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The Mukai pairing—II: the Hochschild–Kostant–Rosenberg isomorphism

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Abstract

We continue the study of the Hochschild structure of a smooth space that we began in our previous paper, examining implications of the Hochschild–Kostant–Rosenberg theorem. The main contributions of the present paper are:

- we introduce a generalization of the usual notions of Mukai vector and Mukai pairing on differential forms that applies to arbitrary manifolds;
- we give a proof of the fact that the natural Chern character map $K_0(X) \to HH_0(X)$ becomes, after the HKR isomorphism, the usual one $K_0(X) \to \bigoplus H^i(X, \Omega^i_X)$; and
- we present a conjecture that relates the Hochschild and harmonic structures of a smooth space, similar in spirit to the Tsygan formality conjecture.

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1. Introduction

1.1. In [5], we introduced the Hochschild structure $(HH^*(X), HH_*(X))$ of a smooth space X, which consists of:

- a graded ring $HH^*(X)$, the Hochschild cohomology ring, defined as

$$HH^i(X) = \mathrm{Hom}_{\mathbf{D}^b_{\mathrm{coh}}(X\times X)}(\mathcal{O}_\Delta, \mathcal{O}_\Delta[i]),$$

  where $\mathcal{O}_\Delta = \Delta_*\mathcal{O}_X$ is the structure sheaf of the diagonal in $X \times X$;
- a graded left $HH^*(X)$-module $HH_*(X)$, the Hochschild homology module, defined as

$$HH_i(X) = \mathrm{Hom}_{\mathbf{D}^b_{\mathrm{coh}}(X\times X)}(\Delta_!\mathcal{O}_X[i], \mathcal{O}_\Delta),$$

  where $\Delta_!$ is the left adjoint of Δ^* defined by Grothendieck–Serre duality [5, 3.3];
- a non-degenerate pairing $\langle\cdot,\cdot\rangle$ defined on $HH_*(X)$, the *generalized Mukai pairing* (for the definition see [5]).

1.2. Following ideas of Markarian [15] we also introduced the Chern character map

$$\mathrm{ch} : K_0(X) \to HH_0(X)$$

by setting $\mathrm{ch}(\mathcal{F})$ for $\mathcal{F} \in \mathbf{D}^b_{\mathrm{coh}}(X)$ to be the unique element of $HH_0(X)$ such that

$$\mathrm{Tr}_{X\times X}(\mu \circ \mathrm{ch}(\mathcal{F})) = \mathrm{Tr}_X(\Phi^\mu_{X\to X}(\mathcal{F})) = \mathrm{Tr}_X(\pi_{2,*}(\pi_1^*\mathcal{F} \otimes \mu))$$

for every $\mu \in \mathrm{Hom}_{\mathbf{D}^b_{\mathrm{coh}}(X\times X)}(\mathcal{O}_\Delta, S_\Delta)$.

Here Tr is the Serre duality trace [5, 2.3], $S_X = \omega_X[\dim X]$ is the dualizing object of $\mathbf{D}^b_{\mathrm{coh}}(X)$ (also to be thought of as the functor $-\otimes_X S_X$), $S_\Delta = \Delta_* S_X$ is the object whose associated integral transform is S_X, and $\Phi^\mu_{X\to X}$ is the natural transformation $1_X \Rightarrow S_X$ associated to μ (2.2).

It is worth pointing out that $\mu \circ \mathrm{ch}(\mathcal{F})$ is a morphism $\Delta_!\mathcal{O}_X \to S_\Delta$, so using the definition of $\Delta_! = S^{-1}_{X\times X}\Delta_* S_X$ it follows that $\mu \circ \mathrm{ch}(\mathcal{F})$ is in fact a morphism

$$S^{-1}_{X\times X} S_\Delta \to S_\Delta$$

and thus it makes sense to take its trace on $X \times X$. For more details see [5].

1.3. The Hochschild structure satisfies the following properties [5]:

- to every integral functor $\Phi : \mathbf{D}^b_{\mathrm{coh}}(X) \to \mathbf{D}^b_{\mathrm{coh}}(Y)$ there is a naturally associated map of graded vector spaces $\Phi_* : HH_*(X) \to HH_*(Y)$. This association is functorial, commutes with ch, and if Ψ is a left adjoint to Φ, then Ψ_* is a left adjoint to Φ_* with respect to the Mukai pairings on X and Y, respectively, i.e.,

$$\langle v, \Phi_* w\rangle_Y = \langle \Psi_* v, w\rangle_X$$

  for $v \in HH_*(Y)$, $w \in HH_*(X)$;
- the Mukai pairing is a generalization of the Euler pairing on $K_0(X)$,

$$\langle \mathrm{ch}(\mathscr{E}), \mathrm{ch}(\mathscr{F})\rangle = \chi(\mathscr{E}, \mathscr{F}) = \sum_i (-1)^i \dim \mathrm{Ext}^i_X(\mathscr{E}, \mathscr{F})$$

  for any $\mathscr{E}, \mathscr{F} \in \mathbf{D}^b_{\mathrm{coh}}(X)$;
- the Hochschild structure is invariant under derived equivalences given by Fourier–Mukai transforms; in other words, if $\Phi_{X\to Y} : \mathbf{D}^b_{\mathrm{coh}}(X) \to \mathbf{D}^b_{\mathrm{coh}}(Y)$ is a Fourier–Mukai transform, then there are induced isomorphisms $HH^*(X) \cong HH^*(Y)$ (as graded rings), $HH_*(X) \cong HH_*(Y)$ (as graded modules over the corresponding cohomology rings) and this isomorphism is an isometry with respect to the generalized Mukai pairings on X and on Y, respectively.

1.4. The purpose of this paper is to study the similarities between the Hochschild structure and the *harmonic structure* $(HT^*(X), H\Omega_*(X))$ of X, whose vector space structure is defined as

$$HT^i(X) = \bigoplus_{p+q=i} H^p(X, \bigwedge^q T_X),$$

$$H\Omega_i(X) = \bigoplus_{q-p=i} H^p(X, \Omega^q_X).$$

These vector spaces carry the same structures as $(HH^*(X), HH_*(X))$, namely $HT^*(X)$ is a ring, with multiplication induced by the exterior product on polyvector fields; $H\Omega_*(X)$ is a module over $HT^*(X)$, via contraction of polyvector fields with forms; and in Section 3 we shall define a pairing on $H\Omega_*(X)$ which is a modification of the usual pairing of forms given by cup product and integration on X (This modified inner product is a more concrete generalization of the Mukai product in [17].) The generalized Mukai pairing can be thought of as the *mirror* of the usual polarization of the Hodge structure on X.

1.5. In Section 2, we explain how to associate to an integral transform $\Phi : \mathbf{D}^b_{\mathrm{coh}}(X) \to \mathbf{D}^b_{\mathrm{coh}}(Y)$ a map of graded vector spaces

$$\Phi_* : H\Omega_*(X) \to H\Omega_*(Y)$$

and we prove in Section 3 that this association satisfies the same adjointness properties as the similar association for Hochschild homology discussed above. The same construction as in [5] gives us a construction of a generalized Mukai vector map

$$v : \mathbf{D}^b_{\mathrm{coh}}(X \times Y) \to H^*(X \times Y, \mathbf{Q})$$

for any pair of spaces X and Y. When X is a point, we recover (a small modification of) Mukai's original definition,

$$v : \mathbf{D}^b_{\mathrm{coh}}(\mathrm{pt} \times Y) \to H^*(\mathrm{pt} \times Y, \mathbf{Q}).$$

The formula we get is

$$v(\mathscr{E}) = \mathrm{ch}(\mathscr{E}).\hat{A}^{1/2},$$

where $\hat{A}_X$ is the $\hat{A}_X$ class of X (see below and Section 2). Mukai's original definition used td_X instead of $\hat{A}_X$; the two are the same in the case considered by Mukai, namely when $c_1(X) = 0$.

The unexpected surprise here is the fact that the Mukai vector is not symmetric: v is not the same if we replace $X \times Y$ by $Y \times X$. In a sense, the Mukai vector associates an element of $H^*(X \times Y, \mathbf{Q})$ to a *functor* $\mathbf{D}^b_{\mathrm{coh}}(X) \to \mathbf{D}^b_{\mathrm{coh}}(Y)$, and there is no reason why this should be symmetric. The Mukai vector $v(\mathscr{E})$ of $\mathscr{E} \in \mathbf{D}^b_{\mathrm{coh}}(X)$ is obtained by considering the functor $\Phi^{\mathscr{E}}_{\mathrm{pt} \to X} : \mathbf{D}(\mathrm{pt}) \to \mathbf{D}(X)$, $\Phi^{\mathscr{E}}_{\mathrm{pt} \to X} - = - \otimes_{\mathbf{C}} \mathscr{E}$.

The modification of Mukai's original definition that we alluded to above has to do with the difference between td_X and $\hat{A}_X$: while the Todd class of X is obtained from the power series associated to

$$\frac{x}{1 - e^{-x}},$$

the $\hat{A}_X$ class is obtained from

$$\frac{x}{e^{x/2} - e^{-x/2}}.$$

They are related by the formula

$$\frac{x}{e^{x/2} - e^{-x/2}} = \frac{x}{1 - e^{-x}} \cdot e^{-x/2}$$

and thus

$$\hat{A}_X = \mathrm{td}_X . \mathrm{ch}(\omega_X)^{1/2}$$

for an appropriately defined square root (see Section 2). The use of the $\hat{A}$ genus instead of td_X in the definition of the Mukai vector ties in well with Kontsevich's claim (5.1).

