THE MUKAI PAIRING, I: THE HOCHSCHILD STRUCTURE

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Abstract. We study the Hochschild structure of a smooth space or orbifold, emphasizing the importance of a pairing defined on Hochschild homology which generalizes a similar pairing introduced by Mukai on the cohomology of a K3 surface. We discuss those properties of the structure which can be derived without appealing to the Hochschild-Kostant-Rosenberg isomorphism and Kontsevich formality, namely:

– functoriality of homology, commutation of push-forward with the Chern character, and adjointness with respect to the generalized pairing;
– formal Hirzebruch-Riemann-Roch and the Cardy condition from physics;
– invariance of the full Hochschild structure under Fourier-Mukai transforms.

Connections with homotopy theory and TQFT's are discussed in an appendix. A separate paper [9] treats consequences of the HKR isomorphism. Applications of these results to the study of a mirror symmetric analogue of Chen-Ruan's orbifold product will be presented in a future paper.

Contents

1. Introduction

1.1. The present work is the first in a series of three papers dedicated to the study of the Hochschild structure of smooth spaces, laying out the foundational material used in the other two [9], [10]. The Hochschild structure $(HH^*(X), HH_*(X))$ is defined for a space X, and its fundamental properties are studied. The space X can be an ordinary compact complex manifold, or more generally a global quotient compact orbifold, a proper Deligne-Mumford stack for which Serre duality holds, or a compact "twisted space" in the sense of [8].

1.2. The *Hochschild structure* of X consists of

– a graded ring $HH^*(X)$, the Hochschild cohomology ring, whose i-th graded piece is defined as
$$HH^i(X) = \mathrm{Hom}_{\mathbf{D}^b_{\mathrm{coh}}(X\times X)}(\mathscr{O}_\Delta, \mathscr{O}_\Delta[i]),$$
where $\mathscr{O}_\Delta = \Delta_* \mathscr{O}_X$ is the structure sheaf of the diagonal in $X \times X$;

– a graded left $HH^*(X)$-module $HH_*(X)$, the Hochschild homology module, defined as
$$HH_i(X) = \mathrm{Hom}_{\mathbf{D}^b_{\mathrm{coh}}(X\times X)}(\Delta_! \mathscr{O}_X[i], \mathscr{O}_\Delta),$$
where $\Delta_!$ is the left adjoint of Δ^* defined by Grothendieck-Serre duality (3.3);

– a non-degenerate graded pairing $\langle \cdot, \cdot \rangle$ on $HH_*(X)$, the *generalized Mukai pairing*.

The Hochschild cohomology ring has a rich and developed theory ([11], [21]). The above definition of homology is, to the author's knowledge, new (but see [34] for an alternative equivalent definition, and [24] for a different attempt). The last important ingredient of the structure, the Mukai pairing, has not been studied previously from the perspective of Hochschild theory.

1.3. In his groundbreaking work [26] Mukai studied the relationship between the derived category and the cohomology of K3 surfaces X and Y by defining

– a map $v : \mathbf{D}^b_{\mathrm{coh}}(X) \to H^*(X, \mathbf{C})$ given by
$$v(\mathscr{F}) = \mathrm{ch}(\mathscr{F}).\, \mathrm{td}(X)^{1/2},$$
where $\mathrm{td}(X)$ is the Todd genus of X ($v(\mathscr{F})$ is called the *Mukai vector* of $\mathscr{F}$);

– an association $\Phi \mapsto \Phi_*$ which maps the integral transform
$$\Phi : \mathbf{D}^b_{\mathrm{coh}}(X) \to \mathbf{D}^b_{\mathrm{coh}}(Y), \qquad \Phi(\,\cdot\,) = \pi_{Y,*}(\pi_X^*(\,\cdot\,) \otimes \mathscr{E})$$
defined by an object $\mathscr{E}$ in $\mathbf{D}^b_{\mathrm{coh}}(X \times Y)$ to the map on cohomology
$$\Phi_* : H^*(X, \mathbf{C}) \to H^*(Y, \mathbf{C}), \qquad \Phi_*(\,\cdot\,) = \pi_{Y,*}(\pi_X^*(\,\cdot\,).v(\mathscr{E}));$$

– a pairing $\langle \cdot, \cdot \rangle$ on the cohomology $H^*(X, \mathbf{C})$, given by the formula
$$\langle v, w \rangle = \int_X v_0.w_4 - v_2.w_2 + v_4.w_0,$$
where for a vector $v \in H^*(X, \mathbf{C})$, v_i is the component of v in $H^i(X, \mathbf{C})$.

It is worth emphasizing that the map Φ_* does not respect the usual grading on the cohomology $H^*(X, \mathbf{C})$.

1.4. Mukai argued that the following properties are satisfied for K3 surfaces X and Y:

a. **Functoriality:** The association of maps on cohomology to integral transforms is functorial, in the sense that $\mathrm{Id}_{\mathbf{D}^b_{\mathrm{coh}}(X)} \mapsto \mathrm{Id}_{H^*(X,\mathbf{C})}$, and $(\Phi \circ \Psi)_* = \Phi_* \circ \Psi_*$.

b. **Commutation with v:** The following diagram commutes
$$\begin{tikzcd} \mathbf{D}^b_{\mathrm{coh}}(X) \arrow[r, "\Phi"] \arrow[d, "v"] & \mathbf{D}^b_{\mathrm{coh}}(Y) \arrow[d, "v"] \\ H^*(X, \mathbf{C}) \arrow[r, "\Phi_*"] & H^*(Y, \mathbf{C}). \end{tikzcd}$$

c. **Adjointness:** If $\Phi : \mathbf{D}^b_{\mathrm{coh}}(Y) \to \mathbf{D}^b_{\mathrm{coh}}(X)$ is left adjoint to $\Psi : \mathbf{D}^b_{\mathrm{coh}}(X) \to \mathbf{D}^b_{\mathrm{coh}}(Y)$ then

$$\langle \Phi_* v, w \rangle_X = \langle v, \Psi_* w \rangle_Y$$

for $v \in H^*(Y, \mathbf{C})$, $w \in H^*(X, \mathbf{C})$.

d. **Hirzebruch-Riemann-Roch:** For $\mathscr{E}, \mathscr{F} \in \mathbf{D}^b_{\mathrm{coh}}(X)$ we have

$$\langle v(\mathscr{E}), v(\mathscr{F}) \rangle = \chi(\mathscr{E}, \mathscr{F}),$$

where $\chi(\,\cdot\,,\,\cdot\,)$ is the Euler pairing on $K_0(X)$,

$$\chi(\mathscr{E}, \mathscr{F}) = \sum_i (-1)^i \dim \operatorname{Ext}^i_X(\mathscr{E}, \mathscr{F}).$$

It follows immediately from these properties that if Φ is an equivalence of triangulated categories, then Φ_* is an isometry between the corresponding cohomology groups, endowed with the Mukai pairing.

1.5. This paper is devoted to generalizing Mukai's results to a wide class of compact spaces, including in particular smooth compact complex manifolds, twisted spaces in the sense of [8], and certain orbifolds or Deligne-Mumford stacks for which Serre duality holds. The main point we want to emphasize is that the natural target for defining Mukai's structure is not singular cohomology but rather Hochschild homology. Replacing singular cohomology by Hochschild homology, we shall obtain all of Mukai's results for the wide class of spaces above.

The first observation that hints to the fact that ordinary cohomology is not the right target for the definition of the maps Φ_* is the observation that in the case of a smooth compact complex manifold these maps do not respect the usual grading on singular cohomology. The correct grading that is preserved is the one given by the verticals, and not the horizontals of the Hodge diamond of the space, which is precisely the grading on Hochschild homology.

1.6. To relate our approach to the original one of Mukai observe that the Hochschild-Kostant-Rosenberg theorem asserts the existence of an isomorphism ([9])

$$I_{\mathrm{HKR}} : HH_i(X) \cong \bigoplus_{q-p=i} H^p(X, \Omega^q_X) \overset{\mathrm{def}}{=\!=} H\Omega_i(X)$$

between the i-th Hochschild homology of a smooth projective manifold X and the $n+i$-th column of the Hodge diamond of X.

It would seem natural to expect that, in the case of a K3 surface X, the I_{HKR} isomorphism will match the abstract structures that we shall define on $HH_*(X)$ with the original structure of Mukai. However, we believe that a correction is needed for that: in [9] we conjecture that we need to adjust the I_{HKR} isomorphism by multiplying it by $\mathrm{td}(X)^{1/2}$ before the abstract structure we define will yield Mukai's original one.

1.7. Let us now present our results. After some generalities on integral transforms and Serre duality in Section 2, we discuss the construction of left-right adjoint functors in Section 3. This will be the basis for all our results. Then in Section 4 we define Hochschild homology and cohomology, as well as the generalized Mukai product. In Section 5 we consider an integral transform $\Phi : \mathbf{D}^b_{\mathrm{coh}}(X) \to \mathbf{D}^b_{\mathrm{coh}}(Y)$ between two spaces X and Y and define a natural map of graded vector spaces $\Phi_* : HH_*(X) \to HH_*(Y)$. Using this construction we present in Section 6 a definition of a Chern character map

$$\mathrm{ch} : K_0(X) \to HH_0(X)$$

which agrees, under I_{HKR}, with the usual Chern character map ([9])

$$\mathrm{ch} : K_0(X) \to \bigoplus_p H^p(X, \Omega_X^p).$$

(And which, under the corrected isomorphism $HH_0(X) \cong \bigoplus H^{p,p}(X)$, yields the Mukai vector. It is worth emphasizing that on the level of Hochschild homology, no correction by the Todd genus is needed; this correction appears in the usual statements because of the "wrong" choice of HKR isomorphism.) We also discuss a definition equivalent to ours given by Markarian [24].

1.8. The formal properties a — d of (1.4) can now be proven to hold in full generality, using $HH_*(X)$ instead of $H^*(X, \mathbf{C})$ and ch instead of v. The corresponding results are Theorems 5.3, 7.1, 7.3, and 7.6. A slightly more general version of the Hirzebruch-Riemann-Roch theorem can be stated in this context (Theorem 7.9). Its origins can be traced to the Cardy condition in physics. It turns out in fact that properties a and c are truly fundamental, while b and d are easy consequences of them.

1.9. The final result of the paper is a proof, in Section 8, of the fact that the full Hochschild structure is invariant with respect to Fourier-Mukai transforms. The main result is:

Theorem 8.1. [1] *Let X and Y be spaces whose derived categories are equivalent via a Fourier-Mukai transform (i.e., the equivalence is given by an integral transform). Then there exists a natural isomorphism of Hochschild structures*

$$(HH^*(X), HH_*(X)) \cong (HH^*(Y), HH_*(Y)).$$

More precisely, there exists an isomorphism of graded rings $HH^*(X) \cong HH^*(Y)$, an isomorphism $HH_*(X) \cong HH_*(Y)$ of graded modules over the corresponding cohomology rings, and the latter isomorphism is an isometry with respect to the Mukai pairings on $HH_*(X)$ and on $HH_*(Y)$, respectively.

1.10. Throughout the paper there will be a certain tension between the "Ext" interpretation of the Hochschild structure given in (1.2) and a parallel categorical interpretation. The point is that there are alternative ways, outlined in Appendices A and B, to regard elements of $HH^*(X)$ and $HH_*(X)$ not as morphisms in $\mathbf{D}^b_{\mathrm{coh}}(X \times X)$, but rather as natural transformations between certain functors $\mathbf{D}^b_{\mathrm{coh}}(X) \to \mathbf{D}^b_{\mathrm{coh}}(X)$. Unfortunately, despite our best efforts, we have been unable to make these ideas fully precise. This appears to be primarily caused by certain known technical problems with the definition of the derived category [13]. However the intuition behind the categorical interpretation is most often the correct one, and as a compromise we have decided to steer a middle course: we presented our results in mathematically correct form in the "Ext" interpretation, and we gave the intuitive ideas in the categorical context. We highly recommend the reader to read the appendices for gaining intuition into the proofs.

The current state of affairs is somewhat unsatisfactory, as several of the proofs appear unnecessarily complicated. We can only hope that future developments of category theory will enable us to rewrite this paper at a later date in the "correct" (categorical) language.

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[1] Parts of this theorem were also proven independently by Orlov [29].

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Conventions. Throughout the paper a *space* will be a compact complex manifold or proper algebraic variety over an algebraically closed field of characteristic zero, possibly endowed with an Azumaya algebra, or a smooth compact Deligne-Mumford stack which satisfies Serre duality. The derived category of a space will refer to the bounded derived category of coherent sheaves on the underlying space (which, in the case of the existence of an Azumaya algebra $\mathscr{A}$, shall mean coherent sheaves of modules over $\mathscr{A}$). Functors between derived categories shall always be implicitly derived, but we shall keep clear the distinction between Hom and $\mathbf{R}$Hom. Whenever we write $\mathscr{F} \otimes \mu$ where $\mathscr{F}$ is an object and μ is a morphism, we mean $\mathrm{Id}_{\mathscr{F}} \otimes \mu$.

2. Preliminaries

In this section we set up the basic context and notation. We also provide a brief introduction to Serre functors and Grothendieck-Serre duality. Our basic reference for these results is [4]. We discuss a trace map that arises from the existence of Serre functors and which is intimately related to one studied by Illusie [16] and Artamkin [1].

2.1. Let X and Y be spaces, and let $\mathscr{E}$ be an object in $\mathbf{D}^b_{\mathrm{coh}}(X \times Y)$. If π_X and π_Y are the projections from $X \times Y$ to X and Y, respectively, define the functor

$$\Phi^{\mathscr{E}}_{X \to Y} : \mathbf{D}^b_{\mathrm{coh}}(X) \to \mathbf{D}^b_{\mathrm{coh}}(Y) \quad \Phi^{\mathscr{E}}_{X \to Y}(\,\cdot\,) = \mathbf{R}\pi_{Y,*}(\pi_X^*(\,\cdot\,) \overset{\mathbf{L}}{\otimes} \mathscr{E}),$$

which will be called the *integral functors* (on derived categories) associated to $\mathscr{E}$ (or with kernel $\mathscr{E}$).

