mathbb{P}^1})) = \mathbf{k}^2,$$

so it suffices to show that $\dim H^1(X, i^*\Omega^1_Y) < 2$. On $Y = \mathbb{P}^5$, consider the exact sequence

$$0 \to \Omega^1_{\mathbb{P}^5} \to \mathscr{O}_{\mathbb{P}^5}(-1)^6 \to \mathscr{O}_{\mathbb{P}^5} \to 0.$$

Applying i^*, we obtain the sequence

$$0 \to i^*\Omega_Y \to (\mathscr{O}_{\mathbb{P}^1}(-1) \boxtimes \mathscr{O}_{\mathbb{P}^1}(-2))^6 \to \mathscr{O}_X \to 0.$$

Looking at the corresponding long exact sequence of cohomology groups, we see that

$$\dim H^k(X, i^*\Omega_Y) = \begin{cases} 0, & k \neq 1 \\ 1, & k = 1 \end{cases},$$

as required. □

2.31. Notice that for a line bundle L on X its Atiyah class at_L equals its first Chern class $c_1(L) \in H^1(X, \Omega^1_X)$. Therefore

$$\alpha_L = \eta \circ \mathsf{at}_L = H^1(\eta)(c_1(L)) \in H^2(X, E),$$

where the latter is interpreted as the value of the map $H^1(\eta)$ evaluated on $c_1(L) \in H^1(X, \Omega^1_X)$. It is clear that α_L is additive in L.

2.32. Lemma. *For the line bundle* $L = \bigwedge^3 E$ *we have* $\alpha_L \neq 0$.

Proof. Set

$$\ell = i^*\mathscr{O}_{\mathbb{P}^5}(1) = \mathscr{O}_{\mathbb{P}^1}(1) \boxtimes \mathscr{O}_{\mathbb{P}^1}(2).$$

Since ℓ is a line bundle on X that extends to Y, $\alpha_\ell = 0$. On the other hand,

$$L = i^*\omega_Y \otimes \omega_X^{-1} \simeq i^*\mathscr{O}_{\mathbb{P}^5}(-6) \otimes (\mathscr{O}_{\mathbb{P}^1}(2) \boxtimes \mathscr{O}_{\mathbb{P}^1}(2)) = \mathscr{O}_{\mathbb{P}^1}(-4) \boxtimes \mathscr{O}_{\mathbb{P}^1}(-10).$$

In particular, we see that $c_1(L)$ and $c_1(\ell)$ form a basis of $H^1(X, \Omega^1_X)$. Now Lemma 2.30 implies that

$$\alpha_L = H^1(\eta)(c_1(L)) \neq 0,$$

since otherwise the map $H^1(\eta)$ would be identically zero. □

2.33. Corollary. *The class* $\alpha = \alpha_E \in H^2(X, E)$ *is not zero.*

Proof. Equivalently, we have to show that the vector bundle E does not extend to the first infinitesimal neighborhood $X' \subset Y$. This is true because by Lemma 2.32 its determinant L does not extend to X'. □

2.34. We conclude this section by speculating how Theorem 0.7 should be understood from the point of view of the L_∞-coalgebra structure on the cotangent complex.

Recall from (2.23) that the construction of the cotangent complex endowed the explicit complex $M_X^\bullet[1]$ with an L_∞-coalgebra structure. This is encoded either as a sequence of maps

$$d_k : M_X^\bullet[1] \to \wedge^k(M_X^\bullet[1])[2-k]$$

for $k \geq 1$, or as a degree one differential d on $\mathbb{S}(M_X^\bullet)$. The complex $(\mathbb{S}(M_X^\bullet), d)$ represents the derived tensor product $\mathscr{O}_X \otimes_{\varphi^{-1}\mathscr{O}_Y} \mathscr{O}_X$ or, in the case φ is an affine morphism of schemes, the object $\varphi^*\varphi_*\mathscr{O}_X$ of $\mathbf{D}(X)$.

2.35. Now consider the case of a closed embedding i of an lci subscheme X of a smooth scheme Y. Then the cohomology of the complex $(M_X^\bullet[1], d_1)$ is concentrated in degree -2, where it equals E. Assume that we can overcome the technical difficulties associated with applying the homological perturbation lemma to the coalgebra $M_X^\bullet[1]$. (Since we work in the category of sheaves on X which has positive homological dimension the usual theory can not be applied directly.) Then by transfer of structure we get an L_∞-coalgebra structure on $E[2]$ which is encoded by structure maps (after adjusting degrees)

$$d'_k : E \to \wedge^k E[k].$$

These maps can further be assembled to yield a square-zero, degree one "differential" in $\mathbf{D}(X)$

$$d' : \mathbb{S}(E[1]) \to \mathbb{S}(E[1])[1].$$

The main feature of homological perturbation theory is that $(\mathbb{S}(E[1]), d')$ shall be homotopic to $(M_X^\bullet[1], d)$, so in particular they will represent the same object of $\mathbf{D}(X)$ which we saw earlier is $i^*i_*\mathscr{O}_X$. (Note however that while the map d' has all the classical properties of a differential, it is not an actual chain map, so the resulting object is not a complex and the notion of quasi-isomorphism does not make sense without further developments of homological algebra.)

The above considerations combined with Theorem 0.7 motivate us to state the following conjecture.

2.36. Conjecture. *The* L_∞*-coalgebra* $E[2]$ *has the property that if the classical co-bracket* $d_2' = \alpha_E$ *vanishes, then all the higher co-brackets* d_k' *vanish for* $k \geq 2$ *as well.*

2.37. Note that in the spectral sequence of Corollary 0.8, if the differentials up to the $(k-1)$-st page vanish, the differentials at the k-th page are maps

$$H^p(X, \wedge^q E^\vee) \to H^{p+k}(X, \wedge^{q-k+1} E^\vee).$$

It seems reasonable to guess that these maps are contraction with d_k', with d_k' regarded as an element of $H^k(X, \wedge^k E \otimes E)$.

If we believe this then Conjecture 2.36 immediately implies Corollary 0.8. It also implies Theorem 0.7 as well: $\alpha_E = 0$ implies $d_2' = 0$, hence by the Conjecture $d_k' = 0$ for all $k \geq 2$, so $d' = 0$. Since the complex $i^* i_* \mathscr{O}_X = (\mathbb{S}(M_X^\bullet), d)$ is quasi-isomorphic to

$$(\mathbb{S}(E[1]), d) = (\mathbb{S}(E[1]), 0) = \mathbb{S}(E[1]),$$

Theorem 0.7 follows.

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