1.6. The connection between the Hochschild and harmonic structures is provided by the Hochschild–Kostant–Rosenberg (HKR) isomorphism, which in modern language can be written as a specific quasi-isomorphism

$$I : \Delta^* \mathcal{O}_\Delta \xrightarrow{\sim} \bigoplus_i \Omega_X^i[i],$$

where Δ^* is the left derived functor of the usual pull-back functor, and the right-hand side of the quasi-isomorphism is the complex which has Ω_X^i in the $-i$th position, and all differentials are zero. The isomorphism I induces isomorphisms of graded vector spaces (Corollary 4.2)

$$I^{\mathrm{HKR}} : HH^*(X) \xleftarrow{\sim} HT^*(X),$$
$$I_{\mathrm{HKR}} : HH_*(X) \xrightarrow{\sim} H\Omega_*(X).$$

Theorem 4.5. *The composition*

$$K_0(X) \xrightarrow{\mathrm{ch}} HH_0(X) \xrightarrow{I_{\mathrm{HKR}}} \bigoplus_i H^i(X, \Omega_X^i)$$

agrees with the usual Chern character map.

This result was originally stated without proof and in an incomplete form in a preprint by Markarian [15].

As part of our proof of this theorem we prove the following result, which provides an interesting interpretation of the Atiyah class in view of the HKR isomorphism:

Proposition 4.4. *The exponential of the universal Atiyah class is precisely the map*

$$\mathcal{O}_\Delta \xrightarrow{\eta} \Delta_*\Delta^*\mathcal{O}_\Delta \xrightarrow{\Delta_* I} \bigoplus_i \Delta_*\Omega_X^i[i],$$

where η is the unit of the adjunction $\Delta^* \dashv \Delta_*$.

1.7. While the HKR isomorphism is well-behaved with respect to the Chern character (in fact one can take Theorem 4.5 as a definition of the differential forms-valued Chern character), it was argued by Kontsevich [12] and Shoikhet [18] that I^{HKR}, I_{HKR} do not respect the Hochschild and harmonic structures. Specifically, I^{HKR} is *not* a ring isomorphism. However, Kontsevich argued that as a consequence of his proof of the formality conjecture, modifying I^{HKR} by $\hat{A}_X^{-1/2}$ does in fact yield a ring isomorphism. More precisely, denote by I^K the isomorphism

$$I^K : HH^*(X) \xrightarrow{(I^{\mathrm{HKR}})^{-1}} HT^*(X) \xrightarrow{\lrcorner \hat{A}_X^{-1/2}} HT^*(X),$$

where the second map is given by the contraction of a polyvector field with $\hat{A}_X^{-1/2}$. Then I^K is a ring isomorphism [12, Claim 8.4].

1.8. A similar phenomenon can be seen on the level of homology theories: the Mukai product that we define in (3.7) does *not* satisfy

$$\langle \mathrm{ch}(\mathscr{E}), \mathrm{ch}(\mathscr{F}) \rangle = \chi(\mathscr{E}, \mathscr{F})$$

as would have been expected from the similar property of Hochschild homology. The correct statement (already known to Mukai in the case of K3 surfaces) is that

$$\langle v(\mathscr{E}), v(\mathscr{F}) \rangle = \chi(\mathscr{E}, \mathscr{F}),$$

where

$$v(\mathscr{E}) = \mathrm{ch}(\mathscr{E}).\hat{A}^{1/2}.$$

These observations lead to the following conjecture:

Conjecture 5.2. *The maps*

$$I^K : HH^*(X) \to HT^*(X), \quad I_K : HH_*(X) \to H\Omega_*(X),$$

where I^K *is the composition*

$$I^K : HH^*(X) \xrightarrow{(I^{\mathrm{HKR}})^{-1}} HT^*(X) \xrightarrow{\lrcorner \hat{A}_X^{1/2}} HT^*(X),$$

and I_K *is given by*

$$I_K : HH_*(X) \xrightarrow{I_{\mathrm{HKR}}} H\Omega_*(X) \xrightarrow{\wedge \hat{A}_X^{1/2}} H\Omega_*(X),$$

induce an isomorphism between the Hochschild and the harmonic structures of X. Concretely, I^K *is a ring isomorphism,* I_K *is an isometry with respect to the generalized Mukai product, and the two isomorphisms are compatible with the module structures on* $H\Omega_*(X)$ *and* $HH_*(X)$*, respectively.*

It is worthwhile observing that both I^K and I_K arise from the same modification of the HKR isomorphism I (5.3). Similar conjectures (without involving the Mukai pairing) have been stated by Tsygan and are usually referred to as Tsygan formality [20].

1.9. The main reason these results are interesting is because it has been conjectured by Kontsevich [13] that, in the case of a Calabi–Yau manifold, $HH^*(X)$ should be closely related to the ordinary cohomology ring $H^*(\check{X}, \mathbf{C})$ of the *mirror* $\check{X}$ of X. In a future paper we shall expand this idea further, introducing a product structure on the Hochschild homology of a Calabi–Yau orbifold and arguing that its properties make it a good candidate for the mirror of Chen–Ruan's [6] orbifold cohomology theory.

Another application of the results in this paper, also to appear in the future, is a conceptual explanation of the results of the computations of Fantechi and Göttsche [7], which show that the orbifold cohomology of a symmetric product of abelian or K3 surfaces agrees with the cohomology of the Hilbert scheme of points on the surface. This explanation is a combination of the main result of Bridgeland et al. [3] with ideas of Verbitsky [21] and with the derived category invariance of the Hochschild structure.

1.10. The paper is structured as follows: after an introductory section in which we discuss integral transforms and natural transformations between them and we define the Mukai vector, we turn in Section 3 to a definition of the Mukai pairing on forms and to proofs of its basic functoriality and adjointness properties. Section 4 is devoted to a discussion of the HKR isomorphism and of the compatibility between the Chern character defined in (1.2) and the usual one. We conclude with a discussion of the main conjecture and of possible ways of proving it in Section 5.

Conventions

All the spaces involved are smooth algebraic varieties proper over **C** (or any algebraically closed field of characteristic zero), or compact complex manifolds. We shall

always omit the symbols **L** and **R** in front of push-forward, pull-back and tensor functors, but we shall consider them as derived except where explicitly stated otherwise. We shall write $\mathscr{F} \otimes \mu$ where $\mathscr{F}$ is a sheaf and μ is a morphism and mean by this the morphism $1_{\mathscr{F}} \otimes \mu$. We shall use either $\wedge$ or . for the usual product in cohomology. Serre duality notations and conventions are presented in detail in Section 2.

2. Preliminaries

In this section, we provide a brief introduction to integral functors on the level of derived categories and rational cohomology. The concepts and results are mostly straightforward generalizations of Mukai's original results [16,17]. The new material is in the definition of the directed Mukai vector (2.1).

We also include in this section several results on traces and duality theory that will be needed later on.

2.1. Let X and Y be complex manifolds, and let $\mathscr{E}$ be an object in $\mathbf{D}^b_{\mathrm{coh}}(X \times Y)$. If π_X and π_Y are the projections from $X \times Y$ to X and Y, respectively, we define the integral transform with kernel $\mathscr{E}$ to be the functor

$$\Phi^{\mathscr{E}}_{X\to Y} : \mathbf{D}^b_{\mathrm{coh}}(X) \to \mathbf{D}^b_{\mathrm{coh}}(Y) \quad \Phi^{\mathscr{E}}_{X\to Y}(\cdot) = \pi_{Y,*}(\pi_X^*(\cdot) \otimes \mathscr{E}).$$

Likewise, if μ is any element of the ring $H^*(X \times Y, \mathbf{Q})$, we define the map

$$\varphi^{\mu}_{X\to Y} : H^*(X, \mathbf{Q}) \to H^*(Y, \mathbf{Q}) \quad \varphi^{\mu}_{X\to Y}(\cdot) = \pi_{Y,*}(\pi_X^*(\cdot).\mu)$$

and call it the integral transform (in cohomology) associated to μ.

Note that none of these concepts is symmetric in X and Y: the object $\mathscr{E}$ defines both a functor from X to Y and one from Y to X, and we clearly distinguish between the two.

2.2. The association between objects of $\mathbf{D}^b_{\mathrm{coh}}(X \times Y)$ and integral transforms is functorial: given a morphism $\mu : \mathscr{E} \to \mathscr{F}$ between objects of $\mathbf{D}^b_{\mathrm{coh}}(X \times Y)$, there is an obvious natural transformation

$$\Phi^{\mu}_{X\to Y} : \Phi^{\mathscr{E}}_{X\to Y} \Rightarrow \Phi^{\mathscr{F}}_{X\to Y},$$

given by

$$\Phi^{\mu}_{X\to Y}(\cdot) = \pi_{Y,*}(\pi_X^*(\cdot) \otimes \mu).$$

2.3. There is a natural map between the derived category and the cohomology ring, namely the exponential Chern character, $\mathrm{ch} : \mathbf{D}^b_{\mathrm{coh}}(X) \to H^*(X, \mathbf{Q})$. It commutes with pull-backs, and transforms tensor products into cup products. In an ideal world, it would also commute with push-forwards, and then the diagram

$$\begin{array}{ccc} \mathbf{D}^b_{\mathrm{coh}}(X) & \xrightarrow{\Phi^{\mathscr{E}}_{X\to Y}} & \mathbf{D}^b_{\mathrm{coh}}(Y) \\ \mathrm{ch}\downarrow & & \mathrm{ch}\downarrow \\ H^*(X, \mathbf{Q}) & \xrightarrow{\varphi^{\mathrm{ch}(\mathscr{E})}_{X\to Y}} & H^*(Y, \mathbf{Q}) \end{array}$$

would commute. However, the Grothendieck–Riemann–Roch formula tells us that we need to correct the commutation of push-forward and ch by the Todd classes of the spaces involved; more precisely, if $\pi : X \to Y$ is a locally complete intersection morphism, then

$$\pi_*(\mathrm{ch}(\cdot).\mathrm{td}_X) = \mathrm{ch}(\pi_*(\cdot)).\mathrm{td}_Y.$$

Our purpose is to define a Mukai vector v such that replacing ch by v the above diagram becomes commutative.