The association between objects of $\mathbf{D}^b_{\mathrm{coh}}(X \times Y)$ and integral transforms is functorial: given a morphism $\mu : \mathscr{E} \to \mathscr{F}$ between objects of $\mathbf{D}^b_{\mathrm{coh}}(X \times Y)$, there is an obvious natural transformation

$$\Phi^{\mu}_{X \to Y} : \Phi^{\mathscr{E}}_{X \to Y} \Rightarrow \Phi^{\mathscr{F}}_{X \to Y}$$

given by

$$\Phi^{\mu}_{X \to Y}(\,\cdot\,) = \pi_{Y,*}(\pi_X^*(\,\cdot\,) \otimes \mu).$$

2.2. Given spaces X, Y, Z, and elements $\mathscr{E} \in \mathbf{D}^b_{\mathrm{coh}}(X \times Y)$ and $\mathscr{F} \in \mathbf{D}^b_{\mathrm{coh}}(Y \times Z)$, define $\mathscr{F} \circ \mathscr{E} \in \mathbf{D}^b_{\mathrm{coh}}(X \times Z)$ by

$$\mathscr{F} \circ \mathscr{E} = \pi_{XZ,*}(\pi_{XY}^* \mathscr{E} \otimes \pi_{YZ}^* \mathscr{F}),$$

where $\pi_{XY}, \pi_{YZ}, \pi_{XZ}$ are the projections from $X \times Y \times Z$ to the corresponding factors. The reason behind the notation is the fact that we have ([5, 1.4])

$$\Phi^{\mathscr{F}}_{Y \to Z} \circ \Phi^{\mathscr{E}}_{X \to Y} = \Phi^{\mathscr{F} \circ \mathscr{E}}_{X \to Z}.$$

2.3. Recall the definition of a (right) Serre functor on an additive category $\mathcal{C}$ with finite dimensional Hom spaces from [30] (generalizing slightly the original definition of [4]). A Serre functor is a functor $S : \mathcal{C} \to \mathcal{C}$ together with natural, bifunctorial isomorphisms

$$\eta_{A,B} : \mathrm{Hom}_{\mathcal{C}}(A, B) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(B, SA)^{\vee},$$

for any A, B, where $\cdot^{\vee}$ denotes the dual vector space. For any A in $\mathcal{C}$, define

$$\mathrm{Tr} : \mathrm{Hom}(A, SA) \to \mathbf{C}, \qquad \mathrm{Tr}(f) = \eta_{A,A}(\mathrm{id}_A)(f).$$

The following are easy consequences of the definition of a Serre functor (see [30] for details):

Lemma 2.1. *For $f : A \to B$ and $g : B \to SA$, we have*

$$\eta_{A,B}(f)(g) = \mathrm{Tr}(g \circ f).$$

Lemma 2.2. *For $f : A \to B$ and $g : B \to SA$, we have*

$$\mathrm{Tr}(g \circ f) = \mathrm{Tr}(Sf \circ g).$$

2.4. To connect with more classical approaches to Serre duality, recall that Illusie ([16]) and Artamkin ([1]) construct a trace map

$$\mathrm{Tr}_{\mathscr{E}} : \mathrm{Hom}_X(\mathscr{F} \otimes \mathscr{E}, \mathscr{G} \otimes \mathscr{E}) \to \mathrm{Hom}_X(\mathscr{F}, \mathscr{G})$$

for objects $\mathscr{E}, \mathscr{F}, \mathscr{G}$ in the derived category of a compact, smooth space X, which generalizes the usual trace map on vector spaces. One way to write the definition of this trace map is that if $\mu : \mathscr{F} \otimes \mathscr{E} \to \mathscr{G} \otimes \mathscr{E}$ then $\mathrm{Tr}_{\mathscr{E}}(\mu)$ is the composition

$$\mathscr{F} \xrightarrow{\mathrm{id} \otimes \eta} \mathscr{F} \otimes \mathscr{E} \otimes \mathscr{E}^\vee \xrightarrow{\mu \otimes \mathrm{id}} \mathscr{G} \otimes \mathscr{E} \otimes \mathscr{E}^\vee \overset{\mathrm{id} \otimes \gamma}{=\!=\!=} \mathscr{G} \otimes \mathscr{E}^\vee \otimes \mathscr{E} \xrightarrow{\mathrm{id} \otimes \epsilon} \mathscr{G}.$$

Here $\eta : \mathscr{O}_X \to \mathscr{E} \otimes \mathscr{E}^\vee \cong \mathbf{R}\underline{\mathrm{Hom}}(\mathscr{E}, \mathscr{E})$ is the morphism which sends the section "1" of $\mathscr{O}_X$ to the the identity of $\mathrm{Hom}(\mathscr{E}, \mathscr{E})$, γ is the isomorphism that interchanges the two factors, and ϵ is the original trace map of Illusie and Artamkin. This definition should be compared to the generalized trace map of May [25].

If we consider the functor

$$S_X(-) = \omega_X[\dim X] \otimes -,$$

then in the standard form of Serre duality ([15]) one constructs a trace map

$$\mathrm{Tr}_X : \mathrm{Hom}_X(\mathscr{O}_X, S_X \mathscr{O}_X) \to \mathbf{C}$$

such that for objects $\mathscr{E}$, $\mathscr{F}$ of $\mathbf{D}^b_{\mathrm{coh}}(X)$ the pairing

$$\langle \cdot, \cdot \rangle : \mathrm{Hom}_X(\mathscr{F}, S_X \mathscr{E}) \otimes \mathrm{Hom}_X(\mathscr{E}, \mathscr{F}) \to \mathbf{C}$$

given by

$$\langle f, g \rangle = \mathrm{Tr}_X(\mathrm{Tr}_{\mathscr{E}}(f \circ g))$$

is non-degenerate. This yields isomorphisms

$$\mathrm{Hom}_X(\mathscr{E}, \mathscr{F}) \cong \mathrm{Hom}_X(\mathscr{F}, S_X \mathscr{E})^\vee$$

for every $\mathscr{E}, \mathscr{F}$ which are natural in both variables. Applying [30, I.1.4] it follows immediately that S_X is a Serre functor for $\mathbf{D}^b_{\mathrm{coh}}(X)$. We shall often abuse notation and denote $\mathrm{Tr}_X(\mathrm{Tr}_{\mathscr{E}}(\,\cdot\,))$ by $\mathrm{Tr}_X(\,\cdot\,)$.

2.5. The following generalization of a standard formula from linear algebra is a rather involved result in category theory:

Proposition 2.3. *Assume given a map of triangles*

$$\begin{tikzcd}
\mathscr{E} \arrow[r] \arrow[d, "e"'] & \mathscr{F} \arrow[r] \arrow[d, "f"'] & \mathscr{G} \arrow[r] \arrow[d, "g"'] & \mathscr{E}[1] \arrow[d, "{e[1]}"'] \\
\mathscr{E} \otimes \mathscr{H} \arrow[r] & \mathscr{F} \otimes \mathscr{H} \arrow[r] & \mathscr{G} \otimes \mathscr{H} \arrow[r] & \mathscr{E} \otimes \mathscr{H}[1],
\end{tikzcd}$$

where the bottom triangle is obtained by tensoring the top one with $\mathscr{H}$. Assume furthermore that it is possible to find representatives of all the objects in the diagram as complexes of *sheaves, such that the maps are maps of complexes and the squares commute on the nose (not just up to homotopy), and that g is the natural quotient map obtained from e and f. Then we have*

$$\mathrm{Tr}_{\mathscr{E}}(e) - \mathrm{Tr}_{\mathscr{F}}(f) + \mathrm{Tr}_{\mathscr{G}}(g) = 0$$

as morphisms $\mathscr{O}_X \to \mathscr{H}$.

Proof. This is [25, Theorem 1.9]. The condition about representing the objects as complexes, etc., is precisely what the proof of [loc. cit.] uses. □

The following is an easy exercise in linear algebra:

Lemma 2.4. *Let $\mu : \mathscr{E} \otimes \mathscr{F} \to \mathscr{E} \otimes \mathscr{G}$ and $\nu : \mathscr{G} \to \mathscr{H}$ be morphisms in $\mathbf{D}^b_{\mathrm{coh}}(X)$. Then*

$$\mathrm{Tr}_{\mathscr{E}}((\mathrm{id}_{\mathscr{E}} \otimes \nu) \circ \mu) = \nu \circ \mathrm{Tr}_{\mathscr{E}}(\mu)$$

as morphisms $\mathscr{F} \to \mathscr{H}$.

3. The basic construction

3.1. The following rather innocuous remark about the construction of a right adjoint functor from a left adjoint one is the basis for all the results in this paper. Consider a functor

$$\Phi : \mathbf{D}^b_{\mathrm{coh}}(X) \to \mathbf{D}^b_{\mathrm{coh}}(Y)$$

that admits a left adjoint

$$\Phi^* : \mathbf{D}^b_{\mathrm{coh}}(Y) \to \mathbf{D}^b_{\mathrm{coh}}(X).$$

Let $\mathscr{F}$ and $\mathscr{G}$ be objects in $\mathbf{D}^b_{\mathrm{coh}}(X)$. The fact that Φ is a functor implies that there is a natural map

$$\mathrm{Hom}_X(\mathscr{G}, \mathscr{F}) \xrightarrow{\quad\Phi\quad} \mathrm{Hom}_Y(\Phi\mathscr{G}, \Phi\mathscr{F}).$$

By Serre duality we can construct a left adjoint of this map (with respect to the Serre pairing)

$$\mathrm{Hom}_X(\mathscr{F}, S_X\mathscr{G}) \xleftarrow{\quad\Phi^\dagger\quad} \mathrm{Hom}_Y(\Phi\mathscr{F}, S_Y\Phi\mathscr{G}).$$

The following proposition gives an explicit description of the map $\Phi^\dagger$.

Proposition 3.1. *Let $\Phi^! : \mathbf{D}^b_{\mathrm{coh}}(Y) \to \mathbf{D}^b_{\mathrm{coh}}(X)$ be given by*

$$\Phi^! = S_X \circ \Phi^* \circ S_Y^{-1}.$$

Then $\Phi^!$ is a right adjoint to Φ, and if $\nu \in \mathrm{Hom}_Y(\Phi\mathscr{F}, S_Y\Phi\mathscr{G})$ then $\Phi^\dagger\nu$ is the composition

$$\Phi^\dagger\nu : \mathscr{F} \xrightarrow{\bar\eta} \Phi^!\Phi\mathscr{F} \xrightarrow{\Phi^!\nu} \Phi^! S_Y \Phi\mathscr{G} = S_X\Phi^*\Phi\mathscr{G} \xrightarrow{S_X\epsilon} S_X\mathscr{G},$$

where $\bar\eta$, ϵ are the unit and counit of the adjunctions $\Phi \dashv \Phi^!$, $\Phi^* \dashv \Phi$, respectively.

Explicitly, for $\mu \in \mathrm{Hom}_X(\mathscr{G}, \mathscr{F})$ and $\nu \in \mathrm{Hom}_Y(\Phi\mathscr{F}, S_Y\Phi\mathscr{G})$ we have

$$\mathrm{Tr}_X(\Phi^\dagger\nu \circ \mu) = \mathrm{Tr}_Y(\nu \circ \Phi\mu).$$

Remark 3.2. There is a striking similarity between the definition of $\Phi^\dagger\nu$ and the definition in [25] of the generalized trace maps. It would be interesting to get a good explanation of this similarity.

Proof. Serre duality on X and Y gives the following diagram for $\mathscr{F} \in \mathbf{D}^b_{\mathrm{coh}}(X)$, $\mathscr{H} \in \mathbf{D}^b_{\mathrm{coh}}(Y)$

$$\begin{tikzcd}
\nu \arrow[rr, mapsto] & & \Phi^! \nu \circ \bar{\eta} \\
\mathrm{Hom}_Y(\Phi\mathscr{F}, \mathscr{H}) \arrow[r, "\sim"] \arrow[d, phantom, "\text{dual to}"] & \mathrm{Hom}_X(\mathscr{F}, S_X \Phi^* S_Y^{-1} \mathscr{H}) \arrow[r, equal] \arrow[d, phantom, "\text{dual to}"] & \mathrm{Hom}_X(\mathscr{F}, \Phi^! \mathscr{H}) \\
\mathrm{Hom}_Y(S_Y^{-1}\mathscr{H}, \Phi\mathscr{F}) & \mathrm{Hom}_X(\Phi^* S_Y^{-1}\mathscr{H}, \mathscr{F}) \arrow[l, "\sim"'] & \\
\Phi\bar{\mu} \circ \eta & \bar{\mu}, \arrow[l, mapsto] &
\end{tikzcd}$$

where η is the unit of $\Phi^* \dashv \Phi$, and the top and bottom rows are dual to each other and are given by the adjunctions $\Phi \dashv \Phi^!$, $\Phi^* \dashv \Phi$. It follows that $S_X \Phi^* S_Y^{-1}$ is a right adjoint to Φ (see also [27, Section 2]).

Reading the duality between the top and bottom rows of the above diagram we get

$$\mathrm{Tr}_X(\Phi^! \nu \circ \bar{\eta} \circ \bar{\mu}) = \mathrm{Tr}_Y(\nu \circ \Phi\bar{\mu} \circ \eta).$$

If we take $\mathscr{H} = S_Y \Phi \mathscr{G}$, μ and ν as in the statement of the proposition, and $\bar{\mu} = \mu \circ \epsilon$, then $\Phi\bar{\mu} \circ \eta$ is nothing but $\Phi\mu$, and we conclude that

$$\mathrm{Tr}_X(\Phi^! \nu \circ \bar{\eta} \circ \mu \circ \epsilon) = \mathrm{Tr}_Y(\nu \circ \Phi\mu).$$

By the commutativity property of the trace (Lemma 2.2) this can be rewritten as

$$\mathrm{Tr}_Y(\nu \circ \Phi\mu) = \mathrm{Tr}_X(S_X \epsilon \circ \Phi^! \nu \circ \bar{\eta} \circ \mu) = \mathrm{Tr}_X(\Phi^\dagger \nu \circ \mu).$$

□

3.2. If for some functor $\Psi : \mathbf{D}^b_{\mathrm{coh}}(X) \to \mathbf{D}^b_{\mathrm{coh}}(X)$ we have a natural transformation

$$\nu : \Phi \Longrightarrow S_Y \Phi \Psi$$

then the above construction yields a new natural transformation

$$\Phi^\dagger \nu : 1_X \xRightarrow{\bar{\eta}} \Phi^! \Phi \xRightarrow{\nu} \Phi^! S_Y \Phi \Psi = S_X \Phi^* \Phi \Psi \xRightarrow{\epsilon} S_X \Psi,$$

and thus we can view $\Phi^\dagger$ as a map

$$\Phi^\dagger : \mathrm{Nat}(\Phi, S_Y \Phi \Psi) \to \mathrm{Nat}(1_X, S_X \Psi)$$

where Nat denotes the set of natural transformations between the corresponding functors. By Proposition 3.1 we have for any $\mu : \Psi\mathscr{F} \to \mathscr{F}$

$$\mathrm{Tr}_X((\Phi^\dagger \nu)_{\mathscr{F}} \circ \mu) = \mathrm{Tr}_Y(\nu_{\Phi\mathscr{F}} \circ \Phi\mu).$$

3.3. The same kind of argument can be used to define a left adjoint to Φ when a right adjoint is known. For example,

$$\Delta_! = S^{-1}_{X \times X} \Delta_* S_X$$

is a left adjoint to Δ^*, where $\Delta : X \to X \times X$ is the diagonal embedding.

3.4. A similar kind of construction is the following: assume Φ and Ψ are functors from $\mathbf{D}^b_{\mathrm{coh}}(X)$ to $\mathbf{D}^b_{\mathrm{coh}}(Y)$, which admit right adjoints $\Phi^!$, $\Psi^!$, respectively. Then there exists a natural isomorphism

$$\tau : \mathrm{Nat}(\Phi, \Psi) \xrightarrow{\sim} \mathrm{Nat}(\Psi^!, \Phi^!),$$

which maps $\mu : \Phi \Longrightarrow \Psi$ to the composite

$$\Psi^! \xRightarrow{\bar{\eta}_\Phi} \Phi^!\Phi\Psi^! \xRightarrow{\mu} \Phi^!\Psi\Psi^! \xRightarrow{\bar{\epsilon}_\Psi} \Phi^!.$$

Indeed, an inverse to τ is given by mapping $\nu : \Psi^! \Longrightarrow \Phi^!$ to the natural transformation

$$\Phi \xRightarrow{\bar{\eta}_\Psi} \Phi\Psi^!\Psi \xRightarrow{\nu} \Phi\Phi^!\Psi \xRightarrow{\bar{\epsilon}_\Phi} \Psi.$$

4. The Hochschild structure: definition and basic properties

4.1. In this section we define Hochschild homology and cohomology for a space. The Mukai product is also introduced, together with its categorical interpretation. For simplicity of exposition we present everything for a smooth compact scheme with no group action; the case of an orbifold (or Deligne-Mumford stack) is obtained by thinking of all the objects involved as equivariant. For example the diagonal in $X \times X$ will be viewed as an equivariant subvariety of $X \times X$, all Ext's are computed in the category of equivariant sheaves, etc. (see [7, Section 4] for details). We give some hints on how to deal with general orbifolds in (4.4). Similarly, the case of twisted spaces will be obtained by working in a twisted derived category (with the observation that the diagonal can also be viewed as an (α, α)-twisted sheaf, etc.), and Serre functors make sense [8].