2.4. It is easy to see that there exists a unique formal series expansion $\sqrt{1 + c_1 + c_2 + \ldots}$ in the symbols $c_1, c_2, \ldots$, such that

$$\sqrt{1} = 1,$$
$$\sqrt{\mu v} = \sqrt{\mu} \cdot \sqrt{v}$$

and

$$(\sqrt{\mu})^2 = \mu$$

for every space X and any $\mu, v \in H^{\mathrm{even}}(X, \mathbf{Q})$ with constant term 1. Its first three terms are

$$\sqrt{1 + c_1 + c_2 + \ldots} = 1 + \tfrac{1}{2}c_1 + \tfrac{1}{8}(4c_2 - c_1^2) + \tfrac{1}{16}(8c_3 - 4c_1c_2 + c_1^3) + \ldots .$$

A similar definition enables us to define a unique fourth-order root,

$$\sqrt[4]{1 + c_1 + c_2 + \ldots} = 1 + \tfrac{1}{4}c_1 + \tfrac{1}{32}(8c_2 - 3c_1^2) + \ldots .$$

For any smooth space X let $\hat{A}_X \in H^*(X, \mathbf{C})$ be the characteristic class associated to T_X, the tangent bundle of X, via the power series of

$$\frac{x}{e^{x/2} - e^{-x/2}}.$$

It is related to the usual Todd class of X by the formula

$$\hat{A}_X = \mathrm{td}_X . \sqrt{\mathrm{ch}(\omega_X)}$$

where ω_X is the canonical line bundle of X.

2.5. Recall from [5] that the Chern character of an object $\mathscr{E} \in \mathbf{D}^b_{\mathrm{coh}}(X)$ was obtained by thinking of $\mathscr{E}$ as giving a functor $\Phi^{\mathscr{E}}_{\mathrm{pt}\to X} : \mathbf{D}^b_{\mathrm{coh}}(\mathrm{pt}) \to \mathbf{D}^b_{\mathrm{coh}}(X)$. This directed point of view explains the asymmetry in the following definition:

Definition 2.1. The *directed* Mukai vector of an element $\mathscr{E} \in \mathbf{D}^b_{\mathrm{coh}}(X \times Y)$ is defined by

$$v(\mathscr{E}, X \to Y) = \mathrm{ch}(\mathscr{E}) . \sqrt{\mathrm{td}_{X\times Y}} . \sqrt[4]{\frac{\mathrm{ch}(\omega_Y)}{\mathrm{ch}(\omega_X)}}.$$

Whenever the direction is obvious, we shall omit the $X \to Y$ and just write $v(\mathscr{E})$ instead. (We abuse notation slightly, and write ω_X for $\pi_X^* \omega_X$, etc.)

Taking the first space to be a point we obtain the definition of the Mukai vector of an object $\mathscr{E} \in \mathbf{D}^b_{\mathrm{coh}}(X) = \mathbf{D}^b_{\mathrm{coh}}(\mathrm{pt} \times X)$:

$$v(\mathscr{E}) = \mathrm{ch}(\mathscr{E}) . \sqrt{\hat{A}_X}.$$

2.6. A straightforward calculation shows that the diagram

$$\begin{array}{ccc} \mathbf{D}^b_{\mathrm{coh}}(X) & \xrightarrow{\Phi^{\mathscr{E}}_{X\to Y}} & \mathbf{D}^b_{\mathrm{coh}}(Y) \\ \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle v} \\ H^*(X, \mathbf{Q}) & \xrightarrow{\varphi^{v(\mathscr{E})}_{X\to Y}} & H^*(Y, \mathbf{Q}) \end{array}$$

commutes. (This is a direct analogue of [5, Theorem 7.1].) We shall denote the map $\varphi^{v(\mathscr{E})}_{X\to Y}$ by Φ_*, where $\Phi = \Phi^{\mathscr{E}}_{X\to Y}$.

2.7. Given complex manifolds X, Y, Z, and elements $\mathscr{E} \in \mathbf{D}^b_{\mathrm{coh}}(X \times Y)$ and $\mathscr{F} \in \mathbf{D}^b_{\mathrm{coh}}(Y \times Z)$, define $\mathscr{F} \circ \mathscr{E} \in \mathbf{D}^b_{\mathrm{coh}}(X \times Z)$ by

$$\mathscr{F} \circ \mathscr{E} = \pi_{XZ,*}(\pi^*_{XY}\mathscr{E} \otimes \pi^*_{YZ}\mathscr{F}),$$

where $\pi_{XY}, \pi_{YZ}, \pi_{XZ}$ are the projections from $X \times Y \times Z$ to $X \times Y$, $Y \times Z$ and $X \times Z$, respectively. Similarly, if $\mu \in H^*(X \times Y, \mathbf{Q})$, $v \in H^*(Y \times Z, \mathbf{Q})$, consider $v \circ \mu \in H^*(X \times Z, \mathbf{Q})$ given by

$$v \circ \mu = \pi_{XZ,*}(\pi^*_{XY}\mu.\pi^*_{YZ}v).$$

The reason behind the notation is the fact that

$$\Phi^{\mathscr{F}}_{Y\to Z} \circ \Phi^{\mathscr{E}}_{X\to Y} = \Phi^{\mathscr{F}\circ\mathscr{E}}_{X\to Z}$$

and

$$\varphi^{v}_{Y\to Z} \circ \varphi^{\mu}_{X\to Y} = \varphi^{v\circ\mu}_{X\to Z}.$$

(The second result is standard; for a proof of the first one see [1, 1.4].) Furthermore, it is a straightforward calculation to check that

$$v(\mathscr{F} \circ \mathscr{E}, X \to Z) = v(\mathscr{F}, Y \to Z) \circ v(\mathscr{E}, X \to Y)$$

[4, 3.1.10]. It follows that if $\Psi : \mathbf{D}^b_{\mathrm{coh}}(X) \to \mathbf{D}^b_{\mathrm{coh}}(Y)$ and $\Phi : \mathbf{D}^b_{\mathrm{coh}}(Y) \to \mathbf{D}^b_{\mathrm{coh}}(Z)$ are integral transforms, then we have

$$(\Phi \circ \Psi)_* = \Phi_* \circ \Psi_*$$

(compare also to [5, Theorem 6.3]). Since it can be easily checked that $\mathrm{Id}_* = \mathrm{Id}$, it follows that if Φ is an equivalence of derived categories, then Φ_* is an isomorphism $H^*(X, \mathbf{Q}) \to H^*(Y, \mathbf{Q})$.

2.8. The map Φ_* does not respect the usual grading on the cohomology rings of X and Y, nor does it respect Hodge decompositions. However, it does respect the decomposition of $H^*(X)$ by *columns* of the Hodge diamond: for every i, Φ_* maps $H\Omega_i(X)$ to $H\Omega_i(Y)$,

$$\Phi_* = \varphi_{X\to Y}^{v(\mathscr{E})} : H\Omega_i(X) = \bigoplus_{q-p=i} H^{p,q}(X) \to H\Omega_i(Y) = \bigoplus_{q-p=i} H^{p,q}(Y),$$

because $v(\mathscr{E})$ consists only of classes of type $H^{p,p}(X \times Y)$, and pushing-forward to Y maps a class of type (p, q) to a class of type $(p - \dim X, q - \dim X)$.

This statement is the harmonic structure analogue of the fact that the push-forward on Hochschild homology preserves the grading.

3. The Mukai pairing on cohomology

In Section 2, we defined a grated vector space map $\Phi_* : H^*(X, \mathbf{Q}) \to H^*(Y, \mathbf{Q})$ associated to an integral transform $\Phi : \mathbf{D}^b_{\mathrm{coh}}(X) \to \mathbf{D}^b_{\mathrm{coh}}(Y)$. In the case of K3 surfaces, Mukai proved that when Φ is an equivalence, Φ_* is an isometry with respect to a modified version of the usual pairing on the total cohomology rings of X and Y. He did this by showing the more powerful result that maps on cohomology associated to adjoint functors are themselves adjoint with respect to this modified pairing. In this section, we generalize this result to arbitrary complex manifolds (not necessarily of dimension 2 or with trivial canonical class), by defining a suitable generalization of Mukai's pairing.

3.1. The reason behind Φ_* being an isometry for the Mukai product is the fact that an equivalence $\Phi : \mathbf{D}^b_{\mathrm{coh}}(X) \to \mathbf{D}^b_{\mathrm{coh}}(Y)$ must satisfy

$$\begin{aligned}\chi_X(\mathscr{F}, \mathscr{G}) &= \sum_i (-1)^i \dim \mathbf{R}\mathrm{Hom}^i(\mathscr{F}, \mathscr{G}) \\ &= \sum_i (-1)^i \dim \mathbf{R}\mathrm{Hom}^i(\Phi\mathscr{F}, \Phi\mathscr{G}) \\ &= \chi_Y(\Phi\mathscr{F}, \Phi\mathscr{G}).\end{aligned}$$

Thus, if we define a pairing on the algebraic part of $H^*(X, \mathbf{Q})$ by

$$\langle v(\mathscr{F}), v(\mathscr{G})\rangle = \chi_X(\mathscr{F}, \mathscr{G})$$

for all $\mathscr{F}, \mathscr{G} \in \mathbf{D}^b_{\mathrm{coh}}(X)$, then Φ_* is an isometry between the algebraic subrings of $H^*(X, \mathbf{Q})$ and $H^*(Y, \mathbf{Q})$ (because v commutes with Φ).

3.2. There are two problems with this definition: one is whether the above pairing is well defined, another if we can extend it to a pairing on the whole cohomology ring of X. For K3 surfaces we have

$$\begin{aligned}
\chi_X(\mathscr{F}, \mathscr{G}) &= \chi_X(\mathscr{F}^\vee \otimes \mathscr{G}) \\
&= \int_X \mathrm{ch}(\mathscr{F}^\vee).\mathrm{ch}(\mathscr{G}).\mathrm{td}_X \\
&= \int_X \mathrm{ch}(\mathscr{F}^\vee).\sqrt{\mathrm{td}_X}.\mathrm{ch}(\mathscr{G}).\sqrt{\mathrm{td}_X} \\
&= \int_X v(\mathscr{F}^\vee).v(\mathscr{G}) \\
&= \int_X v(\mathscr{F})^\vee.v(\mathscr{G}),
\end{aligned}$$

where $\mathscr{F}^\vee = \mathbf{R}\mathrm{Hom}(\mathscr{F}, \mathscr{O}_X)$, and for a vector

$$v = (v_0, v_2, v_4) \in H^0(X, \mathbf{Q}) \oplus H^2(X, \mathbf{Q}) \oplus H^4(X, \mathbf{Q})$$

$v^\vee$ is defined to equal $(v_0, -v_2, v_4)$. Thus the pairing is well defined in the K3 case (it only depends on the Mukai vectors of $\mathscr{F}$ and $\mathscr{G}$, and not on $\mathscr{F}$ and $\mathscr{G}$ themselves). Note that for a K3 surface we have $c_1(\omega_X) = 0$, therefore the $\hat{A}$ class agrees with the Todd class and thus

$$v(\mathscr{F}) = \mathrm{ch}(\mathscr{F}).\sqrt{\mathrm{td}_X}.$$

3.3. Our goal is to define $v^\vee$ for any X and any $v \in H^{\mathrm{even}}(X, \mathbf{Q})$ (and eventually for any $v \in H^*(X, \mathbf{Q})$), such that we have the equality

$$\chi_X(\mathscr{F}, \mathscr{G}) = \int_X v(\mathscr{F})^\vee \cdot v(\mathscr{G}).$$

The definition of $\cdot^\vee$ will arise from the conceptual description of the Hirzebruch–Riemann–Roch formula given in [5].

Recall that in [loc.cit.] we obtained the Riemann–Roch formula by using the adjunction

$$\Phi^{\mathscr{F}}_{\mathrm{pt}\to X} \dashv \Phi^{\mathscr{F}^\vee}_{X\to \mathrm{pt}},$$

through the equalities

$$\begin{aligned}\langle \mathrm{ch}(\mathscr{F}),\ \mathrm{ch}(\mathscr{G})\rangle &= \langle (\Phi^{\mathscr{F}}_{\mathrm{pt}\to X})_*\mathbf{1},\ (\Phi^{\mathscr{G}}_{\mathrm{pt}\to X})_*\mathbf{1}\rangle \\ &= \langle \mathbf{1},\ (\Phi^{\mathscr{F}^\vee}_{X\to \mathrm{pt}})_*(\Phi^{\mathscr{G}}_{\mathrm{pt}\to X})_*\mathbf{1}\rangle \\ &= \langle \mathbf{1},\ (\Phi^{\mathscr{F}^\vee\circ\mathscr{G}}_{\mathrm{pt}\to \mathrm{pt}})_*\mathbf{1}\rangle = \chi_X(\mathscr{F},\mathscr{G}).\end{aligned}$$

This clearly suggests that we should define

$$v(\mathscr{F})^\vee = v(\mathscr{F}, \mathrm{pt}\to X)^\vee = v(\mathscr{F}^\vee, X\to \mathrm{pt}),$$

because then we will get

$$\begin{aligned}\chi_X(\mathscr{F},\mathscr{G}) &= v(\mathscr{F}^\vee\circ\mathscr{G}, \mathrm{pt}\to \mathrm{pt}) = v(\mathscr{F}^\vee, X\to \mathrm{pt})\circ v(\mathscr{G}, \mathrm{pt}\to X) \\ &= \int_X v(\mathscr{F})^\vee . v(\mathscr{G}).\end{aligned}$$

3.4. More generally, we are led to requiring $\cdot^\vee$ to satisfy

$$v(\mathscr{E}, X\to Y)^\vee = v(\mathscr{E}^*, Y\to X)$$

where

$$\mathscr{E}^* = \mathscr{E}^\vee \otimes \pi_X^*\omega_X[\dim X]$$

is the object on $Y\times X$ which gives the adjunction [2, Lemma 4.5]

$$\Phi^{\mathscr{E}}_{X\to Y} \dashv \Phi^{\mathscr{E}^*}_{Y\to X}.$$

Consider the involution

$$\tau : H^{\mathrm{even}}(X,\mathbf{Q}) \to H^{\mathrm{even}}(X,\mathbf{Q})$$

given by

$$\tau(v_0, v_2, \ldots, v_{2n}) = (v_0, -v_2, v_4, \ldots, (-1)^n v_{2n}).$$

It is easy to check that τ satisfies $\tau(vw) = \tau(v)\tau(w)$, and it is well known that $\mathrm{ch}(\mathscr{E}^\vee) = \tau(\mathrm{ch}(\mathscr{E}))$. Thus

$$\begin{aligned}
\tau(v(\mathscr{E}, X \to Y)) &= \tau(\mathrm{ch}(\mathscr{E})).\tau(\sqrt{\mathrm{td}_{X\times Y}}).\tau\left(\sqrt[4]{\frac{\mathrm{ch}(\omega_Y)}{\mathrm{ch}(\omega_X)}}\right) \\
&= \mathrm{ch}(\mathscr{E}^\vee).\sqrt{\mathrm{td}(T^\vee_{X\times Y})}.\sqrt[4]{\frac{\mathrm{ch}(\omega_X)}{\mathrm{ch}(\omega_Y)}} \\
&= \mathrm{ch}(\mathscr{E}^\vee).\sqrt{\mathrm{td}_{X\times Y}}.\sqrt{\mathrm{ch}(\omega_{X\times Y})}.\sqrt[4]{\frac{\mathrm{ch}(\omega_X)}{\mathrm{ch}(\omega_Y)}} \\
&= (-1)^{\dim X}\mathrm{ch}(\mathscr{E}^\vee \otimes \omega_X[\dim_X]).\sqrt{\mathrm{td}_{X\times Y}}.\sqrt[4]{\frac{\mathrm{ch}(\omega_Y)}{\mathrm{ch}(\omega_X)}} \\
&= (-1)^{\dim X} v(\mathscr{E}^*, X \to Y) \\
&= (-1)^{\dim X} v(\mathscr{E}^*, Y \to X).\sqrt{\frac{\mathrm{ch}(\omega_Y)}{\mathrm{ch}(\omega_X)}},
\end{aligned}$$

where the third equality is an immediate consequence of the formula [8, I.5.2]

$$\mathrm{td}(T^\vee_X) = \mathrm{td}(T_X).\exp(-c_1(T_X)) = \mathrm{td}(T_X).\mathrm{ch}(\omega_X)$$

(We have abused notation slightly, and we wrote ω_X for $\pi_X^*\omega_X$, etc.)

3.5. The above calculation motivates the following definition:

Definition 3.1. For $e \in H^*(X \times Y, \mathbf{Q})$ set

$$e^\vee = (-1)^{\dim X}\tau(e).\sqrt{\frac{\mathrm{ch}(\omega_X)}{\mathrm{ch}(\omega_Y)}}.$$

In particular, for a single space X (considered as $\mathrm{pt} \times X$) and $v \in H^*(X, \mathbf{Q})$ let

$$v^\vee = \tau(v).\frac{1}{\sqrt{\mathrm{ch}(\omega_X)}}.$$

The calculations in (3.3) now show that we have

$$\chi_X(\mathscr{F}, \mathscr{G}) = \int_X v(\mathscr{F})^\vee . v(\mathscr{G}).$$

3.6. To obtain a full generalization of the Mukai product we need to extend $\cdot^\vee$ to all of $H^*(X, \mathbf{Q})$. A natural extension of the involution τ is the map $\tau : H^*(X, \mathbf{C}) \to H^*(X, \mathbf{C})$ given by

$$\tau(v_0, v_1, v_2, \ldots, v_{2n}) = (v_0, i v_1, -v_2, \ldots, i^{2n} v_{2n}),$$

where $i = \sqrt{-1}$. Its main properties are

(1) $\tau(vw) = \tau(v).\tau(w)$;
(2) $\tau(\sqrt{v}) = \sqrt{\tau(v)}$ for any v with leading term equal to 1;
(3) $\tau(\tau(v)) = v$ for any $v \in H^{\text{even}}(X, \mathbf{C})$;
(4) $\tau(\text{ch}(\mathscr{L})) = \text{ch}(\mathscr{L}^{-1}) = \text{ch}(\mathscr{L})^{-1}$ for any line bundle $\mathscr{L}$;
(5) $\tau(f^*(v)) = f^*(\tau(v))$;
(6) $f_*(\tau(v)) = (-1)^{\dim_{\mathbf{C}} X - \dim_{\mathbf{C}} Y} \tau(f_* v)$,

where $f : X \to Y$ is any proper morphism of complex manifolds. The proof of all these properties is immediate.

Thus, defining

$$\cdot^\vee : H^*(X \times Y, \mathbf{C}) \to H^*(X \times Y, \mathbf{C})$$

by

$$e^\vee = (-1)^{\dim X} \tau(e). \sqrt{\frac{\text{ch}(\omega_X)}{\text{ch}(\omega_Y)}}$$

extends in a natural way the operator $\cdot^\vee$ previously defined.