4.2. Let X be a smooth, proper variety of dimension n over $\mathbf{C}$. The following notations will be used throughout the paper:

- $\Delta : X \to X \times X$ is the diagonal embedding;
- ω_X is the canonical bundle of X;
- $S_X = \omega_X[n]$ as an object of $\mathbf{D}^b_{\mathrm{coh}}(X)$; often we shall also think of S_X as the Serre functor $S_X \otimes -$;
- $S_{X\times X} = \omega_{X\times X}[2n]$ in $\mathbf{D}^b_{\mathrm{coh}}(X \times X)$;
- $\mathscr{O}_\Delta = \Delta_*\mathscr{O}_X$, $S_\Delta = \Delta_* S_X$, $S_\Delta^{-1} = \Delta_* S_X^{-1}$;
- $\Delta_! : \mathbf{D}^b_{\mathrm{coh}}(X) \to \mathbf{D}^b_{\mathrm{coh}}(X \times X)$ is the left adjoint of Δ^*,

$$\Delta_! = S^{-1}_{X\times X}\Delta_* S_X.$$

Note that $\Delta_!\mathscr{O}_X \cong S_\Delta^{-1}$.

Definition 4.1. The Hochschild cohomology of X is defined to be

$$HH^i(X) = \mathrm{Hom}_{\mathbf{D}^b_{\mathrm{coh}}(X\times X)}(\mathscr{O}_\Delta, \mathscr{O}_\Delta[i]) = \mathrm{Ext}^i_{X\times X}(\mathscr{O}_\Delta, \mathscr{O}_\Delta),$$

and the Hochschild homology is defined as

$$\begin{aligned} HH_i(X) &= \mathrm{Hom}_{\mathbf{D}^b_{\mathrm{coh}}(X)}(\Delta_!\mathscr{O}_X[i], \mathscr{O}_\Delta) = \mathrm{Ext}^{-i}_{X\times X}(\Delta_!\mathscr{O}_X, \mathscr{O}_\Delta) \\ &= \mathrm{Ext}^{-i}_{X\times X}(S^{-1}_{X\times X} \otimes S_\Delta, \mathscr{O}_\Delta) \\ &= \mathrm{Ext}^{-i}_{X\times X}(S_\Delta^{-1}, \mathscr{O}_\Delta). \end{aligned}$$

4.3. This is a compact definition of the Hochschild groups, but for completeness we include a discussion of the relationship of our definition with the classical definitions of Weibel [34]. For further details on the cohomology side see [32].

The bar resolution is defined to be the complex of quasi-coherent sheaves of $\mathscr{O}_{X\times X}$-modules

$$\cdots \to \mathscr{O}_X^{\otimes n} \to \cdots \to \mathscr{O}_X \otimes_{\mathbf{C}} \mathscr{O}_X \otimes_{\mathbf{C}} \mathscr{O}_X \to \mathscr{O}_X \otimes_{\mathbf{C}} \mathscr{O}_X \to 0,$$

with $\mathscr{O}_{X\times X}$-module structure on $\mathscr{O}_X^{\otimes n}$ given by multiplication in the first and last factors, and with differential

$$\begin{aligned} & d(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \\ & \qquad a_0 a_1 \otimes a_2 \otimes \cdots \otimes a_n \,-\, a_0 \otimes a_1 a_2 \otimes \cdots \otimes a_n \,+\, \cdots \,+ \\ & \qquad (-1)^{n-1} a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} a_n. \end{aligned}$$

It is a resolution of $\mathscr{O}_\Delta$ in $\mathfrak{Qcoh}(X)$ [23, 1.1.12].

Hochschild cohomology is defined by Weibel by taking this resolution, applying the functor $\underline{\mathrm{Hom}}_{X\times X}(\,\cdot\,, \mathscr{O}_\Delta)$, and then taking hypercohomology of the resulting complex. Since the bar resolution is a resolution by free $\mathscr{O}_X$-modules, applying $\underline{\mathrm{Hom}}_{X\times X}(\,\cdot\,, \mathscr{O}_\Delta)$ and taking hypercohomology amounts to computing the complex

$$\mathbf{R}\Gamma(X \times X, \mathbf{R}\underline{\mathrm{Hom}}_{X\times X}(\mathscr{O}_\Delta, \mathscr{O}_\Delta)) = \mathbf{R}\mathrm{Hom}_{X\times X}(\mathscr{O}_\Delta, \mathscr{O}_\Delta),$$

whose i-th cohomology group is precisely $\mathrm{Ext}^i_{X\times X}(\mathscr{O}_\Delta, \mathscr{O}_\Delta)$, which is our definition of $HH^i(X)$.

Similarly, $HH_i(X)$ is usually defined by taking the bar resolution, applying the functor $-\otimes_{X\times X} \mathscr{O}_\Delta$, and then taking hypercohomology of the resulting complex, thought of as a complex of $\mathscr{O}_X$-modules by multiplication in the $\mathscr{O}_\Delta$ factor. (The complex obtained by tensoring the bar resolution with $\mathscr{O}_\Delta$ is usually referred to as the *bar complex*.) In derived category language this is equivalent to computing

$$\mathbf{R}\Gamma(X, \Delta^*\mathscr{O}_\Delta) = \mathbf{R}\mathrm{Hom}_X(\mathscr{O}_X, \Delta^*\mathscr{O}_\Delta) = \mathbf{R}\mathrm{Hom}_{X\times X}(\Delta_!\mathscr{O}_X, \mathscr{O}_\Delta).$$

Hence the i-th homology group of $\mathbf{R}\Gamma(X, \Delta^*\mathscr{O}_\Delta)$ (which is the classic definition of $HH_i(X)$) is naturally isomorphic to the i-th homology (or $(-i)$-th cohomology) group of

$$\mathbf{R}\mathrm{Hom}_{X\times X}(\Delta_!\mathscr{O}_X, \mathscr{O}_\Delta).$$

which is our definition of $HH_i(X)$.

4.4. An alternative way of defining $HH_*(X)$ is to take the exact category $\mathfrak{Coh}(X)$ of coherent sheaves on X and to apply Keller's construction [19], which yields Hochschild homology. This provides an alternative easy way to define Hochschild homology for arbitrary orbifolds: the usual notion of an orbibundle generalizes immediately to that of a coherent orbisheaf, and these form an abelian category. Applying Keller's construction to this abelian category yields a definition of Hochschild homology for an arbitrary orbifold. A similar approach also works for the abelian category of twisted sheaves.

4.5. In the affine case, the idea of defining Hochschild homology as an Ext group appears also in [33] (where it is applied to the study of Gorenstein rings, which are precisely the rings for which Serre duality works as for smooth schemes).

4.6. **Degree Bounds.** The following result shows that homology and cohomology are non-zero only in certain dimensions.

Lemma 4.2. *If Δ is a locally complete intersection in $X \times X$ (in particular, if X is smooth), then*

$$\mathscr{H}_i(\Delta^* \mathscr{O}_\Delta) = 0$$

for $i < 0$ or $i > \dim X$, where $\mathscr{H}_i(\Delta^* \mathscr{O}_\Delta)$ denotes the i-th homology sheaf of the complex $\Delta^* \mathscr{O}_\Delta$.

Proof. The sheaf $\mathscr{H}_i(\Delta^* \mathscr{O}_\Delta)$ can be identified with $\mathrm{Tor}_i^{X \times X}(\mathscr{O}_\Delta, \mathscr{O}_\Delta)$. If $\mathscr{O}_\Delta$ is a locally complete intersection, this can be computed from the Koszul resolution, which has length $\dim X$. The result follows. □

From the lemma it follows immediately that cohomology lives in degrees $0 \leq i \leq 2n$ and homology can be non-zero only for $-n \leq i \leq n$, where $n = \dim X$. Indeed,

$$\mathbf{R}\mathrm{Hom}_{X \times X}(\mathscr{O}_\Delta, \mathscr{O}_\Delta) \cong \mathbf{R}\mathrm{Hom}_X(\Delta^* \mathscr{O}_\Delta, \mathscr{O}_X),$$

and the Grothendieck spectral sequence computing the right-hand side of the above equality will only have non-zero terms ${}^2E^{pq}$ in the square $0 \leq p, q \leq n$. Similarly, the spectral sequence computing

$$\mathbf{R}\mathrm{Hom}_X(\mathscr{O}_X, \Delta^* \mathscr{O}_\Delta)$$

(which yields Hochschild homology) will only have non-zero terms for $0 \leq p \leq n$, $-n \leq q \leq 0$.

4.7. **Ring-Module Structure.** Cohomology is naturally a graded ring, with product given by Yoneda composition, and homology is a graded left $HH^*(X)$-module with the same action. The graded structure is given by the composition maps

$$HH^i(X) \otimes HH_j(X) \to HH_{j-i}(X).$$

4.8. **Mukai Product.** Homology is equipped with a non-degenerate inner product (the *Mukai product*)

$$\langle \cdot, \cdot \rangle : HH_*(X) \otimes HH_*(X) \to \mathbf{C},$$

which pairs $HH_i(X)$ with $HH_{-i}(X)$. In order to define it, consider the contravariant functor

$${}^! : \mathbf{D}^b_{\mathrm{coh}}(X \times X) \to \mathbf{D}^b_{\mathrm{coh}}(X \times X)$$

given by

$$\mathscr{F} \mapsto \mathscr{F}^! = \rho(\mathscr{F}^\vee \otimes \pi_1^* S_X),$$

where

$$\mathscr{F}^\vee = \mathbf{R}\underline{\mathrm{Hom}}_{X \times X}(\mathscr{F}, \mathscr{O}_{X \times X}).$$

and ρ is the involution on $X \times X$ that interchanges the two factors. Since every object in $\mathbf{D}^b_{\mathrm{coh}}(X \times X)$ is quasi-isomorphic to a finite complex of locally free sheaves of finite rank, [15, II.5.16] shows that the functor ${}^!$ induces an isomorphism

$$\tau : \mathrm{Hom}_{X \times X}(\mathscr{F}, \mathscr{G}) \xrightarrow{\sim} \mathrm{Hom}_{X \times X}(\mathscr{G}^!, \mathscr{F}^!),$$

for $\mathscr{F}, \mathscr{G} \in \mathbf{D}^b_{\mathrm{coh}}(X \times X)$.

4.9. If we take $\mathscr{F} = S_\Delta^{-1}[i]$ and $\mathscr{G} = \mathscr{O}_\Delta$, then $\mathscr{G}^! = \mathscr{O}_\Delta$ and $\mathscr{F}^! = S_\Delta[-i]$. Indeed, we have

$$\mathscr{O}_\Delta^\vee = \mathbf{R}\underline{\mathrm{Hom}}_{X\times X}(\Delta_*\mathscr{O}_X, \mathscr{O}_{X\times X}) = \Delta_*\mathbf{R}\underline{\mathrm{Hom}}_X(\mathscr{O}_X, \Delta^!\mathscr{O}_{X\times X}) = \Delta_* S_X^{-1},$$

where $\Delta^! = S_X \Delta^* S_{X\times X}^{-1}$ is the right adjoint of Δ_* ([15, III.11.1]), and thus

$$\mathscr{O}_\Delta^! = \rho(\Delta_* S_X^{-1} \otimes \pi_1^* S_X) = \rho(\mathscr{O}_\Delta) = \mathscr{O}_\Delta,$$

and similarly for $\mathscr{F}^!$.

Thus τ is an isomorphism between

$$HH_i(X) = \mathrm{Hom}_{X\times X}(S_\Delta^{-1}[i], \mathscr{O}_\Delta)$$

and $\mathrm{Hom}_{X\times X}(\mathscr{O}_\Delta, S_\Delta[-i])$, which is the Serre dual of $HH_{-i}(X)$,

$$\begin{aligned}\mathrm{Hom}_{X\times X}(\mathscr{O}_\Delta, S_\Delta[-i]) &= \mathrm{Hom}_{X\times X}(\mathscr{O}_\Delta, S_{X\times X} S_\Delta^{-1}[-i]) = \mathrm{Hom}_{X\times X}(S_\Delta^{-1}[-i], \mathscr{O}_\Delta)^\vee \\ &= HH_{-i}(X)^\vee.\end{aligned}$$

Definition 4.3. The non-degenerate pairing

$$HH_i(X) \otimes HH_{-i}(X) \to \mathbf{C}$$

given by

$$\langle v, w\rangle = \mathrm{Tr}_{X\times X}(\tau(v)\circ w).$$

is called the *generalized Mukai pairing*. Note that it is not symmetric in general.

4.10. For a more intuitive (but not fully precise) introduction to the Hochschild structure, the reader is suggested to consult Appendices A and B.

4.11. In the particular case $\mathscr{F} = S_\Delta^{-1}$, $\mathscr{G} = \mathscr{O}_\Delta$, (3.4) provides a better understanding of the isomorphism

$$\tau : \mathrm{Hom}_{X\times X}(S_\Delta^{-1}, \mathscr{O}_\Delta) \xrightarrow{\sim} \mathrm{Hom}_{X\times X}(\mathscr{O}_\Delta, S_\Delta).$$

Indeed, τ will map $\mu : S_\Delta^{-1}[i] \to \mathscr{O}_\Delta$ to

$$\tau(\mu) : \mathscr{O}_\Delta \xrightarrow{\bar\eta} S_\Delta[-i]\circ S_\Delta^{-1}[i]\circ \mathscr{O}_\Delta \xrightarrow{\mu} S_\Delta[-i]\circ\mathscr{O}_\Delta\circ\mathscr{O}_\Delta \xrightarrow{\bar\epsilon} S_\Delta[-i],$$

where $\bar\eta$ and $\bar\epsilon$ are the "unit" and "counit" of the "adjunctions" $S_\Delta^{-1} \dashv S_\Delta$, $\mathscr{O}_\Delta \dashv \mathscr{O}_\Delta$, respectively (see Proposition 5.1 for the precise meaning of this unit and counit). But we have

$$S_\Delta[-i]\circ S_\Delta^{-1}[i]\circ\mathscr{O}_\Delta \cong \mathscr{O}_\Delta,$$

and thus $\bar\eta$ is a map $\mathscr{O}_\Delta \to \mathscr{O}_\Delta$, which is obviously the identity. Similarly $\bar\epsilon$ is seen to be the identity under the identification

$$S_\Delta[-i]\circ\mathscr{O}_\Delta\circ\mathscr{O}_\Delta \cong S_\Delta[-i].$$

We conclude that $\tau(\mu)$ is nothing else than $\mu\otimes\pi_2^* S_X$, under the obvious identifications. (We use π_2 because all the S_Δ's appear on the left.)

Observe that an identification similar to the one in (3.4) could be made using *left* adjoints instead of right ones. This gives another isomorphism

$$\bar\tau : \mathrm{Hom}_{X\times X}(S_\Delta^{-1}[i], \mathscr{O}_\Delta) \to \mathrm{Hom}_{X\times X}(\mathscr{O}_\Delta, S_\Delta[-i]),$$

which is easily seen to be multiplication by $\pi_1^* S_X$.

5. Functoriality of homology

We present in this section the construction of a map of graded vector spaces

$$\Phi_* : HH_*(X) \to HH_*(Y)$$

associated to an integral transform $\Phi : \mathbf{D}^b_{\mathrm{coh}}(X) \to \mathbf{D}^b_{\mathrm{coh}}(Y)$. This construction is natural in the sense that to the identity functor we associate the identity map on homology, and $(\Phi \circ \Psi)_* = \Phi_* \circ \Psi_*$ for composable integral transforms Φ and Ψ. It is worth pointing out that, despite its name, Hochschild cohomology is not functorial in any reasonable sense.

5.1. Let $\Phi : \mathbf{D}^b_{\mathrm{coh}}(X) \to \mathbf{D}^b_{\mathrm{coh}}(Y)$ be an exact functor which admits a left adjoint (for example, any integral transform). Given an element $\mu \in HH_*(X)$ we want to define $\Phi_*\mu$ in a way that would be natural with respect to Φ.