3.7. We can now tackle the generalized Mukai product:

Definition 3.2. Let X be a complex manifold, and let $v, w \in H^*(X, \mathbf{C})$. Define the product $\langle v, w \rangle$ by the formula

$$\langle v, w \rangle = \int_X v^\vee . w,$$

where $v^\vee$ is defined above. This product will be called the *generalized Mukai product*.

3.8. It is interesting to compare this definition with a similar one that appears in Hodge theory. Define the Weyl operator, $\bar{\tau}$, by $\bar{\tau}(v) = i^{p-q}v$ for $v \in H^{p,q}(X)$. The pairing

$$\langle v, w\rangle = \int_X \bar{\tau}(v).w$$

is the standard one that appears in the definition of a polarized Hodge structure. Observe that the analogy between the Mukai pairing as a mirror to the usual Poincaré pairing holds, if we take this in the sense of matching polarizations: the map τ is formally the mirror of $\bar{\tau}$ (if we mirror the Hodge diamond, τ gets transformed into $\bar{\tau}$). We do not have a good understanding of the $1/\sqrt{\mathrm{ch}(\omega_X)}$ term that appears in the definition of the Mukai pairing.

Proposition 3.3. *Let X and Y be complex manifolds, and $\Phi : \mathbf{D}^b_{\mathrm{coh}}(X) \to \mathbf{D}^b_{\mathrm{coh}}(Y)$ and $\Psi : \mathbf{D}^b_{\mathrm{coh}}(Y) \to \mathbf{D}^b_{\mathrm{coh}}(X)$ be adjoint integral transforms (Ψ is a left adjoint to Φ). Then Ψ_* is a left adjoint to Φ_* with respect to the generalized Mukai product; in other words, we have*

$$\langle v, \Phi_* w\rangle_Y = \langle \Psi_* v, w\rangle_X$$

for all $v \in H^*(Y, \mathbf{C})$, $w \in H^*(X, \mathbf{C})$.

Remark 3.4. When v and w are Mukai vectors of elements in $\mathbf{D}^b_{\mathrm{coh}}(Y)$ and $\mathbf{D}^b_{\mathrm{coh}}(X)$, the result is a trivial consequence of the discussion in (3.1). The actual content is that the result holds for all v, w.

Corollary 3.5. *Under the hypotheses of Proposition* 3.3, *assume furthermore that Φ is an equivalence of categories. Then $\Phi_* : H^*(X, \mathbf{C}) \to H^*(Y, \mathbf{C})$ is an isometry with respect to the generalized Mukai product.*

Proof. See the proof of [5, Corollary 7.5]. □

Proof of Proposition 3.3. Assume $\Phi = \Phi^{\mathscr{E}}_{X\to Y}$, and let $\mathscr{E}^* = \mathscr{E}^\vee \otimes \pi_Y^* \omega_Y[\dim Y]$, so that $\Psi = \Phi^{\mathscr{E}^*}_{Y\to X}$. Define $e = v(\mathscr{E}, X \to Y)$ and $e^* = v(\mathscr{E}^*, Y \to X)$. A computation entirely similar to the one in (3.4) yields

$$e^* = (-1)^{\dim Y} \tau(e) \frac{\pi_Y^* \sqrt{\mathrm{ch}(\omega_Y)}}{\pi_X^* \sqrt{\mathrm{ch}(\omega_X)}}$$

and thus

$$\tau(e^*) = (-1)^{\dim Y} e.\frac{\pi_X^*\sqrt{\mathrm{ch}(\omega_X)}}{\pi_Y^*\sqrt{\mathrm{ch}(\omega_Y)}}.$$

We then have

$$\begin{aligned}
\langle \Psi_* v, w\rangle &= \langle \varphi^{e^*}_{Y\to X}(v), w\rangle = \int_X \varphi^{e^*}_{Y\to X}(v)^\vee w = \int_X \tau(\varphi^{e^*}_{Y\to X}(v)).\frac{1}{\sqrt{\mathrm{ch}(\omega_X)}}\, w\\
&= \int_X \tau(\pi_{X,*}(\pi_Y^* v e^*)).\frac{1}{\sqrt{\mathrm{ch}(\omega_X)}}\, w\\
&= (-1)^{\dim Y}\int_X \pi_{X,*}(\tau(\pi_Y^* v).\tau(e^*)).\frac{1}{\sqrt{\mathrm{ch}(\omega_X)}} w\\
&= (-1)^{\dim Y}\int_{X\times Y} \tau(\pi_Y^* v).\tau(e^*).\frac{1}{\pi_X^*\sqrt{\mathrm{ch}(\omega_X)}}.\pi_X^* w\\
&= (-1)^{\dim Y}\int_{X\times Y} \tau(\pi_Y^* v).(-1)^{\dim Y}.e.\frac{\pi_X^*\sqrt{\mathrm{ch}(\omega_X)}}{\pi_Y^*\sqrt{\mathrm{ch}(\omega_Y)}}.\frac{1}{\pi_X^*\sqrt{\mathrm{ch}(\omega_X)}}.\pi_X^* w\\
&= \int_{X\times Y} \pi_Y^*(\tau(v)).\frac{1}{\pi_Y^*\sqrt{\mathrm{ch}(\omega_Y)}}.e.\pi_X^* w\\
&= \int_Y \tau(v).\frac{1}{\sqrt{\mathrm{ch}(\omega_Y)}}.\pi_{Y,*}(e.\pi_X^* w)\\
&= \int_Y v^\vee.\varphi^e_{X\to Y}(w) = \langle v, \varphi^e_{X\to Y}(w)\rangle\\
&= \langle v, \Phi_* w\rangle. \qquad \square
\end{aligned}$$

4. The Hochschild–Kostant–Rosenberg theorem and the Chern character

In this section, we study the relationship between the Hochschild and harmonic structures. We provide a discussion of the connection between the usual Chern character and the one introduced in [5].

4.1. The starting point of our analysis is the following theorem:

Theorem 4.1 *(Hochschild–Kostant–Rosenberg [9], Kontsevich [12], Swan [19] and Yekutieli [22]). Let X be a smooth, quasi-projective variety, and let $\Delta : X \to X \times X$*

be the diagonal embedding. Then there exists a quasi-isomorphism

$$I : \Delta^* \mathcal{O}_\Delta \xrightarrow{\sim} \bigoplus_i \Omega_X^i[i],$$

where the right-hand side denotes the complex whose $-i$-th term is Ω_X^i, and all differentials are zero.

Proof. (This is nothing but a brief recounting of the results in [22], and the reader should consult [loc.cit.] for more details.) Recall that if R is a commutative $\mathbf{C}$-algebra there exists a standard resolution of R as an $R^e = R \otimes_{\mathbf{C}} R$-module. For $i \geqslant 0$ let

$$\mathcal{B}_i(R) = R^{\otimes(i+2)},$$

where the tensor product is taken over $\mathbf{C}$. It is an R^e-module by multiplication in the first and last factor. The bar resolution is defined to be the complex of R^e-modules

$$\cdots \to \mathcal{B}_i(R) \to \cdots \to \mathcal{B}_1(R) \to \mathcal{B}_0(R) \to 0,$$

with differential

$$\begin{aligned} &d(a_0 \otimes a_1 \otimes \cdots \otimes a_i) \\ &\quad = a_0 a_1 \otimes a_2 \otimes \cdots \otimes a_i \ - \ a_0 \otimes a_1 a_2 \otimes \cdots \otimes a_i \ + \ \cdots \\ &\qquad + (-1)^{i-1} a_0 \otimes a_1 \otimes \cdots \otimes a_{i-1} a_i . \end{aligned}$$

It is an exact complex, except at the last step where the cohomology is R. Thus it is a resolution of R in R^e-$\mathfrak{Mod}$ [14, 1.1.12].

If X were affine, $X = \operatorname{Spec} R$, we could use the above resolution to compute $\Delta^* \mathcal{O}_\Delta$: indeed, $\mathcal{O}_\Delta$ is nothing but R viewed as an $R^e = \mathcal{O}_{X \times X}$-module, and the modules $\mathcal{B}_i$ are R^e-flat. The complex obtained by tensoring the bar resolution over R^e with R is called the Hochschild chain complex:

$$\cdots \to \mathcal{C}_i(R) \to \cdots \to \mathcal{C}_1(R) \to \mathcal{C}_0(R) \to 0,$$

where

$$\mathcal{C}_i(R) = \mathcal{B}_i(R) \otimes_{R^e} R,$$

and the differential is obtained from the differential of $\mathcal{B}_\cdot(R)$.

Problems arise when one tries to sheafify the bar resolution to obtain a complex of sheaves on a scheme: the resulting sheaves are ill-behaved (in particular, not quasi-coherent). As a replacement, Yekutieli proposed to use the complete bar resolution, which he defined in [22]. For $i \geqslant 0$, let $\mathfrak{X}^i$ be the formal completion of the scheme $X^i = X \times \cdots \times X$ along the small diagonal. Define

$$\widehat{\mathscr{B}_i}(X) = \mathscr{O}_{\mathfrak{X}^{i+2}},$$

which is a sheaf of abelian groups on the topological space X. Yekutieli argued that one can formally complete and sheafify the original bar resolution to get the complete bar resolution

$$\cdots \to \widehat{\mathscr{B}_i}(X) \to \cdots \to \widehat{\mathscr{B}_1}(X) \to \widehat{\mathscr{B}_0}(X) \to 0,$$

where the maps are locally obtained from the maps of the original bar resolution, by noting that these are continuous for the topologies with respect to which we are completing. The complete bar resolution is an exact resolution of $\mathscr{O}_\Delta$ by sheaves of flat $\mathscr{O}_{X\times X}$-modules (see remark following Proposition 1.4 and proof of Proposition 1.5 in [22]). Over an affine open set $U = \operatorname{Spec} R$ of X, $\Gamma(U, \widehat{\mathscr{B}_i}(X))$ is the completion $\widehat{\mathscr{B}_i}(R)$ of $\mathscr{B}_i(R)$ at the ideal I_i which is the kernel of the multiplication map $\mathscr{B}_i(R) = R^{\otimes i} \to R$.