Let us begin with the categorical interpretation, where things are easier. Recall that in Section 3 we constructed a natural map

$$\Phi^\dagger : \mathrm{Nat}(\Phi, S_Y\Phi\Psi) \to \mathrm{Nat}(1_X, S_X\Psi).$$

If we take Ψ to be the shift functor $[i]$, there is a natural restriction map

$$\mathrm{Nat}(1_Y, S_Y[i]) \to \mathrm{Nat}(\Phi, S_Y\Phi[i]),$$

and composing we get a map

$$\Phi^\dagger : \mathrm{Nat}(1_Y, S_Y[i]) \to \mathrm{Nat}(1_X, S_X[i]).$$

The defining property of $\Phi^\dagger$ is the equality

$$\mathrm{Tr}_X((\Phi^\dagger\nu)_{\mathscr{F}} \circ \mu) = \mathrm{Tr}_Y(\nu_{\Phi\mathscr{F}} \circ \Phi\mu)$$

for any $\nu \in \mathrm{Nat}(1_Y, S_Y[i])$ and $\mu : \mathscr{F}[i] \to \mathscr{F}$. In particular, if $\nu \in \mathrm{Nat}(1_Y, S_Y)$ and we take $\mu = \mathrm{id}_{\mathscr{F}}$, we have

$$\mathrm{Tr}_X((\Phi^\dagger\nu)_{\mathscr{F}}) = \mathrm{Tr}_Y(\nu_{\Phi\mathscr{F}}).$$

To construct the map $\Phi_* : HH_i(X) \to HH_i(Y)$ we would want to take the adjoint of $\Phi^\dagger$ (recall that homology is thought of as the dual of $\mathrm{Nat}(1_Y, S_Y)$ with respect to Serre duality of natural transformations). Unfortunately we do not know how to make this precise, hence we need to switch to the "Ext" interpretation.

5.2. We want to use Proposition 3.1 to rewrite the above definition in a way that generalizes to the "Ext" interpretation. Indeed, we want to find a map

$$\Phi_* : \mathrm{Hom}_{X\times X}(S^{-1}_{\Delta_X}[i], \mathscr{O}_{\Delta_X}) \to \mathrm{Hom}_{Y\times Y}(S^{-1}_{\Delta_Y}[i], \mathscr{O}_{\Delta_Y}),$$

and not just a map on natural transformations.

By Proposition 3.1 we see that for $\nu \in \mathrm{Nat}(1_Y, S_Y[i])$, $\Phi^\dagger\nu$ can be written as the composite

$$\Phi^\dagger\nu : 1_X \overset{\bar\eta}{\Longrightarrow} \Phi^!\Phi \overset{\nu}{\Longrightarrow} \Phi^! S_Y\Phi[i] = S_X\Phi^*\Phi[i] \overset{\epsilon}{\Longrightarrow} S_X[i],$$

where $\bar\eta$ and ϵ are the unit and counit of the respective adjunctions.

Assume that Φ is an integral transform, given by an object $\mathscr{F} \in \mathbf{D}^b_{\mathrm{coh}}(X \times Y)$, and define

$$\mathscr{G} = \mathscr{F}^\vee \otimes \pi_Y^* S_Y, \quad \mathscr{H} = \mathscr{F}^\vee \otimes \pi_X^* S_X.$$

Then by [6, Lemma 4.5],

$$\Phi^* = \Phi^{\mathscr{G}}_{Y\to X}, \quad \Phi^! = \Phi^{\mathscr{H}}_{Y\to X}$$

are left and right adjoints of Φ, respectively.

Proposition 5.1. *There exist natural morphisms* $\bar{\eta} : \mathscr{O}_{\Delta_X} \to \mathscr{H} \circ \mathscr{F}$ *and* $\epsilon : \mathscr{F} \circ \mathscr{G} \to \mathscr{O}_{\Delta_Y}$ *that correspond to*

$$\bar{\eta} : 1_X \Rightarrow \Phi^! \circ \Phi \quad \text{ and } \quad \epsilon : \Phi^* \circ \Phi \Rightarrow 1_Y$$

under the correspondence between morphisms between objects on a product and natural transformations of the underlying functors.

Proof. Let π_{ij} be the projection from $X \times Y \times X$ onto the i-th and j-th factors, so that

$$\mathscr{H} \circ \mathscr{F} = \pi_{13,*}(\pi_{12}^* \mathscr{F} \otimes \pi_{23}^* \mathscr{H}).$$

Also, let

$$\Delta : X \times Y \to X \times Y \times X$$

be the map that on points is given by

$$(x, y) \mapsto (x, y, x).$$

Then we have

$$\begin{aligned}
\mathrm{Hom}_{X\times X}(\mathscr{O}_{\Delta_X}, \mathscr{H} \circ \mathscr{F}) &= \mathrm{Hom}_{X\times X}(\mathscr{O}_{\Delta_X}, \pi_{13,*}(\pi_{12}^* \mathscr{F} \otimes \pi_{23}^* \mathscr{H})) \\
&= \mathrm{Hom}_{X\times Y\times X}(\pi_{13}^* \mathscr{O}_{\Delta_X}, \pi_{12}^* \mathscr{F} \otimes \pi_{23}^* \mathscr{H}) \\
&= \mathrm{Hom}_{X\times Y\times X}(\Delta_* \mathscr{O}_{X\times Y}, \pi_{12}^* \mathscr{F} \otimes \pi_{23}^* \mathscr{H}) \\
&= \mathrm{Hom}_{X\times Y}(\mathscr{O}_{X\times Y}, \Delta^!(\pi_{12}^* \mathscr{F} \otimes \pi_{23}^*(\mathscr{F}^\vee \otimes \pi_X^* S_X))) \\
&= \mathrm{Hom}_{X\times Y}(\mathscr{O}_{X\times Y}, \mathscr{F} \otimes \mathscr{F}^\vee) \\
&= \mathrm{Hom}_{X\times Y}(\mathscr{F}, \mathscr{F}),
\end{aligned}$$

and we take $\bar{\eta}$ to be the image of the identity morphism of $\mathscr{F}$ under the above isomorphism.

The construction of ϵ is entirely similar and will be left to the reader. □

Definition 5.2. Given a morphism $\nu : \mathscr{O}_{\Delta_Y} \to S_{\Delta_Y}[i]$ define $\Phi^\dagger \nu : \mathscr{O}_{\Delta_X} \to S_{\Delta_X}[i]$ to be the composite morphism

$$\mathscr{O}_{\Delta_X} \xrightarrow{\bar{\eta}} \mathscr{H} \circ \mathscr{F} = \mathscr{H} \circ \mathscr{O}_{\Delta_Y} \circ \mathscr{F} \xrightarrow{\nu} \mathscr{H} \circ S_{\Delta_Y}[i] \circ \mathscr{F} = S_{\Delta_X}[i] \circ \mathscr{G} \circ \mathscr{F} \xrightarrow{\epsilon} S_{\Delta_X}[i],$$

where $\bar{\eta}$ and ϵ are the maps defined in Proposition 5.1. Define

$$\Phi_* : HH_*(X) \to HH_*(Y)$$

as the right adjoint to the map $\Phi^\dagger$ with respect to Serre duality on $X \times X$ and on $Y \times Y$, i.e., for $\mu \in HH_*(X)$, $\Phi_*\mu$ is the unique element in $HH_*(Y)$ such that

$$\mathrm{Tr}_{X\times X}(\Phi^\dagger \nu \circ \mu) = \mathrm{Tr}_{Y\times Y}(\nu \circ \Phi_* \mu)$$

for every $\nu \in \mathrm{Hom}^*_{Y\times Y}(\mathscr{O}_{\Delta_Y}, S_{\Delta_Y})$.

5.3. The following theorem summarizes the functoriality properties of this construction:

Theorem 5.3. *The map on homology associated to the identity functor is the identity, and if* $\Psi : \mathbf{D}^b_{\mathrm{coh}}(X) \to \mathbf{D}^b_{\mathrm{coh}}(Y)$ *and* $\Phi : \mathbf{D}^b_{\mathrm{coh}}(Y) \to \mathbf{D}^b_{\mathrm{coh}}(Z)$ *are integral transforms then we have*

$$(\Phi \circ \Psi)_* = \Phi_* \circ \Psi_*.$$

Proof. Follows easily from the observation that if Φ^*, Ψ^* are left adjoints to Φ and Ψ, then $\Psi^* \circ \Phi^*$ is a left adjoint to $\Phi \circ \Psi$, and similarly for right adjoint. Also, the obvious relations between units and counits hold. This proves the result at a categorical level, and we leave to the patient reader the task of checking that the corresponding compatibilities hold in the Ext interpretation. □

6. The Chern character

In this section we define the Chern character map $\mathrm{ch} : K_0(X) \to HH_0(X)$. We also discuss an equivalent construction of Markarian [24, Definition 2].

6.1. Let $p : X \to \mathrm{pt}$ be the structure map of X, and let $\mathscr{F}$ be any object in $\mathbf{D}^b_{\mathrm{coh}}(X)$. Consider the functor $\Phi = \Phi^{\mathscr{F}}_{\mathrm{pt}\to X}$ defined by $\mathscr{F}$, and observe that we have $\Phi(\mathscr{O}_{\mathrm{pt}}) = \mathscr{F}$. By the results in Section 5, the integral transform Φ induces a map on homology

$$\Phi_* : HH_*(\mathrm{pt}) \to HH_*(X).$$

Observe that $HH_0(\mathrm{pt}) = \mathrm{Hom}_{\mathrm{pt}\times\mathrm{pt}}(\mathscr{O}_{\mathrm{pt}}, \mathscr{O}_{\mathrm{pt}})$ has a distinguished element $\mathbf{1}$ given by the identity (we use the fact that $S_{\mathrm{pt}} = \mathscr{O}_{\mathrm{pt}}$).

Definition 6.1. Define the Chern character of $\mathscr{F}$, $\mathrm{ch}(\mathscr{F})$, by

$$\mathrm{ch}(\mathscr{F}) = (\Phi^{\mathscr{F}}_{\mathrm{pt}\to X})_*(\mathbf{1}) \in HH_0(X).$$

6.2. Since this definition is slightly hard to work with, we unravel it to a more usable version. Recall that in (5.1) we defined a map

$$\Phi^\dagger : \mathrm{Hom}_{X\times X}(\mathscr{O}_{\Delta_X}, S_{\Delta_X}) \to \mathrm{Hom}_{\mathrm{pt}\times\mathrm{pt}}(\mathscr{O}_{\Delta_{\mathrm{pt}}}, S_{\Delta_{\mathrm{pt}}}).$$

Applying its defining property with $\mu = \mathrm{id}_{\mathscr{O}_{\mathrm{pt}}}$, we get

$$\mathrm{Tr}_{\mathrm{pt}\times\mathrm{pt}}(\Phi^\dagger\nu) = \mathrm{Tr}_X(\nu_{\Phi\mathscr{O}_{\mathrm{pt}}}) = \mathrm{Tr}_X(\nu_{\mathscr{F}})$$

for any $\nu \in \mathrm{Hom}_{X\times X}(\mathscr{O}_\Delta, S_\Delta)$. (Here $\nu_{\mathscr{F}} : \mathscr{F} \to S_X\mathscr{F}$ is the value at $\mathscr{F}$ of the natural transformation induced by ν.) The map Φ_* that we are interested in is the adjoint of $\Phi^\dagger$ with respect to Serre duality on $X \times X$ and $\mathrm{pt} \times \mathrm{pt}$, respectively. Explicitly, we must have the equality

$$\mathrm{Tr}_{X\times X}(\nu \circ \Phi_*\mathbf{1}) = \mathrm{Tr}_{\mathrm{pt}\times\mathrm{pt}}(\Phi^\dagger\nu \circ \mathbf{1})$$

and thus since $\mathbf{1}$ is nothing but the identity, we conclude that we must have

$$\mathrm{Tr}_{X\times X}(\nu \circ \mathrm{ch}(\mathscr{F})) = \mathrm{Tr}_{\mathrm{pt}\times\mathrm{pt}}(\Phi^\dagger\nu) = \mathrm{Tr}_X(\nu_{\mathscr{F}}).$$

6.3. We rewrite the above definition as follows: a homomorphism $\nu : \mathscr{O}_\Delta \to S_\Delta$ in $\mathbf{D}^b_{\mathrm{coh}}(X \times X)$ induces a natural transformation

$$\iota(\nu) : 1_X \Longrightarrow S_X$$

between the identity functor and the Serre functor on $\mathbf{D}^b_{\mathrm{coh}}(X)$. Thus for every $\mathscr{F} \in \mathbf{D}^b_{\mathrm{coh}}(X)$ we get a map

$$\mathrm{Hom}_{X\times X}(\mathscr{O}_\Delta, S_\Delta) \xrightarrow{\iota_{\mathscr{F}}} \mathrm{Hom}_X(\mathscr{F}, S_X\mathscr{F}),$$

whose left adjoint with respect to the Serre duality pairing we denote by $\iota^{\mathscr{F}}$:

$$HH_0(X) = \mathrm{Hom}_{X\times X}(S_\Delta \otimes S^{-1}_{X\times X}, \mathscr{O}_\Delta) \xleftarrow{\iota^{\mathscr{F}}} \mathrm{Hom}_X(\mathscr{F}, \mathscr{F}).$$

With these notations, the above calculations reduce to the following equivalent definition of $\mathrm{ch}(\mathscr{F})$, similar to one given by Markarian [24]:

Definition 6.2. The Chern character of $\mathscr{F}$ is defined as the image

$$\operatorname{ch}(\mathscr{F}) = \iota^{\mathscr{F}}(\mathrm{id}_{\mathscr{F}}) \in HH_0(X)$$

of the identity morphism of $\mathscr{F}$ in $HH_0(X)$ under $\iota^{\mathscr{F}}$. Explicitly, $\operatorname{ch}(\mathscr{F})$ is the unique element of $HH_0(X)$ such that

$$\operatorname{Tr}_{X\times X}(\nu \circ \operatorname{ch}(\mathscr{F})) = \operatorname{Tr}_X(\iota_{\mathscr{F}}(\nu)) = \operatorname{Tr}_X(\pi_{2,*}(\pi_1^*\mathscr{F} \otimes \nu))$$

for all $\nu \in \operatorname{Hom}_{\mathbf{D}^b_{\mathrm{coh}}(X\times X)}(\mathscr{O}_\Delta, S_\Delta)$.

Under the Hochschild-Kostant-Rosenberg isomorphism this definition of $\operatorname{ch}(\mathscr{F})$ agrees with the usual one [9, Theorem 4.5].

6.4. The following proposition shows that the map $\operatorname{ch} : \mathbf{D}^b_{\mathrm{coh}}(X) \to HH_0(X)$ factors through $\mathbf{D}^b_{\mathrm{coh}}(X) \to K_0(X)$ to yield the desired Chern character map

$$\operatorname{ch} : K_0(X) \to HH_0(X).$$

Proposition 6.3. *If $\mathscr{F} \to \mathscr{G} \to \mathscr{H} \to \mathscr{F}[1]$ is an exact triangle in $\mathbf{D}^b_{\mathrm{coh}}(X)$, then*

$$\operatorname{ch}(\mathscr{F}) - \operatorname{ch}(\mathscr{G}) + \operatorname{ch}(\mathscr{H}) = 0$$

in $HH_0(X)$.