One can take the complete bar resolution as a flat resolution of $\mathscr{O}_\Delta$ on $X \times X$, and use it to compute $\Delta^* \mathscr{O}_\Delta$. This is the same as tensoring the complete bar resolution over $\mathscr{O}_{X\times X}$ with $\mathscr{O}_\Delta$. The resulting complex is called the complex of complete Hochschild chains of X (see [22, Definition 1.3] for details),

$$\cdots \to \widehat{\mathscr{C}}_i(X) \to \cdots \to \widehat{\mathscr{C}}_1(X) \to \widehat{\mathscr{C}}_0(X) \to 0,$$

where

$$\widehat{\mathscr{C}}_i(X) = \widehat{\mathscr{B}_i}(X) \otimes_{\mathscr{O}_{X\times X}} \mathscr{O}_\Delta.$$

Over an affine open set $U = \operatorname{Spec} R$, $\Gamma(U, \widehat{\mathscr{C}}_i(X))$ is the completion $\widehat{\mathscr{C}}_i(R)$ of $\mathscr{C}_i(R)$ at I_i (as a $\mathscr{B}_i(R)$-module).

Over any affine open $U = \operatorname{Spec} R$ define

$$I_i : \mathscr{C}_i(R) \to \Omega^i_{R/k}$$

by setting

$$I_i((1 \otimes a_1 \otimes \cdots \otimes a_i \otimes 1) \otimes_{R^e} 1) = \frac{1}{i!}\, da_1 \wedge da_2 \wedge \cdots \wedge da_i.$$

These maps are continuous with respect to the topology that is used for completing [22, Lemma 4.1], so they can be completed and sheafified to maps

$$I_i : \widehat{\mathscr{C}}_i(X) \to \Omega_X^i.$$

They also commute with the zero differentials of the complex $\oplus_i \Omega_X^i$, so they assemble to a morphism of complexes

$$I : \Delta^* \mathcal{O}_\Delta \to \bigoplus_i \Omega_X^i[i]$$

which can be seen to be a quasi-isomorphism in characteristic 0 [12, Theorem 4.6.1.1, 22, Proposition 4.4]. In the affine case this is essentially the Hochschild–Kostant–Rosenberg theorem [9]. □

Corollary 4.2. *The Hochschild–Kostant–Rosenberg isomorphism I induces isomorphisms of graded vector spaces*

$$I^{\mathrm{HKR}} : HH^*(X) \xleftarrow{\sim} HT^*(X),$$
$$I_{\mathrm{HKR}} : HH_*(X) \xrightarrow{\sim} H\Omega_*(X).$$

Proof.

$$\begin{aligned} HH^k(X) &= \mathrm{Hom}_{X\times X}(\mathcal{O}_\Delta, \mathcal{O}_\Delta[k]) \cong \mathrm{Hom}_X(\Delta^*\mathcal{O}_\Delta, \mathcal{O}_X[k]) \\ &\cong \mathrm{Hom}_X\left(\bigoplus_i \Omega_X^i[i], \mathcal{O}_X[k]\right) = \bigoplus_i H^{k-i}(X, \bigwedge^i T_X) = HT^k(X) \end{aligned}$$

and

$$\begin{aligned} HH_k(X) &= \mathrm{Hom}_{X\times X}(\Delta_!\mathcal{O}_X[k], \mathcal{O}_\Delta) \cong \mathrm{Hom}_X(\mathcal{O}_X[k], \Delta^*\mathcal{O}_\Delta) \\ &\cong \mathrm{Hom}_X\left(\mathcal{O}_X[k], \bigoplus_i \Omega_X^i[i]\right) = \bigoplus_i H^{i-k}(X, \Omega_X^i) = H\Omega_k(X). \qquad \square \end{aligned}$$

4.2. We are now interested in understanding how the above isomorphisms relate the Chern character $K_0(X) \to HH_0(X)$ defined in the introduction to the usual Chern character.

Let $\Omega_{\Delta}^{\otimes i}$ and Ω_{Δ}^{i} denote the push-forwards by Δ of $\Omega_X^{\otimes i}$ and Ω_X^i, respectively (Here the tensor product is taken over $\mathcal{O}_X$.) Let

$$\varepsilon : \Omega_X^{\otimes i} \to \Omega_X^i$$

be the natural projection map. By an abuse of notation, we shall also denote by ε the push-forward

$$\Delta_* \varepsilon : \Omega_{\Delta}^{\otimes i} \to \Omega_{\Delta}^{i}.$$

Definition 4.3. Define the universal Atiyah class to be the class

$$\alpha_1 \in \mathrm{Ext}^1_{X \times X}(\mathcal{O}_{\Delta}, \Omega^1_{\Delta}),$$

of the extension

$$0 \to \Omega^1_{\Delta} \to \mathcal{O}_{\Delta^{(2)}} \to \mathcal{O}_{\Delta} \to 0,$$

where $\mathcal{O}_{\Delta^{(2)}}$ is the second infinitesimal neighborhood of the diagonal in $X \times X$. Furthermore, define α_i for $i \geqslant 0$ by the formula

$$\alpha_i = \varepsilon \circ (\pi_2^* \Omega_X^{\otimes (i-1)} \otimes \alpha_1) \circ (\pi_2^* \Omega_X^{\otimes (i-2)} \otimes \alpha_1) \circ \cdots \circ \alpha_1 : \mathcal{O}_{\Delta} \to \Omega_{\Delta}^i[i].$$

The exponential Atiyah class $\exp(\alpha)$ is defined by the formula below, where $n = \dim X$

$$\exp(\alpha) = 1 + \alpha_1 + \frac{1}{2!}\alpha_2 + \cdots + \frac{1}{n!}\alpha_n : \mathcal{O}_{\Delta} \to \bigoplus_i \Delta_* \Omega_X^i[i].$$

This definition requires a short explanation. Recall that given an object $\mathscr{E} \in \mathbf{D}^b_{\mathrm{coh}}(X)$, the Atiyah class of $\mathscr{E}$ is the class

$$\alpha_1(\mathscr{E}) \in \mathrm{Ext}^1_X(\mathscr{E}, \mathscr{E} \otimes \Omega^1_X)$$

of the extension on X

$$0 \to \mathscr{E} \otimes \Omega^1_X \to J^1(\mathscr{E}) \to \mathscr{E} \to 0$$

where $J^1(\mathscr{E})$ is the first jet bundle of $\mathscr{E}$ [11, 1.1]. A natural way to construct this extension is to consider the natural transformation $\Phi^{\alpha_1}_{X\to X}$ associated to the universal Atiyah class

$$\alpha_1 : \mathscr{O}_\Delta \to \Omega^1_\Delta[1]$$

between the identity functor and the "tensor by $\Omega^1_X[1]$" functor. The value $\Phi^{\alpha_1}_{X\to X}(\mathscr{E})$ of this natural transformation on $\mathscr{E}$ is precisely the Atiyah class $\alpha_1(\mathscr{E})$ of $\mathscr{E}$ (see, for example, [10, 10.1.5]). The ith component of the Chern character of $\mathscr{E}$ is then obtained as

$$\mathrm{ch}_i(\mathscr{E}) = \frac{1}{i!}\,\mathrm{Tr}_{\mathscr{E}}(\alpha_i(\mathscr{E}))$$

where

$$\alpha_i(\mathscr{E}) = \varepsilon \circ (\Omega_X^{\otimes(i-1)} \otimes \alpha_1(\mathscr{E})) \circ (\Omega_X^{\otimes(i-2)} \otimes \alpha_1(\mathscr{E})) \circ \cdots \circ \alpha_1(\mathscr{E}) : \mathscr{E} \to \mathscr{E} \otimes \Omega^i_X[i].$$

(See [10, 10.1.6] for details.) Our definition of $\alpha_i : \mathscr{O}_\Delta \to \Delta_*\Omega^i_X[i]$ has been tailored to mimic this definition: $\alpha_i(\mathscr{E})$ will be precisely the value on $\mathscr{E}$ of the natural transformation associated to the morphism α_i. Therefore, if we consider the natural transformation $\Phi^{\exp(\alpha)}_{X\to X}$ associated to $\exp(\alpha)$, its value

$$\Phi^{\exp(\alpha)}(\mathscr{E}) : \mathscr{E} \to \bigoplus_i \mathscr{E} \otimes \Omega^i_X[i]$$

on $\mathscr{E}$ will satisfy

$$\mathrm{ch}_{\mathrm{orig}}(\mathscr{E}) = \mathrm{Tr}_{\mathscr{E}}(\Phi^{\exp(\alpha)}(\mathscr{E})),$$

where $\mathrm{ch}_{\mathrm{orig}}(\mathscr{E})$ is the usual Chern character of $\mathscr{E}$.

Proposition 4.4. *The exponential* $\exp(\alpha)$ *of the universal Atiyah class is precisely the map*

$$\mathscr{O}_\Delta \xrightarrow{\eta} \Delta_*\Delta^*\mathscr{O}_\Delta \xrightarrow{\Delta_* I} \bigoplus_i \Delta_*\Omega^i_X[i],$$

where η *is the unit of the adjunction* $\Delta^* \dashv \Delta_*$.

Proof. We divide the proof of this proposition into several steps, to make it more manageable. We will use the notations used in the proof of Theorem 4.1.