Proof. For any $\nu \in \operatorname{Hom}_{X\times X}(\mathscr{O}_\Delta, S_\Delta)$, $\iota(\nu)$ is a natural transformation, and as such it gives a map of triangles

$$\begin{tikzcd}
\mathscr{F} \arrow[r] \arrow[d, "\iota_{\mathscr{F}}(\nu)"] & \mathscr{G} \arrow[r] \arrow[d, "\iota_{\mathscr{G}}(\nu)"] & \mathscr{H} \arrow[r] \arrow[d, "\iota_{\mathscr{H}}(\nu)"] & \mathscr{F}[1] \arrow[d, "\iota_{\mathscr{F}}(\nu)[1]"] \\
S_X\mathscr{F} \arrow[r] & S_X\mathscr{G} \arrow[r] & S_X\mathscr{H} \arrow[r] & S_X\mathscr{F}[1].
\end{tikzcd}$$

Observe that if we represent ν by an actual map of complexes of injectives, and $\mathscr{F}$, $\mathscr{G}$, $\mathscr{H}$ by complexes of locally free sheaves, then the resulting maps in the above diagram commute on the nose (no further injective or locally free resolutions are needed), so we can apply Proposition 2.3 to get

$$\operatorname{Tr}_X(\iota_{\mathscr{F}}(\nu)) - \operatorname{Tr}_X(\iota_{\mathscr{G}}(\nu)) + \operatorname{Tr}_X(\iota_{\mathscr{H}}(\nu)) = 0.$$

Therefore

$$\operatorname{Tr}_{X\times X}(\nu \circ (\operatorname{ch}(\mathscr{F}) - \operatorname{ch}(\mathscr{G}) + \operatorname{ch}(\mathscr{H}))) = 0$$

for any $\nu \in \operatorname{Hom}_{X\times X}(\mathscr{O}_\Delta, S_\Delta)$; since the Serre duality pairing between $\operatorname{Hom}_{X\times X}(\mathscr{O}_\Delta, S_\Delta)$ and $HH_0(X)$ is non-degenerate, we conclude that

$$\operatorname{ch}(\mathscr{F}) - \operatorname{ch}(\mathscr{G}) + \operatorname{ch}(\mathscr{H}) = 0.$$

□

Example 6.4. To have a non-commutative example at hand, consider the case when G is a finite group, acting trivially on a point. The resulting orbifold $BG = [\cdot/G]$ can be thought of as $\operatorname{Spec} R$ where $R = \mathbf{C}[G]$, the group ring of G. Indeed, a coherent sheaf on BG, which by definition is a finite-dimensional representation of G, is precisely the same thing as a module over $\mathbf{C}[G]$. The Serre functor on BG is trivial, and $BG \times BG$ should be thought of as $\operatorname{Spec}(R\otimes R^{\mathrm{o}})$, with $\mathscr{O}_\Delta$ represented by R as a module over $R \otimes R^{\mathrm{o}}$ by left and right multiplication (R^{o} denotes the opposite ring of R). Thus

$$HH_0(BG) = \operatorname{Hom}_{R\otimes R^{\mathrm{o}}}(R, R) = Z(R),$$

where $Z(R)$ represents the center of R (composition of morphisms in $\mathrm{Hom}(R,R)$ is the same as multiplication in $Z(R)$ under the identification). Thus the Chern character is a map from $K_0(\mathrm{Rep}(G))$ to the center $Z(R)$ of the group ring.

To understand this map, let $f \in Z(R)$ and V be a representation of G (i.e., a right R-module). The map $\iota^V(f) : V \to V$ is multiplication by f on the right. The Chern character of V, $\mathrm{ch}(V)$, is by definition the unique element $e_V \in Z(R)$ such that

$$\mathrm{Tr}_{BG\times BG}(e_V \cdot f) = \mathrm{Tr}_{BG}(\iota^V(f))$$

for all $f \in Z(R)$. The left hand side is $\chi_{V_{\mathrm{reg}}}(e_V \cdot f)$, the value at $e_V \cdot f$ of the character $\chi_{V_{\mathrm{reg}}}$ of the regular representation $V_{\mathrm{reg}} = R$ of G, and the right hand side is the value of χ_V at f. Recall that R, being semisimple, is isomorphic to the direct sum of the endomorphism algebras $\mathrm{End}(V_i)$ over a set of representatives $\{V_i\}$ of isomorphism classes of irreducible representations of G (Wedderburn's theorem). Let $\{e_i\}$ be the orthogonal set of idempotents corresponding to this decomposition. Then it is obvious from the fact that multiplication by e_i is the projection on the $\mathrm{End}(V_i)$ component that we have

$$\chi_{V_{\mathrm{reg}}}(e_i \cdot f) = \chi_{V_i}(f),$$

for any $f \in Z(R)$, and thus it follows that

$$\mathrm{ch}(V_i) = e_i.$$

By semisimplicity this computes the value of the Chern character of any representation.

The explicit value of $\mathrm{ch}(V_i)$ can be found in [17, 2.12]:

$$\mathrm{ch}(V_i) = \frac{1}{|G|}\sum_{g\in G} \chi_{V_i}(1)\chi_{V_i}(g^{-1})g.$$

7. Properties of the structure

In this section we argue that properties b, c and d of the original Mukai construction hold if we replace $H^*(X,\mathbf{C})$ with Hochschild homology and v by ch.

7.1. The commutativity of Φ_* and ch is the content of the following theorem:

Theorem 7.1. *The following diagram commutes for any integral transform $\Phi : \mathbf{D}^b_{\mathrm{coh}}(X) \to \mathbf{D}^b_{\mathrm{coh}}(Y)$:*

$$\begin{array}{ccc}
\mathbf{D}^b_{\mathrm{coh}}(X) & \xrightarrow{\Phi} & \mathbf{D}^b_{\mathrm{coh}}(Y) \\
\downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} \\
HH_0(X) & \xrightarrow{\Phi_*} & HH_0(Y).
\end{array}$$

Proof. We use the first of the two equivalent definitions of ch given in Section 6. Let $\mathscr{E}$ be an object of $\mathbf{D}^b_{\mathrm{coh}}(X)$, and let $\mathscr{F} = \Phi\mathscr{E}$. Observe that we have

$$\Phi \circ \Phi^{\mathscr{E}}_{\mathrm{pt}\to X} = \Phi^{\mathscr{F}}_{\mathrm{pt}\to Y},$$

since any functor from $\mathbf{D}^b_{\mathrm{coh}}(\mathrm{pt})$ is determined by its value at $\mathscr{O}_{\mathrm{pt}}$. By Theorem 5.3 we have

$$\begin{aligned}
\Phi_*\,\mathrm{ch}(\mathscr{E}) &= \Phi_*[(\Phi^{\mathscr{E}}_{\mathrm{pt}\to X})_*\mathbf{1}] = (\Phi \circ \Phi^{\mathscr{E}}_{\mathrm{pt}\to X})_*\mathbf{1} \\
&= (\Phi^{\mathscr{F}}_{\mathrm{pt}\to Y})_*\mathbf{1} = \mathrm{ch}(\mathscr{F}) = \mathrm{ch}(\Phi\mathscr{E}).
\end{aligned}$$

∎

7.2. We now move on to adjoint properties of maps on homology induced by adjoint functors.

Proposition 7.2. *Let $\mu : \mathscr{F} \to S_{X\times Y}\mathscr{F} = S_{\Delta_Y} \circ \mathscr{F} \circ S_{\Delta_X}$, and let μ' be the composite morphism*

$$\mu' : \mathscr{O}_{\Delta_X} \xrightarrow{\bar{\eta}} \mathscr{H} \circ \mathscr{F} \xrightarrow{\mathscr{H}\circ\mu} \mathscr{H} \circ S_{\Delta_Y} \circ \mathscr{F} \circ S_{\Delta_X} \cong S_{\Delta_X} \circ \mathscr{G} \circ \mathscr{F} \circ S_{\Delta_X} \xrightarrow{S_{\Delta_X}\circ\epsilon\circ S_{\Delta_X}} \\ \longrightarrow S_{\Delta_X} \circ S_{\Delta_X} = S_{X\times X}\mathscr{O}_{\Delta_X}.$$

Then

$$\mathrm{Tr}_{X\times Y}(\mu) = \mathrm{Tr}_{X\times X}(\mu').$$

Proof. Follows from a calculation entirely similar to that of Proposition 3.1 which is left to the reader. □

Theorem 7.3. *Let $\Psi : \mathbf{D}^b_{\mathrm{coh}}(Y) \to \mathbf{D}^b_{\mathrm{coh}}(X)$ be a left adjoint to Φ. Then we have*

$$\langle v, \Phi_* w\rangle = \langle \Psi_* v, w\rangle$$

for $v \in HH_*(Y)$, $w \in HH_*(X)$.

Proof. We begin with the observation that it is enough to show that

$$\tau\Psi_* = \Phi^\dagger\tau : HH_*(Y) \to \mathrm{Hom}^*_{X\times X}(\mathscr{O}_{\Delta_X}, S_{\Delta_X}),$$

where τ is the map defined in (4.8). Indeed, if this equality holds, we have

$$\begin{aligned} \langle v, \Phi_* w\rangle &= \mathrm{Tr}_{Y\times Y}(\tau(v)\circ\Phi_* w) = \mathrm{Tr}_{X\times X}(\Phi^\dagger(\tau(v))\circ w) \\ &= \mathrm{Tr}_{X\times X}(\tau(\Psi_*(v))\circ w) = \langle \Psi_* v, w\rangle. \end{aligned}$$

We have observed in (4.11) that

$$\tau : \mathrm{Hom}_{X\times X}(S_\Delta^{-1}, \mathscr{O}_\Delta) \to \mathrm{Hom}_{X\times X}(\mathscr{O}_\Delta, S_\Delta)$$

is the isomorphism given by

$$\mu \mapsto \tau(\mu) = \mu \otimes \pi_2^* S_X,$$

and we considered a similar isomorphism

$$\bar{\tau} : \mathrm{Hom}_{X\times X}(S_\Delta^{-1}, \mathscr{O}_\Delta) \to \mathrm{Hom}_{X\times X}(\mathscr{O}_\Delta, S_\Delta)$$

given by

$$\mu \mapsto \bar{\tau}(\mu) = \mu \otimes \pi_1^* S_X,$$

which corresponds to choosing left adjoints in the definition of the Mukai product, instead of right adjoints, as we did in Definition 4.3. We extend this notation to simply mean that τ is the operation of tensoring with $\pi_2^* S_X$, and $\bar{\tau}$ is the similar operation that corresponds to $\pi_1^* S_X$.

Let

$$\begin{aligned} \alpha &= \tau(v) : \mathscr{O}_{\Delta_Y} \to S_{\Delta_Y}, \\ \beta &= \bar{\tau}(w) : \mathscr{O}_{\Delta_X} \to S_{\Delta_X}, \end{aligned}$$

and consider the morphisms

$$\begin{aligned} \mu_1 &: \mathscr{F} \xrightarrow{\mathscr{F}\circ\beta} \mathscr{F} \circ S_{\Delta_X} \xrightarrow{\alpha\circ\mathscr{F}\circ S_{\Delta_X}} S_{\Delta_Y} \circ \mathscr{F} \circ S_{\Delta_X}, \\ \mu_2 &: \mathscr{F} \xrightarrow{\alpha\circ\mathscr{F}} S_{\Delta_Y} \circ \mathscr{F} \xrightarrow{S_{\Delta_Y}\circ\mathscr{F}\circ\beta} S_{\Delta_Y} \circ \mathscr{F} \circ S_{\Delta_X}. \end{aligned}$$

By the commutativity of the trace (Lemma 2.2) it follows that

$$\mathrm{Tr}_{X\times Y}(\mu_1) = \mathrm{Tr}_{X\times Y}(\mu_2).$$

Now consider the commutative diagrams

$$
\begin{tikzcd}
1_X \arrow[r, "\bar\eta"] \arrow[d, "\beta"'] & \mathscr{H}\mathscr{F} \arrow[d, "\mathscr{H}\mathscr{F}\beta"] \\
S_X \arrow[r, "\bar\eta S_X"] \arrow[ddd, "(\Phi^\dagger\alpha)S_X"'] & \mathscr{H}\mathscr{F}S_X \arrow[d, "\mathscr{H}\alpha\mathscr{F}S_X"] \\
 & \mathscr{H}S_Y\mathscr{F}S_X \arrow[d, equal] \\
 & S_X\mathscr{G}\mathscr{F}S_X \arrow[d, "S_X\epsilon S_X"] \\
S_XS_X \arrow[r, equal] & S_XS_X
\end{tikzcd}
\qquad
\begin{tikzcd}
1_Y \arrow[r, "\eta"] \arrow[d, "\alpha"'] & \mathscr{F}\mathscr{G} \arrow[d, "\alpha\mathscr{F}\mathscr{G}"] \\
S_Y \arrow[r, "S_Y\eta"] \arrow[ddd, "S_Y(\Psi^\dagger\beta)"'] & S_Y\mathscr{F}\mathscr{G} \arrow[d, "S_Y\mathscr{F}\beta\mathscr{G}"] \\
 & S_Y\mathscr{F}S_X\mathscr{G} \arrow[d, equal] \\
 & S_Y\mathscr{F}\mathscr{H}S_Y \arrow[d, "S_Y\bar\epsilon S_Y"] \\
S_YS_Y \arrow[r, equal] & S_YS_Y,
\end{tikzcd}
$$

where we have omitted the $\circ$ signs, and we wrote S_X for S_{Δ_X} and 1_X for $\mathscr{O}_{\Delta_X}$.

Reading around the diagrams and using Proposition 7.2 we see that

$$
\begin{aligned}
\mathrm{Tr}_{X\times X}((\Phi^\dagger\alpha)S_X\circ\beta) &= \mathrm{Tr}_{X\times X}(S_X\epsilon S_X\circ\mathscr{H}\mu_1\circ\bar\eta)\\
&= \mathrm{Tr}_{X\times Y}(\mu_1) = \mathrm{Tr}_{X\times Y}(\mu_2)\\
&= \mathrm{Tr}_{Y\times Y}(S_Y\bar\epsilon S_Y\circ\mu_2\mathscr{G}\circ\eta)\\
&= \mathrm{Tr}_{Y\times Y}(S_Y(\Psi^\dagger\beta)\circ\alpha).
\end{aligned}
$$

Reverting to the τ, $\bar\tau$ notation (where multiplication by S_X or S_Y on the left corresponds to τ, and on the right to $\bar\tau$), we conclude that

$$
\mathrm{Tr}_{X\times X}(\bar\tau\Phi^\dagger\tau v\circ\bar\tau w) = \mathrm{Tr}_{Y\times Y}(\tau\Psi^\dagger\bar\tau w\circ\tau v),
$$

or, since τ, $\bar\tau$ are simply multiplication by a line bundle

$$
\begin{aligned}
\mathrm{Tr}_{X\times X}(\Phi^\dagger\tau v\circ w) &= \mathrm{Tr}_{Y\times Y}(\Psi^\dagger\bar\tau w\circ v)\\
&= \mathrm{Tr}_{Y\times Y}(\bar\tau w\circ\Psi_* v)\\
&= \mathrm{Tr}_{Y\times Y}(\tau\Psi_* v\circ w),
\end{aligned}
$$

where the second equality is the definition of Ψ_*, and the third one follows from the fact that $\tau\bar\tau = S_{Y\times Y}$ and Lemma 2.2.

Since w was arbitrary and the pairings are non-degenerate, we conclude that $\Phi^\dagger\tau = \tau\Psi_*$, and this completes the proof. □

Remark 7.4. Although we have

$$
\tau(\mathscr{O}_{\Delta_X}) = \bar\tau(\mathscr{O}_{\Delta_X}) = S_{\Delta_X},
$$

and

$$
\tau(S_{\Delta_X}) = \bar\tau(S_{\Delta_X}) = S_{X\times X}\mathscr{O}_{\Delta_X},
$$

the two isomorphisms

$$
\tau,\ \bar\tau : \mathrm{Hom}_{X\times X}(\mathscr{O}_{\Delta_X}, S_{\Delta_X}) \to \mathrm{Hom}_{X\times X}(S_{\Delta_X}, S_{X\times X}\mathscr{O}_{\Delta_X})
$$

are *different*. This can essentially be seen by looking at Chern characters. This is the reason we need to be careful about the distinction between τ and $\bar{\tau}$.