Step 1: Consider the exact sequence

$$0 \to \Omega^1_\Delta \to \mathcal{O}_{\Delta^{(2)}} \to \mathcal{O}_\Delta \to 0$$

which defines the universal Atiyah class α_1. Tensoring it by the locally free sheaf $\pi_2^*\Omega_X^{\otimes i}$ yields the exact sequence

$$0 \to \Omega_\Delta^{\otimes(i+1)} \to \mathcal{O}_{\Delta^{(2)}} \otimes \pi_2^*\Omega_X^{\otimes i} \to \Omega_\Delta^{\otimes i} \to 0.$$

Stringing together these exact sequences for successive values of i we construct the exact sequence

$$0 \to \Omega_\Delta^{\otimes i} \to \mathcal{O}_{\Delta^{(2)}} \otimes \pi_2^*\Omega_X^{\otimes(i-1)} \to \mathcal{O}_{\Delta^{(2)}} \otimes \pi_2^*\Omega_X^{\otimes(i-2)} \to \cdots \to \mathcal{O}_{\Delta^{(2)}} \to \mathcal{O}_\Delta \to 0,$$

whose extension class is precisely

$$(\pi_2^*\Omega_X^{\otimes(i-1)} \otimes \alpha_1) \circ (\pi_2^*\Omega_X^{\otimes(i-2)} \otimes \alpha_1) \circ \cdots \circ \alpha_1 : \mathcal{O}_\Delta \to \Omega_\Delta^{\otimes i}[i].$$

Step 2: We claim that there exists a map $\varphi_\cdot$ of exact sequences

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & \widehat{\mathscr{B}_i}(X) & \longrightarrow & \widehat{\mathscr{B}_{i-1}}(X) & \longrightarrow & \cdots & \longrightarrow & \widehat{\mathscr{B}_0}(X) & \longrightarrow & \mathcal{O}_\Delta \longrightarrow 0 \\
 & & \downarrow{\scriptstyle \varphi_i'} & & \downarrow{\scriptstyle \varphi_{i-1}} & & & & \downarrow{\scriptstyle \varphi_0} & & \| \\
0 & \longrightarrow & \Omega_\Delta^{\otimes i} & \longrightarrow & \mathcal{O}_{\Delta^{(2)}} \otimes \pi_2^*\Omega_X^{\otimes(i-1)} & \longrightarrow & \cdots & \longrightarrow & \mathcal{O}_{\Delta^{(2)}} & \longrightarrow & \mathcal{O}_\Delta \longrightarrow 0,
\end{array}$$

where the top row is the (augmented) completed bar resolution defined in the proof of Theorem 4.1, and the bottom row is the one defined in Step 1. It is sufficient to define the maps in a local patch $U = \operatorname{Spec} R$. Let $I = I_2 = \ker(R \otimes R \to R)$ be the ideal defining the diagonal in $U \times U$, and identify $\Omega^1_{R/\mathbf{C}}$ with I/I^2 via the differential map

$$R \to \Omega^1_{R/\mathbf{C}} = I/I^2, \quad r \mapsto dr = r \otimes 1 - 1 \otimes r + I^2.$$

Consider the maps

$$\varphi_i : \mathscr{B}_i(R) = R^{\otimes(i+2)} \to (R \otimes R)/I^2 \otimes_R \Omega_R^{\otimes Ri}$$

defined by

$$\varphi_i(a_0 \otimes a_1 \otimes \cdots \otimes a_{i+1}) = (a_0 \otimes a_{i+1} + I^2) \otimes_R da_1 \otimes_R da_2 \otimes_R \cdots \otimes_R da_i$$

(we write Ω_R on the right because we use π_2^*). The same argument as the one in the proof of [22, Lemma 4.1] shows that these maps are continuous with respect to the adic topology used to complete $\mathscr{B}_i(R) = R^{\otimes(i+2)}$, thus the maps φ_i descend to maps

$$\varphi_i : \widehat{\mathscr{B}_i}(R) \to (R \otimes R)/I^2 \otimes_R \Omega_R^{\otimes_R i},$$

which then sheafify to give the desired maps

$$\varphi_i : \widehat{\mathscr{B}_i}(X) \to \mathscr{O}_{\Delta^{(2)}} \otimes \pi_2^* \Omega_X^{\otimes i}.$$

The map φ_i' is the composition

$$\widehat{\mathscr{B}_i}(X) \xrightarrow{\varphi_i} \mathscr{O}_{\Delta^{(2)}} \otimes \pi_2^* \Omega_X^{\otimes i} \to \mathscr{O}_{\Delta} \otimes \pi_2^* \Omega_X^{\otimes i} = \Omega_{\Delta}^{\otimes i}.$$

Step 3: We now need to check the commutativity of the squares in the above diagram. Note that since everything is local, we can assume we are in an open patch $U = \operatorname{Spec} R$, $U \times U = \operatorname{Spec} R \otimes R$. The ideal I in $R \otimes R$ is generated by expressions of the form $r \otimes 1 - 1 \otimes r$ for $r \in R$. Then a relevant square in the above diagram (before completing) is

$$\begin{array}{ccc} R \otimes R \otimes R \otimes R & \xrightarrow{h_1} & R \otimes R \otimes R \\ \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_1} \\ (R \otimes R)/I^2 \otimes_R I/I^2 \otimes_R I/I^2 & \xrightarrow{h_1'} & (R \otimes R)/I^2 \otimes_R I/I^2, \end{array}$$

where $(R \otimes R)/I^2$ is considered a right R-module by multiplication in the second factor, and I/I^2 is considered an R-module by multiplication in either factor (the two module structures are the same). The maps in this diagram are:

$$h_1(1 \otimes b \otimes c \otimes 1) = b \otimes c \otimes 1 - 1 \otimes bc \otimes 1 + 1 \otimes b \otimes c,$$

the Hochschild differential

$$h_1'((1 \otimes 1 + I^2) \otimes_R db \otimes_R dc) = db \otimes_R dc = (b \otimes 1 - 1 \otimes b + I^2) \otimes_R dc,$$

$$\varphi_1(a \otimes b \otimes c) = (a \otimes c + I^2) \otimes_R db,$$

$$\varphi_2(1 \otimes b \otimes c \otimes 1) = (1 \otimes 1 + I^2) \otimes_R db \otimes_R dc.$$

By direct computation we have

$$\begin{aligned}\varphi_1(h_1(1\otimes b\otimes c\otimes 1)) &= \varphi_1(b\otimes c\otimes 1 - 1\otimes bc\otimes 1 + 1\otimes b\otimes c)\\ &= (b\otimes 1 + I^2)\otimes_R dc - (1\otimes 1 + I^2)\otimes_R d(bc)\\ &\quad +(1\otimes c + I^2)\otimes_R db\end{aligned}$$

which, using $d(bc) = b\,dc + c\,db$, equals

$$\begin{aligned}&= (b\otimes 1 + I^2)\otimes_R dc - (1\otimes b + I^2)\otimes_R dc - (1\otimes c + I^2)\otimes_R db\\ &\quad +(1\otimes c + I^2)\otimes_R db\\ &= (b\otimes 1 - 1\otimes b + I^2)\otimes_R dc\\ &= h_1'((1\otimes 1)\otimes_R db\otimes_R dc)\\ &= h_1'(\varphi_2(1\otimes b\otimes c\otimes 1)).\end{aligned}$$

Similar computations ensure the commutativity of the other squares.

Step 4: Observe that there exists a natural map η from the bar resolution $\widehat{\mathscr{B}}.(X)$ to the bar complex $\widehat{\mathscr{C}}.(X) = \widehat{\mathscr{B}}.(X)\otimes_{X\times X}\mathscr{O}_\Delta$, simply given by $1\otimes\mu$ where $\mu : \mathscr{O}_{X\times X}\to\mathscr{O}_\Delta$ is the natural projection. This map is immediately seen to be precisely the unit η of the adjunction $\Delta^*\dashv\Delta_*$.

It is now obvious that multiplying by $1/i!$ the composite map

$$\widehat{\mathscr{B}}_i(X)\xrightarrow{\varphi_i'}\Omega_\Delta^{\otimes i}\xrightarrow{\varepsilon}\Omega_\Delta^i$$

yields precisely the map

$$\widehat{\mathscr{B}}_i(X)\xrightarrow{\eta_i}\widehat{\mathscr{C}}_i(X)\xrightarrow{\Delta_* I_i}\Omega_\Delta^i,$$

where η_i is the ith component of η, locally (before completion) given by

$$a_0\otimes a_1\otimes\cdots\otimes a_{i+1}\mapsto a_0a_{i+1}\otimes a_1\otimes\cdots\otimes a_i$$

and $\Delta_* I_i$ is the ith component of the HKR isomorphism.