Corollary 7.5. *If $\Phi : \mathbf{D}^b_{\mathrm{coh}}(X) \to \mathbf{D}^b_{\mathrm{coh}}(Y)$ is an equivalence, then $\Phi_* : HH_*(X) \to HH_*(Y)$ is an isometry with respect to the generalized Mukai product.*

Proof. We have

$$\langle\, \Phi_* x, \Phi_* y \,\rangle = \langle\, \Psi_* \Phi_* x, y \,\rangle = \langle\, (\Psi \circ \Phi)_* x, y \,\rangle = \langle\, x, y \,\rangle,$$

where the last equality follows from the fact that if Φ is an equivalence, then its left adjoint Ψ is an inverse to it. □

7.3. The Hirzebruch-Riemann-Roch theorem is a consequence of the other properties:

Theorem 7.6. *For $\mathscr{E}, \mathscr{F} \in \mathbf{D}^b_{\mathrm{coh}}(X)$ we have*

$$\langle \mathrm{ch}(\mathscr{E}),\, \mathrm{ch}(\mathscr{F}) \rangle = \chi(\mathscr{E}, \mathscr{F}),$$

where $\chi(\,\cdot\,,\,\cdot\,)$ is the Euler pairing on $K_0(X)$,

$$\chi(\mathscr{E}, \mathscr{F}) = \sum_i (-1)^i \dim \mathrm{Ext}^i_X(\mathscr{E}, \mathscr{F}).$$

Proof. Let $p : X \to \mathrm{pt}$ be the structure morphism of X, and observe that $\mathscr{O}_X = p^* \mathscr{O}_{\mathrm{pt}}$. The functor p^* is left adjoint to p_*, and if Φ is the functor $\mathscr{E} \otimes -$, then its right adjoint Ψ is given by $\mathscr{E}^\vee \otimes -$.

Using the properties of the Mukai product and Chern character we get

$$\begin{aligned}
\langle \mathrm{ch}(\mathscr{E}),\, \mathrm{ch}(\mathscr{F}) \rangle &= \langle \mathrm{ch}(\mathscr{E} \otimes \mathscr{O}_X),\, \mathrm{ch}(\mathscr{F}) \rangle \\
&= \langle \mathrm{ch}(\Phi \mathscr{O}_X),\, \mathrm{ch}(\mathscr{F}) \rangle \\
&= \langle \Phi_* \,\mathrm{ch}(\mathscr{O}_X),\, \mathrm{ch}(\mathscr{F}) \rangle \\
&= \langle \mathrm{ch}(\mathscr{O}_X),\, \Psi_* \,\mathrm{ch}(\mathscr{F}) \rangle \\
&= \langle \mathrm{ch}(\mathscr{O}_X),\, \mathrm{ch}(\Psi \mathscr{F}) \rangle \\
&= \langle \mathrm{ch}(\mathscr{O}_X),\, \mathrm{ch}(\mathscr{E}^\vee \otimes \mathscr{F}) \rangle \\
&= \langle \mathrm{ch}(p^* \mathscr{O}_{\mathrm{pt}}),\, \mathrm{ch}(\mathscr{E}^\vee \otimes \mathscr{F}) \rangle \\
&= \langle (p^*)_* \,\mathrm{ch}(\mathscr{O}_{\mathrm{pt}}),\, \mathrm{ch}(\mathscr{E}^\vee \otimes \mathscr{F}) \rangle \\
&= \langle \mathrm{ch}(\mathscr{O}_{\mathrm{pt}}),\, (p_*)_* \,\mathrm{ch}(\mathscr{E}^\vee \otimes \mathscr{F}) \rangle \\
&= \langle \mathrm{ch}(\mathscr{O}_{\mathrm{pt}}),\, \mathrm{ch}(p_*(\mathscr{E}^\vee \otimes \mathscr{F})) \rangle \\
&= \langle \mathrm{ch}(\mathscr{O}_{\mathrm{pt}}),\, \mathrm{ch}(\mathbf{R}\mathrm{Hom}_X(\mathscr{E}, \mathscr{F})) \rangle.
\end{aligned}$$

Since ch is a map on K-theory, $K_0(\mathrm{pt}) \cong \mathbf{Z}$, and the Mukai product is additive, we see that

$$\langle \mathrm{ch}(\mathscr{O}_{\mathrm{pt}}),\, \mathrm{ch}(\mathbf{R}\mathrm{Hom}_X(\mathscr{E}, \mathscr{F})) \rangle = \chi(\mathbf{R}\mathrm{Hom}_X(\mathscr{E}, \mathscr{F})) \,\cdot\, \langle \mathrm{ch}(\mathscr{O}_{\mathrm{pt}}),\, \mathrm{ch}(\mathscr{O}_{\mathrm{pt}}) \rangle = \chi(\mathscr{E}, \mathscr{F}),$$

as it is a trivial computation to check that

$$\langle \mathrm{ch}(\mathscr{O}_{\mathrm{pt}}),\, \mathrm{ch}(\mathscr{O}_{\mathrm{pt}}) \rangle_{\mathrm{pt}} = 1.$$

□

Remark 7.7. The same proof works in the case of a global quotient orbifold, with some minor corrections. The map $p : X \to \mathrm{pt}$ needs to be replaced by $p : X \to [\mathrm{pt}/G] = BG$. Then we consider $\mathscr{O}_{\mathrm{pt}}$ with the trivial G-representation, and $p^* \mathscr{O}_{\mathrm{pt}}$ is what we use for $\mathscr{O}_X$. The K-theory of BG is not $\mathbf{Z}$, but we are only interested in the part that is generated by $\mathscr{O}_{\mathrm{pt}}$ (with the trivial representation), since we are only interested in G-equivariant Hom groups in the definition of $\chi(\mathscr{E},\mathscr{F})$ (see [7, Section 4] for details).

Remark 7.8. If we define $\mathrm{td}(X) = \tau\,\mathrm{ch}(\mathscr{O}_X)$, we have the following alternative version of the above theorem

$$\chi(\mathscr{F}) = \mathrm{Tr}_{X\times X}(\mathrm{td}(X)\circ \mathrm{ch}(\mathscr{F})),$$

more reminiscent of the classical version of Hirzebruch-Riemann-Roch.

7.4. We conclude with a mention of the following result, inspired by the Cardy condition in physics. We omit the proof, as we shall not use it in the sequel, and it is mainly an exercise in applying several times the basic construction (Proposition 3.1). The interested reader can easily supply the details.

Theorem 7.9 (Cardy condition)**.** *Let $\mathscr{E},\mathscr{F}$ be objects in $\mathbf{D}^b_{\mathrm{coh}}(X)$, and let $e\in \mathrm{Hom}_X(\mathscr{E},\mathscr{E})$ and $f\in \mathrm{Hom}_X(\mathscr{F},\mathscr{F})$. Consider the operator*

$${}_f m_e : \mathrm{Hom}^i_X(\mathscr{E},\mathscr{F}) \to \mathrm{Hom}^i_X(\mathscr{E},\mathscr{F})$$

given by composition by f on the left and by e on the right. Then we have

$$\langle \iota^{\mathscr{E}}(e),\, \iota^{\mathscr{F}}(f)\rangle = \mathrm{Tr}\,{}_f m_e,$$

where $\iota^{\mathscr{E}}$, $\iota^{\mathscr{F}}$ are the maps defined in (6.3), and Tr *denotes the alternating sum of the traces of the action of ${}_f m_e$ on $\mathrm{Hom}^i_X(\mathscr{E},\mathscr{F})$.*

Observe that the Hirzebruch-Riemann-Roch formula is a direct consequence of the Cardy condition, with $e = \mathrm{id}_{\mathscr{E}}$, $f = \mathrm{id}_{\mathscr{F}}$.

8. Derived equivalence invariance

This section is devoted to a discussion of the invariance of the Hochschild structure under derived equivalences. This is the primary reason for our decision to use it instead of the harmonic structure given by cohomology of vector fields and/or forms discussed in [9]. We provide proofs of our statements in the "Ext" interpretation; it is obvious that the proofs in the categorical interpretation would be significantly shorter, perhaps trivial.

8.1. We aim to prove the following result:

Theorem 8.1. *Let X and Y be spaces whose derived categories are equivalent via a Fourier-Mukai transform (i.e., the equivalence is given by an integral transform). Then there exists a natural isomorphism of Hochschild structures*

$$(HH^*(X), HH_*(X)) \cong (HH^*(Y), HH_*(Y)).$$

8.2. There are three statements implicit in the above theorem, which we'll discuss in turn:

a. $HH^*(X)\cong HH^*(Y)$ as graded rings;
b. $HH_*(X)\cong HH_*(Y)$ as graded modules over the cohomology rings;
c. the isomorphism $HH_*(X)\cong HH_*(Y)$ is an isometry with respect to the generalized Mukai product.

Proposition 8.2. *Under the hypothesis of Theorem 8.1 there is an equivalence of derived categories*

$$\mathbf{D}^b_{\mathrm{coh}}(X\times X)\cong \mathbf{D}^b_{\mathrm{coh}}(Y\times Y)$$

which maps $\mathscr{O}_{\Delta_X}$ to $\mathscr{O}_{\Delta_Y}$ and S_{Δ_X} to S_{Δ_Y}.

Again, this statement would be trivial in the "natural transformations" context: $\mathscr{O}_{\Delta_X}$ and $\mathscr{O}_{\Delta_Y}$ correspond to the identity natural transformations, and S_{Δ_X} and S_{Δ_Y} correspond to the Serre functors (intrinsic to any triangulated category that possesses one).

Proof. (Independently also proven in [12], [28].) Begin with the observation that if $\mathscr{E}$ is an object in $\mathbf{D}^b_{\mathrm{coh}}(X \times Y)$ that induces a Fourier-Mukai transform $F = \Phi^{\mathscr{E}}_{X\to Y} : \mathbf{D}^b_{\mathrm{coh}}(X) \to \mathbf{D}^b_{\mathrm{coh}}(Y)$, then $F^\vee = \Phi^{\mathscr{E}^\vee}_{X\to Y}$ is also an equivalence. (We have *not* interchanged X and Y, as expected.) Indeed,

$$F^\vee \mathscr{O}_x = (F\mathscr{O}_x)^\vee$$

for $x \in X$, and thus

$$\begin{aligned}\mathrm{Hom}_Y(F^\vee\mathscr{O}_x, F^\vee\mathscr{O}_{x'}) &= \mathrm{Hom}_Y((F\mathscr{O}_x)^\vee, (F\mathscr{O}_{x'})^\vee)\\ &= \mathrm{Hom}_Y(F\mathscr{O}_{x'}, F\mathscr{O}_x)\end{aligned}$$

for $x, x' \in X$. It follows that the orthogonality condition of [6, Theorem 5.1] is satisfied by $F^\vee$ if it is already satisfied by F. Furthermore, if F is an equivalence

$$F^\vee\mathscr{O}_x = (F\mathscr{O}_x)^\vee \cong (F\mathscr{O}_x \otimes \omega_Y)^\vee = (F\mathscr{O}_x)^\vee \otimes \omega_Y^{-1} = F^\vee\mathscr{O}_x \otimes \omega_Y^{-1}$$

by [6, Theorem 5.4], and therefore

$$F^\vee\mathscr{O}_x \otimes \omega_Y \cong F^\vee\mathscr{O}_x,$$

so we conclude by the same theorem that $F^\vee$ is an equivalence.

Now consider the objects $\mathscr{E}^* = \mathscr{E}^\vee \otimes \pi_Y^* S_Y$ and $\mathscr{E}$ on $X \times Y$. Both induce equivalences $\mathbf{D}^b_{\mathrm{coh}}(X) \to \mathbf{D}^b_{\mathrm{coh}}(Y)$, so by [28, Corollary 1.8] we conclude that if we let

$$\mathscr{H} = \mathscr{E}^* \boxtimes \mathscr{E} = \pi_{12}^*\mathscr{E}^* \otimes \pi_{34}^*\mathscr{E}$$

then $\mathscr{H}$ induces an equivalence of derived categories $\Phi^{\mathscr{H}} : \mathbf{D}^b_{\mathrm{coh}}(X \times X) \to \mathbf{D}^b_{\mathrm{coh}}(Y \times Y)$. (Here, $\pi_{ij} : X \times Y \times X \times Y \to X \times Y$ are the projections onto the corresponding factors.)

We claim that $\Phi^{\mathscr{H}}(\mathscr{O}_{\Delta_X}) = \mathscr{O}_{\Delta_Y}$. Indeed, we have

$$\Phi^{\mathscr{H}}(\mathscr{O}_{\Delta_X}) = \pi_{24,*}(\pi_{13}^*(\mathscr{O}_{\Delta_X}) \otimes \mathscr{H}),$$

and

$$\pi_{13}^*\mathscr{O}_{\Delta_X} = \Delta_{13,*}\mathscr{O}_{Y\times X\times Y},$$

where Δ_{13} is the morphism

$$\Delta_{13} : Y \times X \times Y \mapsto X \times Y \times X \times Y$$

which maps

$$(y_1, x, y_2) \mapsto (x, y_1, x, y_2).$$

Therefore, by the projection formula we have

$$\pi_{13}^*(\mathscr{O}_{\Delta_X}) \otimes \mathscr{H} = \Delta_{13,*}\Delta_{13}^*\mathscr{H},$$

and by the commutative diagram

$$\begin{tikzcd} Y \times X \times Y \arrow[r, "\Delta_{13}"] \arrow[d, "p_{13}"'] & X \times Y \times X \times Y \arrow[dl, "\pi_{24}"] \\ Y \times Y, & \end{tikzcd}$$

we conclude that

$$\begin{aligned}\Phi^{\mathscr{H}}(\mathscr{O}_{\Delta_X}) &= \pi_{24,*}(\pi_{13}^*(\mathscr{O}_{\Delta_X}) \otimes \mathscr{H}) \\ &= \pi_{24,*}\Delta_{13,*}\Delta_{13}^*\mathscr{H} = p_{13,*}\Delta_{13}^*\mathscr{H} \\ &= p_{13,*}(\mathscr{E}^* \boxtimes' \mathscr{E}),\end{aligned}$$

where we have denoted by p_{ij} the projections from $Y \times X \times Y$ onto the corresponding factors, and

$$\mathscr{E}^* \boxtimes' \mathscr{E} = p_{12}^*\mathscr{E}^* \otimes p_{23}^*\mathscr{E}.$$

But with notation as in Section 2,

$$p_{13,*}(\mathscr{E}^* \boxtimes' \mathscr{E}) = \mathscr{E} \circ \mathscr{E}^*,$$

and since $\Phi^{\mathscr{E}}_{X\to Y}$ and $\Phi^{\mathscr{E}^*}_{Y\to X}$ were inverse to one another,

$$\mathscr{E} \circ \mathscr{E}^* \cong \mathscr{O}_{\Delta_Y}$$

by the comments in (2.2). We conclude that

$$\Phi^{\mathscr{H}}(\mathscr{O}_{\Delta_X}) = \mathscr{O}_{\Delta_Y}.$$

The computation of $\Phi^{\mathscr{H}}(S_{\Delta_X})$ is entirely similar, and we shall omit the details. We shall only mention that what one obtains is that

$$\Phi^{\mathscr{H}}(S_{\Delta_X}) = \mathscr{E} \circ S_{\Delta_X} \circ \mathscr{E}^*,$$

and in terms of functors that corresponds to

$$F \circ S_X \circ F^{-1},$$

where $F = \Phi^{\mathscr{E}}$. But since the Serre functor is intrinsic [4], this functor must be isomorphic to S_Y, and hence it must be given by S_{Δ_Y} (an equivalence of derived categories is induced by an object on the product which is unique up to isomorphism [27, Theorem 2.2]). □

Corollary 8.3. *A Fourier-Mukai transform $\mathbf{D}^b_{\mathrm{coh}}(X) \cong \mathbf{D}^b_{\mathrm{coh}}(Y)$ induces an isomorphism of graded rings $HH^*(X) \cong HH^*(Y)$, as well as an isomorphism $HH_*(X) \cong HH_*(Y)$ of graded modules over the corresponding cohomology rings.*

Proof. Follows immediately from Proposition 8.2. It is useful to point out that this isomorphism is independent of the choice of isomorphism $\Phi^{\mathscr{H}}(\mathscr{O}_{\Delta_X}) \cong \mathscr{O}_{\Delta_Y}$: indeed, different choices are conjugate by an element of $\mathrm{Hom}_{Y\times Y}(\mathscr{O}_{\Delta_Y}, \mathscr{O}_{\Delta_Y})$, and these elements are central in $HH^*(Y)$. A similar argument works for HH_*.

In the affine case this theorem has been proven for cohomology even for non-commutative rings by Happel [14] and Rickard [31]. (Rickard even removed the requirement that the equivalence be given by a Fourier-Mukai transform.) □

This completes parts a. and b. of Theorem 8.1. Part c. is just Corollary 7.5.

Appendix A. A categorical approach via topology and TQFT's

In this appendix we present two categorical/topological/TQFT approaches to the Hochschild structure. Some of the ideas will not be completely rigorous, however the intuition behind these approaches is often extremely valuable in understanding the proofs in this paper.

A.1. Consider the weak 2-category $\mathscr{D}$ (the 2-category of "all derived categories"), defined as follows:

– objects of $\mathscr{D}$ are smooth, projective schemes (or compact orbifolds, or twisted spaces, etc.); we'll call the objects of $\mathscr{D}$ *spaces*;
– if X and Y are spaces, $\mathrm{Hom}(X, Y) = \mathrm{Ob}\, \mathbf{D}^b_{\mathrm{coh}}(X \times Y)$;
– if $\mathscr{E}, \mathscr{F}$ are elements of $\mathrm{Hom}(X, Y)$, then $2\text{-}\mathrm{Hom}(\mathscr{E}, \mathscr{F}) = \mathrm{Hom}_{\mathbf{D}^b_{\mathrm{coh}}(X\times Y)}(\mathscr{E}, \mathscr{F})$.

The composition of 1-morphisms is given by convolution of kernels, as in (2.2). The horizontal and vertical composition of 2-morphisms is defined in the obvious way. The (weak) identity 1-morphism $X \to X$ is given by $\mathscr{O}_{\Delta_X} \in \mathrm{Hom}(X, X) = \mathbf{D}^b_{\mathrm{coh}}(X \times X)$. (The reader unfamiliar with 2-categories is referred to [2].)

Observe that $\mathscr{D}$ has a richer structure than just that of a 2-category: if X and Y are spaces, $\mathrm{Hom}(X, Y)$ has a natural structure of a triangulated category which admits a Serre functor.

A.2. It is useful to consider the 2-functor $\Phi : \mathscr{D} \to \mathfrak{Cat}$ between the 2-category $\mathscr{D}$ and the 2-category $\mathfrak{Cat}$ of all categories (where 1-morphisms are functors and 2-morphisms are natural transformations). The functor Φ is defined by setting

– $\Phi(X) = \mathbf{D}^b_{\mathrm{coh}}(X)$ for any space X;
– $\Phi(\mathscr{E}) = \Phi^{\mathscr{E}}_{X\to Y} : \mathbf{D}^b_{\mathrm{coh}}(X) \to \mathbf{D}^b_{\mathrm{coh}}(Y)$ for any object $\mathscr{E} \in \mathbf{D}^b_{\mathrm{coh}}(X \times Y) = \mathrm{Hom}_{\mathscr{D}}(X, Y)$;
– $\Phi(f) = \Phi^f_{X\to Y} : \Phi^{\mathscr{E}}_{X\to Y} \Rightarrow \Phi^{\mathscr{F}}_{X\to Y}$ for any morphism $f : \mathscr{E} \to \mathscr{F}$ between $\mathscr{E}, \mathscr{F} \in \mathbf{D}^b_{\mathrm{coh}}(X \times Y)$.

Note again that the natural properties associated with the triangulated structure are preserved ($\Phi(\mathscr{E})$ is exact, $\Phi(f)$ commutes with translations, etc.) Example B.1 shows that Φ can not be fully faithful, even if we impose extra conditions on what functors or natural transformations we allow in $\mathfrak{Cat}$. In spite of this, we shall think of Φ as being fully faithful, and thus think of morphisms of $\mathscr{D}$ as functors, 2-morphisms of $\mathscr{D}$ as natural transformations. For more on this problem and a possible way around it see Appendix B.

A.3. Our purpose is to find invariants of spaces that are, in a sense to be made precise later, functorial with respect to $\mathscr{D}$. As a first example of such an invariant, consider associating to a space X the group $K_0(X \times X)$. The functoriality of K_0 is expressed by the fact that for a morphism $X \to Y$ in $\mathscr{D}$ given by an object $\mathscr{E} \in \mathbf{D}^b_{\mathrm{coh}}(X \times Y)$, there is a natural map $K_0(X \times X) \to K_0(Y \times Y)$ defined by

$$\mathscr{F} \in \mathbf{D}^b_{\mathrm{coh}}(X \times X) \mapsto \mathscr{E} \circ \mathscr{F} \circ \mathscr{E}^*$$

where

$$\mathscr{E}^* = \mathscr{E}^\vee \otimes \pi_Y^* S_Y,$$

is the object in $\mathbf{D}^b_{\mathrm{coh}}(Y \times X)$ whose associated functor $\mathbf{D}^b_{\mathrm{coh}}(X) \to \mathbf{D}^b_{\mathrm{coh}}(Y)$ is left adjoint of $\Phi^{\mathscr{E}}_{X\to Y}$. It is easy to see that this association of morphisms in $\mathscr{D}$ and maps between K_0 groups is functorial.

A.4. A useful analogy is obtained by considering the 2-category $\mathscr{T}$ associated to a topological space T, where objects of $\mathscr{T}$ are points of T, morphisms between two points $x, y \in T$ are given by paths in T from x to y, and 2-morphisms are given by homotopy equivalence classes of homotopies between paths. Observe that in $\mathscr{T}$ all morphisms are isomorphisms.

The first homotopy invariant of T that one studies is the fundamental group $\pi_1(T, x)$ of homotopy classes of loops based at $x \in T$. Categorically, this can be thought of as the set of morphisms $x \to x$ in $\mathscr{T}$ (loops at x), modulo the equivalence relation induced by 2-isomorphisms (i.e., homotopies). Given a morphism $x \to y$ (a path from x to y), one obtains a natural map $\pi_1(T, x) \to \pi_1(T, y)$ by conjugating a loop at x with the given path $x \to y$. This map only depends on the homotopy class of the path $x \to y$, and the association of maps to paths is functorial.

It is now obvious that the same procedure that was used to construct the fundamental group of T has also been used to construct $K_0(X \times X)$ in $\mathscr{D}$. (In fact, the analogy is imperfect: in constructing $K_0(X \times X)$ we have taken the quotient of $\operatorname{Ob} \mathbf{D}^b_{\mathrm{coh}}(X \times X)$ by a far larger equivalence relation: while in defining π_1 we only identified 1-morphisms that were 2-isomorphic, in constructing $K_0(X \times X)$ we have declared that 1-morphisms that form a triangle should sum to zero. But in the context of the existence of the triangulated structure on $\operatorname{Hom}_{\mathscr{D}}(X, Y)$ it makes sense to consider this coarser equivalence relation to get a more finite invariant.) The analogy also extends to the association of morphisms between K_0-groups to 1-morphisms in the underlying category.

The above discussion also shows one of the most important weaknesses of this analogy: while in $\mathscr{T}$ every morphism $x \to y$ is an isomorphism, and hence it makes sense to talk about its inverse when conjugating a loop at x to get a loop at y, morphisms in $\mathscr{D}$ are not invertible in general. Thus we had to settle for the weaker concept of left adjoint; but there was no particular reason to choose left over right: right adjoints would have worked equally well.

A.5. The procedure just described should be thought of as producing a functor

$$K_0 : 1\text{-}\mathscr{D} \to \mathfrak{Gps}$$

from the decategorification 1-$\mathscr{D}$ of $\mathscr{D}$ (the 1-category that is obtained from $\mathscr{D}$ by forgetting 2-morphisms and setting 1-morphisms that are 2-isomorphic in $\mathscr{D}$ to be equal in 1-$\mathscr{D}$) to the category $\mathfrak{Gps}$ of groups. There is no reason, however, to stop at the π_1 level in our analogy with topological spaces. Indeed, note that $\pi_2(T, x)$, as the space of homotopy classes of maps $(S^2, \mathrm{pt}) \to (T, x)$, can be thought of as the space of homotopy classes of homotopies from the constant path Id_x at x to itself. In other words

$$\pi_2(T, x) = 2\text{-}\operatorname{Hom}_{\mathscr{T}}(\mathrm{Id}_x, \mathrm{Id}_x).$$

Just as in the case of π_1, a path $x \to y$ induces in a functorial way a map $\pi_2(T, x) \to \pi_2(T, y)$.

A.6. By analogy, in the category $\mathscr{D}$ we associate to a space X its Hochschild cohomology

$$HH^*(X) = 2\text{-}\operatorname{Hom}_{\mathscr{D}}(\mathrm{Id}_X, \mathrm{Id}_X) = \operatorname{Hom}_{\mathbf{D}^b_{\mathrm{coh}}(X \times X)}(\mathscr{O}_\Delta, \mathscr{O}_\Delta).$$

However, when we try to mimic the construction of the map $\pi_2(T, x) \to \pi_2(T, y)$ associated to a map $x \to y$ to obtain a map $HH^*(X) \to HH^*(Y)$ associated to a map (in $\mathscr{D}$) $X \to Y$ we hit a major difficulty: in the topological setting, the construction relies on the fact that if $f : x \to y$ is a path in T, then $f^{-1} \circ f$ is homotopic to the constant path Id_x, *and this homotopy can be read as either a 2-morphism* $1 \Rightarrow f^{-1} \circ f$ *or* $f^{-1} \circ f \Rightarrow 1$. In $\mathscr{D}$ this is no longer the case: we have to replace a path $x \to y$ by an object $\mathscr{E} \in \mathbf{D}^b_{\mathrm{coh}}(X \times Y)$, and there is no reason why the functor $\Phi^{\mathscr{E}}_{X \to Y}$ should have an inverse. It will always have left and right adjoints, but these need not be isomorphic in general. In order to compensate for this discrepancy we need to modify the definition of HH^* to

$$HH_*(X) = 2\text{-}\operatorname{Hom}_{\mathscr{D}}(S_X^{-1}, \mathrm{Id}_X) = \operatorname{Hom}_{\mathbf{D}^b_{\mathrm{coh}}(X \times X)}(\Delta_! \mathscr{O}_X, \mathscr{O}_\Delta).$$

As it turned out in Section 5, with this definition it is possible to mimic the construction that we did above for π_2, and to get a functor

$$HH_* : 1\text{-}\mathscr{D} \to \mathfrak{Vect}$$

from the decategorification of $\mathscr{D}$ to the category of graded vector spaces.

One can also see that when dealing with an isomorphism $X \to Y$ in $\mathscr{D}$ (i.e., and equivalence of categories) the above technique can be used to construct an isomorphism $HH^*(X) \cong HH^*(Y)$ (we use the fact that the left and right adjoints of an equivalence are isomorphic). This is precisely what we did in Section 8.

A.7. There is a second approach to Hochschild homology via topology, which in a sense is orthogonal to the above one. In this approach we consider the 2-category $\mathscr{C}$ of 3-cobordisms with corners defined as follows (for details see [22]):

- objects of $\mathscr{C}$ are smooth, oriented 1-manifolds (i.e., disjoint unions of circles);
- 1-morphisms of $\mathscr{C}$ are cobordisms between the objects of $\mathscr{C}$ (smooth oriented surfaces with boundary);
- 2-morphisms of $\mathscr{C}$ are cobordisms between 1-morphisms (3-manifolds with corners).

We are interested in studying 2-functors from $\mathscr{C} \to \mathscr{D}$. Such functors, when they satisfy certain other properties (the list of these properties varies in the literature), are also known as extended TQFT's (or 1+1+1 TQFT's). One of the more common requirements on TQFT's is that they should respect the monoidal structure on objects, given in $\mathscr{C}$ by disjoint union of 1-manifolds and in $\mathscr{D}$ by product of spaces, and we shall search for functors with this property. When a TQFT $F : 1\text{-}\mathscr{C} \to 1\text{-}\mathscr{D}$ is only defined between the decategorifications $1\text{-}\mathscr{C}$, $1\text{-}\mathscr{D}$ of $\mathscr{C}$ and $\mathscr{D}$, it is known as a 1+1 TQFT.

A.8. Building on work of Roberts, Sawon and Willerton (unpublished) we discuss a possible approach to constructing such a functor (their construction is essentially the same as ours, but with a different target category). Due to technical difficulties we shall actually only discuss their construction of a 1-functor $\mathscr{X} : 1\text{-}\mathscr{C} \to 1\text{-}\mathscr{D}$; the main point of our discussion is to argue that despite the fact that we can prove that this functor can not be extended to a 2-functor $\mathscr{C} \to \mathscr{D}$, the functoriality property of Hochschild homology should be viewed as strong evidence that there exists a modification of the category $\mathscr{C}$ for which such a lifting exists. Such a theory should yield interesting invariants of 3-manifolds.

A.9. Let us begin first with the construction of a 1-functor $\mathscr{X} : 1\text{-}\mathscr{C} \to 1\text{-}\mathscr{D}$, which depends on the choice of a space X. Associate to S^1 the space X, and since we want $\mathscr{X}$ to respect the monoidal structure on objects,

$$\mathscr{X}(S^1 \coprod \cdots \coprod S^1) = X \times \cdots \times X,$$

so in particular $\mathscr{X}(\emptyset) = \mathrm{pt}$. To a pair of pants associate $\mathscr{O}_\Delta \in \mathbf{D}^b_{\mathrm{coh}}(X \times X \times X)$, where Δ is the small diagonal

$$\{(x, x, x) \in X \times X \times X \mid x \in X\}$$

in the product $X \times X \times X$ (we think of objects in $\mathbf{D}^b_{\mathrm{coh}}(X \times X \times X)$ as morphisms $X \to X \times X$ in $\mathscr{D}$). As functors, this corresponds either to

$$\Delta_* : \mathbf{D}^b_{\mathrm{coh}}(X) \to \mathbf{D}^b_{\mathrm{coh}}(X \times X)$$

or to

$$\Delta^* : \mathbf{D}^b_{\mathrm{coh}}(X \times X) \to \mathbf{D}^b_{\mathrm{coh}}(X),$$

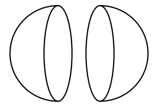

FIGURE 1. Computation of $\mathscr{X}(S^2)$

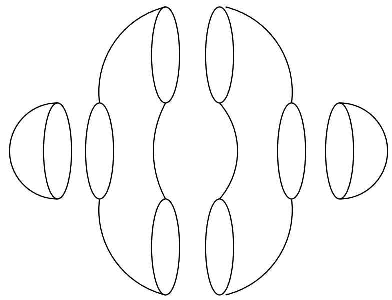

FIGURE 2. Computation of $\mathscr{X}(T^2)$

depending on whether we want to think of the pair of pants as a cobordism from S^1 to $S^1 \coprod S^1$, or the other way around. We associate to the disk (thought of as a cobordism $\emptyset \to S^1$) the object $\mathscr{O}_X \in \mathbf{D}^b_{\mathrm{coh}}(\mathrm{pt} \times X)$. As a functor it corresponds either to p^* or to p_* depending on whether we view it as a map $\emptyset \to S^1$ or $S^1 \to \emptyset$ ($p : X \to \mathrm{pt}$ is the structure map of X).

Knowing this information is enough to determine the value of $\mathscr{X}$ on any 1-cobordism (any oriented surface with boundary can be decomposed into a finite number of pairs of pants and caps). Figures 1 and 2 show how to compute $\mathscr{X}(S^2)$ and $\mathscr{X}(T^2)$ (we think of a closed surface as a cobordism from the empty manifold to itself, and thus $\mathscr{X}$(closed surface) is a map from a point to itself in $\mathscr{D}$, which is just a graded vector space). The results are

$$\begin{aligned}\mathscr{X}(S^2) &= p_*\mathscr{O}_X = H^*(X, \mathscr{O}_X)\\ \mathscr{X}(T^2) &= p_*\Delta^*\Delta_*\mathscr{O}_X = \mathbf{R}\mathrm{Hom}_X(\mathscr{O}_X, \Delta^*\Delta_*\mathscr{O}_X)\\ &= \mathbf{R}\mathrm{Hom}_{X\times X}(\Delta_!\mathscr{O}_X, \Delta_*\mathscr{O}_X) = HH_*(X).\end{aligned}$$

Thus we conclude that this 1+1 TQFT with target X should associate to T^2 the Hochschild homology $HH_*(X)$ of X.

A.10. Unfortunately it is known that $\mathscr{X}$ can not be lifted to a 2-functor $\mathscr{X} : \mathscr{C} \to \mathscr{D}$. The obstruction to doing this has already appeared in the work of Khovanov [20]: his observation is that if one has a 2-functor $F : \mathscr{C} \to \mathfrak{Cat}$, then if S is any 2-dimensional cobordism (1-morphism in $\mathscr{C}$) between the 1-manifolds C_1 and C_2, then $F(S)$, which is a functor between the categories $F(C_1)$ and $F(C_2)$ must have a biadjoint (a functor which is both a left and a right adjoint to $F(S)$). This biadjoint is given by $F(S')$, where $S' : C_2 \to C_1$ is the reverse cobordism, given by the same manifold S.

In particular this shows that the functor $\mathscr{X}$ that we constructed earlier has no hope of lifting to a 2-functor $\mathscr{C} \to \mathscr{D}$ unless the target space is a point: the composition $\Phi \circ \mathscr{X}$

would be a 2-functor $\mathscr{C} \to \mathfrak{Cat}$, which associates to a pair of pants (thought of as a morphism $S^1 \to S^1 \coprod S^1$) the functor $\Delta_* : \mathbf{D}^b_{\mathrm{coh}}(X) \to \mathbf{D}^b_{\mathrm{coh}}(X \times X)$, and to the reverse pair of pants $\Delta^* : \mathbf{D}^b_{\mathrm{coh}}(X \times X) \to \mathbf{D}^b_{\mathrm{coh}}(X)$. Unfortunately, while Δ^* is a left adjoint to Δ_*, it is not a right adjoint unless $X = \mathrm{pt}$.

A.11. Let us, however, assume that an extension of $\mathscr{X}$ to a 2-functor could be found, for every target space X. Observe that once the image of $\mathscr{X}$ on S^1 is fixed, the entire functor $\mathscr{X}$ is essentially fixed. We can expect that a similar statement would hold for natural transformations between such functors: given functors $\mathscr{X}, \mathscr{Y} : \mathscr{C} \to \mathscr{D}$ that correspond to spaces X, Y, respectively, a natural transformation (with certain properties yet to be fixed) should be completely determined by its value on S^1. But its value on S^1 is just a map $\mathscr{X}(S^1) \to \mathscr{Y}(S^1)$ in $\mathscr{D}$, i.e., an object in $\mathbf{D}^b_{\mathrm{coh}}(X \times Y)$. Thus given spaces X and Y, we should get associated 2-functors $\mathscr{X}$ and $\mathscr{Y}$, and given an object $\mathscr{E} \in \mathbf{D}^b_{\mathrm{coh}}(X \times Y)$, we should get an associated natural transformation between them.

Observe that a natural transformation $\eta : \mathscr{X} \to \mathscr{Y}$ of 2-functors between $\mathscr{C}$ and $\mathscr{D}$ is a collection of 1-morphisms $\eta(o) : \mathscr{X}(o) \to \mathscr{Y}(o)$ associated to objects $o \in \mathrm{Ob}\,\mathscr{C}$, as well as a collection of 2-morphisms $\eta(o \to o')$ associated to 1-morphisms in $\mathscr{C}$. The 2-morphism $\eta(o \to o')$ is depicted in the following diagram

$$\begin{tikzcd}
& \mathscr{X}(o') \arrow[dr, "\eta(o')"] \arrow[dd, phantom, "\Downarrow \eta(o\to o')"] & \\
\mathscr{X}(o) \arrow[ur, "\mathscr{X}(o\to o')"] \arrow[dr, "\eta(o)"'] & & \mathscr{Y}(o') \\
& \mathscr{Y}(o). \arrow[ur, "\mathscr{Y}(o\to o')"'] &
\end{tikzcd}$$

Note that in the 1-categorical setting, the commutativity of the above diagram is precisely the condition that η be a natural transformation (the corresponding commutativity conditions on η to be a natural transformation of 2-functors are too complicated to write down here).

In particular, for every morphism $S : \emptyset \to \emptyset$ from the empty 1-manifold to itself (in $\mathscr{C}$) we get a natural transformation

$$\begin{tikzcd}
& \mathrm{pt} \arrow[dr, "\eta(\emptyset)"] \arrow[dd, phantom, "\Downarrow \eta(S)"] & \\
\mathrm{pt} \arrow[ur, "\mathscr{X}(S)"] \arrow[dr, "\eta(\emptyset)"'] & & \mathrm{pt} \\
& \mathrm{pt}. \arrow[ur, "\mathscr{Y}(S)"'] &
\end{tikzcd}$$

Since morphisms in $\mathscr{D}$ between a point and itself are given by $\mathbf{D}^b_{\mathrm{coh}}(\mathrm{pt})$, and it is reasonable to expect that $\eta(\emptyset) = \mathscr{O}_{\mathrm{pt}}$, it follows that $\eta(S)$ should be thought of as a morphism $\mathscr{X}(S) \to \mathscr{Y}(S)$ (in $\mathbf{D}^b_{\mathrm{coh}}(\mathrm{pt})$, i.e., a morphism of graded vector spaces). In particular, taking $S = T^2$, it follows that any such natural transformation will induce a map of graded vector spaces

$$HH_*(X) = \mathscr{X}(T^2) \to \mathscr{Y}(T^2) = HH_*(Y).$$

A.12. To summarize the above discussion, the main conjecture that we make is that every map $\mathscr{E} : \mathscr{X}(S^1) \to \mathscr{Y}(S^1)$ should lift to a natural transformation of 2-functors $\eta_{\mathscr{E}} : \mathscr{X} \Rightarrow \mathscr{Y}$ (in an appropriate sense), which in turn should induce a map of graded vector spaces $\mathscr{X}(S) \to \mathscr{Y}(S)$ for every closed surface S, and in particular the existence of the map

$$(\Phi^{\mathscr{E}}_{X \to Y})_* : HH_*(X) \to HH_*(Y)$$

should be regarded as conjectural evidence for the existence of such a lifting.

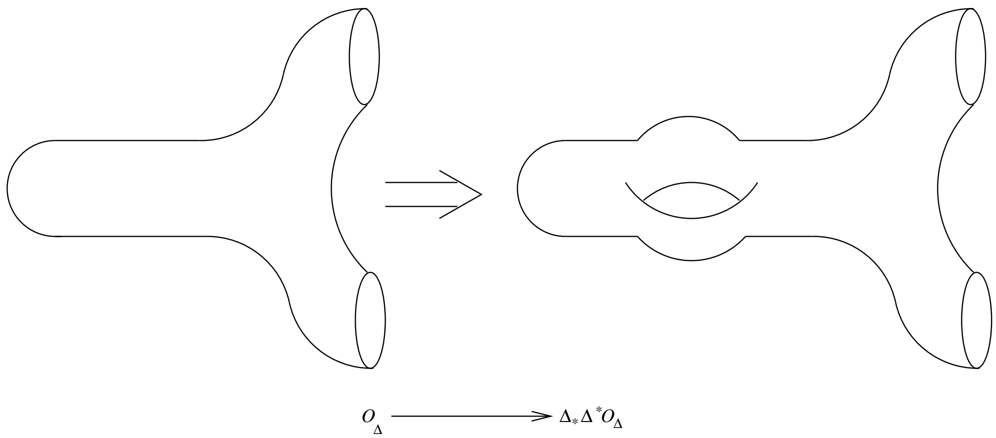

FIGURE 3. The Atiyah class

A.13. As a quick check of this conjecture, note that it also predicts that to an object $\mathscr{E} \in \mathbf{D}^b_{\mathrm{coh}}(X \times Y)$ we should be able to associate in a natural way a map $H^*(X, \mathscr{O}_X) \to H^*(Y, \mathscr{O}_Y)$ (this corresponds to taking $S = S^2$ instead of T^2 above): we believe the map should be given by

$$\begin{aligned} H^*(X, \mathscr{O}_X) = \mathrm{Hom}_X(\mathscr{O}_X, \mathscr{O}_X) &\xrightarrow{\Phi^{\mathscr{E}}_{X\to Y}} \mathrm{Hom}_Y(\Phi^{\mathscr{E}}_{X\to Y}(\mathscr{O}_X), \Phi^{\mathscr{E}}_{X\to Y}(\mathscr{O}_X)) \longrightarrow \\ &\xrightarrow{\mathrm{Tr}} \mathrm{Hom}_Y(\mathscr{O}_Y, \mathscr{O}_Y) = H^*(Y, \mathscr{O}^*_Y). \end{aligned}$$

A particular consequence of this conjecture would thus be that the numbers $h^{i,0}(X)$ should naturally be derived category invariants.

A.14. The problem with making these conjectures precise (and attempting to prove them) is the fact that it is obvious that one needs to change the category $\mathscr{C}$ and the construction of the functor $\mathscr{X}$ associated to a space X so as to incorporate the fact that S_X is nontrivial unless X is a point. For example, note that in our association of functors to cobordisms, the left and right adjoints of such functors only differ by a number of Serre functors (e.g., Δ^* is the left adjoint of Δ_*, while the right adjoint is isomorphic to $S_X^{-1} \circ \Delta^*$, for $\Delta : X \to X \times X$ the diagonal embedding). The category $\mathscr{C}$ needs to be changed in such a way as to break the symmetry given by the fact that a 3-dimensional cobordism $S_1 \Leftrightarrow S_2$ can be read as either $S_1 \Rightarrow S_2$ or $S_2 \Rightarrow S_1$, for example by labeling 2-dimensional cobordisms by integers, and letting 3-cobordisms “flow” from the larger integer to the smaller. Then to a labeled 2-cobordism we would still associate an object of $D(X \times \cdots \times X)$ which would be supported on the small diagonal, but given by $S_X^{\otimes n}$ for some appropriate n.

A.15. We conclude with a remark about the relevance of the Atiyah class in the context of TQFT's. This connects well with Kapranov's approach to the Rozansky-Witten TQFT [18]. The point is that we shall see in [9] that the Atiyah class can be seen (via the HKR isomorphism) to be nothing else but the unit

$$\mathscr{O}_\Delta \to \Delta_*\Delta^*\mathscr{O}_\Delta$$

of the adjunction $\Delta^* \dashv \Delta_*$. Pictorially this corresponds to the cobordism depicted in Figure 3, induced by a standard surgery on a pair of pants, which is one of the two fundamental building blocks of cobordisms of surfaces.

Appendix B. DG-categories versus derived categories

As mentioned in the introduction, working with derived categories as triangulated categories raises several technical problems. We discuss these problems in this appendix, and we point out a possible way of solving them. The main difficulty arises from the fact that the functor Φ that we introduced in Appendix A is not fully faithful, even if restrict it to map into a smaller category than $\mathfrak{Cat}$, as the following example shows:

Example B.1. Let $X = Y =$ an elliptic curve over $\mathbf{C}$, let $\mathscr{F} = \mathscr{O}_\Delta$, the structure sheaf of the diagonal in $X \times X$, and let $\mathscr{G} = \mathscr{O}_\Delta[2]$. Then $\Phi^{\mathscr{F}} = \mathrm{Id}$, $\Phi^{\mathscr{G}}$ is the translation by 2 functor. It is a straightforward calculation to see that

$$\mathrm{Hom}_{X\times Y}(\mathscr{F}, \mathscr{G}) = \mathrm{Ext}^2_{X\times Y}(\mathscr{O}_\Delta, \mathscr{O}_\Delta) = \mathbf{C},$$

but the fact that $\mathfrak{Coh}(X)$ has cohomological dimension 1 implies that any natural transformation between the identity functor and the translation by 2 functor on X must be zero. (One has to use the fact that every complex on X is quasi-isomorphic to the direct sum of its cohomologies.) Therefore the functor Φ can not be faithful in general.

B.1. The trouble with the functor Φ as defined is that for two spaces X and Y, it is trying to be an equivalence between a triangulated category ($\mathrm{Hom}(X, Y)$) and a category which has no obvious triangulated structure (exact functors between $\mathbf{D}^b_{\mathrm{coh}}(X)$ and $\mathbf{D}^b_{\mathrm{coh}}(Y)$). There is another category of functors which is naturally triangulated and which we could use in our situation, namely

$$\mathrm{ExFun}^+(X, Y) = H^0\,\mathrm{Prex}(\tilde{\mathbf{D}}(X), \tilde{\mathbf{D}}(Y)),$$

where Prex is the DG-category of preexact functors between the DG-enhancements $\tilde{\mathbf{D}}(X)$, $\tilde{\mathbf{D}}(Y)$ of $\mathbf{D}^b_{\mathrm{coh}}(X)$, $\mathbf{D}^b_{\mathrm{coh}}(Y)$, in the sense of Bondal and Kapranov [3]. The functor Φ can be defined as before. It seems reasonable to expect that $\mathrm{ExFun}^+(X, Y)$ is independent of the choice of enhancement, carries a Serre functor which on objects is given by

$$F \mapsto S_Y \circ F \circ S_X,$$

and Φ is fully faithful. As pointed out in Appendix A, if this is correct we can think of objects in $\mathbf{D}^b_{\mathrm{coh}}(X \times Y)$ as functors, and of morphisms between objects as natural transformations, in the above sense.

B.2. As an example of this kind of reasoning, the following is a more intuitive "definition" of Hochschild homology and cohomology (see also (A.6)):

"Definition" B.2. Define

$$HH^i(X) = \mathrm{Hom}_{\mathrm{ExFun}^+(X,X)}(1_X, [i]),$$

where 1_X and $[1]$ are the identity and translation functors on the pretriangulated category $\tilde{\mathbf{D}}(X)$ ([3, Section 3]). Similarly, define

$$HH_i(X) = \mathrm{Hom}_{\mathrm{ExFun}^+(X,X)}(S_X^{-1} \circ [i], 1_X),$$

where S_X is the Serre functor of $\mathbf{D}^b_{\mathrm{coh}}(X)$ (which lifts to a functor on $\tilde{\mathbf{D}}(X)$).

Again, cohomology is naturally a ring (with multiplication given by composition of natural transformations) and homology is a module over cohomology.

B.3. The isomorphism τ in the definition of the Mukai pairing has an obvious categorical interpretation: it is just the isomorphism τ in (3.4). The Mukai pairing on $HH_*(X)$ can now be reinterpreted by noting that the Serre dual of $HH_i(X)$ (with respect to the Serre functor on $\mathrm{ExFun}^+(X,X)$) is the space

$$\mathrm{Hom}_{\mathrm{ExFun}^+(X,X)}(\mathrm{Id}, S_X \circ [i]),$$

which by (3.4) is isomorphic to $HH_{-i}(X)$ via τ. The same definition of the Mukai pairing can now be written in the categorical context.

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