Now, chopping off at the last step the two exact sequences we have studied above we get the diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \widehat{\mathscr{B}_i}(X) & \longrightarrow & \widehat{\mathscr{B}_{i-1}}(X) & \longrightarrow \cdots \longrightarrow & \widehat{\mathscr{B}_0}(X) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi_i'} & & \downarrow{\scriptstyle \varphi_{i-1}} & & \downarrow{\scriptstyle \varphi_0} & & \\
0 & \longrightarrow & \Omega_{\Delta}^{\otimes i} & \longrightarrow & \mathscr{O}_{\Delta^{(2)}} \otimes \pi_2^* \Omega_X^{\otimes (i-1)} & \longrightarrow \cdots \longrightarrow & \mathscr{O}_{\Delta^{(2)}} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle p_i} & & & & & & \\
 & & \Omega_{\Delta}^{\otimes i} & & & & & & \\
 & & \downarrow{\scriptstyle \varepsilon} & & & & & & \\
 & & \Omega_{\Delta}^{i}, & & & & & &
\end{array}$$

which can be thought of as a map from the top complex (which represents $\mathscr{O}_{\Delta}$) to $\Omega_{\Delta}^{i}[i]$. In fact what we have is a factoring

$$\mathscr{O}_{\Delta} \overset{p_i \circ \varphi_\cdot}{\longrightarrow} \Omega_{\Delta}^{\otimes i} \overset{\varepsilon}{\longrightarrow} \Omega_{\Delta}^{i}$$

of the map

$$\varepsilon \circ p_i \circ \varphi_\cdot = (i!) \Delta_* I_i \circ \eta,$$

where $\varphi_\cdot$ is the map of complexes appearing at the top of the above diagram. However, note that both the source and the target of $\varphi_\cdot$ are naturally isomorphic (in $\mathbf{D}^b_{\mathrm{coh}}(X \times X)$) to $\mathscr{O}_{\Delta}$, and then $\varphi_\cdot$ can be viewed as the identity map $\mathscr{O}_{\Delta} \to \mathscr{O}_{\Delta}$. Under these identifications we conclude

$$\frac{1}{i!}\, \varepsilon \circ p_i = \Delta_* I_i \circ \eta.$$

But the construction of p_i is such that it is represented by the i-step extension

$$0 \longrightarrow \Omega_{\Delta}^{\otimes i} \longrightarrow \mathscr{O}_{\Delta^{(2)}} \otimes \pi_2^* \Omega_X^{\otimes (i-1)} \longrightarrow \cdots \longrightarrow \mathscr{O}_{\Delta^{(2)}} \longrightarrow \mathscr{O}_{\Delta} \longrightarrow 0,$$

whose class we argued is

$$(\pi_2^* \Omega_X^{\otimes (i-1)} \otimes \alpha_1) \circ (\pi_2^* \Omega_X^{\otimes (i-2)} \otimes \alpha_1) \circ \cdots \circ \alpha_1 : \mathscr{O}_{\Delta} \to \Omega_{\Delta}^{\otimes i}[i].$$

Therefore

$$p_i = (\pi_2^* \Omega_X^{\otimes (i-1)} \otimes \alpha_1) \circ (\pi_2^* \Omega_X^{\otimes (i-2)} \otimes \alpha_1) \circ \cdots \circ \alpha_1 : \mathscr{O}_{\Delta} \to \Omega_{\Delta}^{\otimes i}[i],$$

and hence

$$\frac{1}{i!}\alpha_i = \frac{1}{i!}\varepsilon \circ p_i = \Delta_* I_i \circ \eta.$$

We conclude that

$$\exp(\alpha) = \bigoplus_i \frac{1}{i!}\alpha_i = \bigoplus_i \Delta_* I_i \circ \eta = \Delta_* I \circ \eta. \qquad \square$$

Theorem 4.5. *The composition*

$$K_0(X) \xrightarrow{\text{ch}} HH_0(X) \xrightarrow{I_{\text{HKR}}} \bigoplus_i H^i(X, \Omega_X^i)$$

is the usual Chern character map.

Proof. Let $\mathscr{F} \in K_0(X)$, and let

$$\text{ch}(\mathscr{F}) \in HH_0(X) = \text{Hom}_{X\times X}(\Delta_! \mathscr{O}_\Delta, \mathscr{O}_\Delta)$$

be the Chern character defined in (1.2). Let

$$\text{ch}'(\mathscr{F}) \in \text{Hom}_X(\mathscr{O}_X, \Delta^* \mathscr{O}_\Delta)$$

be the element that corresponds to $\text{ch}(\mathscr{F})$ under the adjunction $\Delta_! \dashv \Delta^*$. If μ' is any element of $\text{Hom}_X(\Delta^* \mathscr{O}_\Delta, S_X)$ and

$$\mu = \Delta_* \mu' \circ \eta$$

is the corresponding element of $\text{Hom}_{X\times X}(\mathscr{O}_\Delta, S_\Delta)$ under the adjunction $\Delta^* \dashv \Delta_*$, the construction of $\Delta_!$ is such that

$$\text{Tr}_X(\mu' \circ \text{ch}'(\mathscr{F})) = \text{Tr}_{X\times X}(\mu \circ \text{ch}(\mathscr{F})).$$

(Here $\eta : \mathscr{O}_\Delta \to \Delta_* \Delta^* \mathscr{O}_\Delta$ is the unit of the adjunction.)

On the other hand, the definition of $\mathrm{ch}(\mathscr{F})$ is such that for any μ,

$$\mathrm{Tr}_{X\times X}(\mu \circ \mathrm{ch}(\mathscr{F})) = \mathrm{Tr}_X(\pi_{2,*}(\pi_1^*\mathscr{F} \otimes \mu)),$$

and $\mathrm{ch}(\mathscr{F})$ is the unique element in $HH_0(X)$ with this property. We then have

$$\begin{aligned}
\mathrm{Tr}_X(\mu' \circ \mathrm{ch}'(\mathscr{F})) &= \mathrm{Tr}_{X\times X}(\mu \circ \mathrm{ch}(\mathscr{F})) = \mathrm{Tr}_X(\pi_{2,*}(\pi_1^*\mathscr{F} \otimes \mu)) \\
&= \mathrm{Tr}_X(\pi_{2,*}(\pi_1^*\mathscr{F} \otimes (\Delta_*\mu' \circ \eta))) \\
&= \mathrm{Tr}_X(\pi_{2,*}(\pi_1^*\mathscr{F} \otimes \Delta_*\mu') \circ \pi_{2,*}(\pi_1^*\mathscr{F} \otimes \eta)) \\
&= \mathrm{Tr}_X(\mathscr{F} \otimes \mu' \circ \Phi^{\eta}(\mathscr{F})) \\
&= \mathrm{Tr}_X(\mu' \circ \mathrm{Tr}_{\mathscr{F}}(\Phi^{\eta}(\mathscr{F}))),
\end{aligned}$$

where the last equality is [5, Lemma 2.4]. Since the trace induces a non-degenerate pairing and the above equalities hold for any μ', it follows that

$$\mathrm{ch}'(\mathscr{F}) = \mathrm{Tr}_{\mathscr{F}}(\Phi^{\eta}(\mathscr{F})).$$

Applying the isomorphism I to both sides we conclude that

$$I_{\mathrm{HKR}}(\mathrm{ch}(\mathscr{F})) = I \circ \mathrm{ch}'(\mathscr{F})) = I \circ \mathrm{Tr}_{\mathscr{F}}(\Phi^{\eta}(\mathscr{F})) = \mathrm{Tr}_{\mathscr{F}}(\Phi^{\exp(\alpha)}(\mathscr{F})) = \mathrm{ch}_{\mathrm{orig}}(\mathscr{F}),$$

where the third equality is Proposition 4.4. □

5. The main conjecture

In this section we discuss the main conjecture and ways to approach its proof.

5.1. It was argued by Kontsevich [12] and Shoikhet [18] that the isomorphisms arising from the Hochschild–Kostant–Rosenberg do *not* respect the natural structures that exist on the Hochschild and harmonic structures, respectively. However, as a consequence of Kontsevich's famous proof of the formality conjecture, he was able to prove that correcting the I^{HKR} isomorphism by a factor of $\hat{A}_X^{-1/2} \in H^*(X, \mathbf{C})$ yields a ring isomorphism:

Claim 5.1 (*Kontsevich [12, Claim 8.4]*). *Let I^K be the composite isomorphism*

$$I^K : HH^*(X) \xrightarrow{(I^{\mathrm{HKR}})^{-1}} HT^*(X) \xrightarrow{\lrcorner \hat{A}_X^{-1/2}} HT^*(X).$$

Then I^K is a ring isomorphism.

5.2. Observe that the way the I^{HKR} isomorphism was defined, I^K can be defined with the same definition, but using a modified Hochschild–Kostant–Rosenberg isomorphism

$$I' : \Delta^* \mathcal{O}_\Delta \xrightarrow{\sim} \bigoplus_i \Omega_X^i[i],$$

given by

$$I' : \Delta^* \mathcal{O}_\Delta \xrightarrow{I} \bigoplus_i \Omega_X^i[i] \xrightarrow{\wedge \hat{A}^{1/2}} \bigoplus_i \Omega_X^i[i].$$

Here, by $\wedge \hat{A}^{1/2}$ we have denoted the morphisms

$$\Omega_X^j[j] \xrightarrow{\Omega_X^j[j] \wedge \hat{A}_X} \bigoplus_i \Omega_X^{i+j}[i+j],$$

where

$$\hat{A}_X^{1/2} : \mathcal{O}_X \to \bigoplus_i \Omega_X^i[i]$$

is the map that corresponds to

$$\hat{A}_X^{1/2} \in \bigoplus_i H^i(X, \Omega_X^i) = \mathrm{Hom}_X(\mathcal{O}_X, \bigoplus_i \Omega_X^i[i]).$$

5.3. The moral of Kontsevich's result is that I is the "wrong" isomorphism to use, and the correct one is I'. With this replacement, I_{HKR} gets replaced by

$$I_K : HH_*(X) \xrightarrow{I_{\mathrm{HKR}}} H\Omega_*(X) \xrightarrow{\wedge \hat{A}^{1/2}} H\Omega_*(X).$$

Not surprisingly, this matches well with the definition of the Mukai vector: if we use I and take Theorem 4.5 as our definition of differential forms-valued Chern character, we get back the classic definition of the Chern character; replacing I by I' replaces this classic Chern character with the Mukai vector

$$v(\mathcal{F}) = \mathrm{ch}(\mathcal{F}) \cdot \hat{A}^{1/2},$$

which we saw in Sections 2 and 3 is better behaved from a functorial point of view.

5.4. These observations, combined with the fact that all the properties of the Hochschild and the harmonic structures appear to match, lead us to state the following conjecture:

Conjecture 5.2. *The maps* (I^K, I_K) *form an isomorphism between the Hochschild and the harmonic structures of a compact smooth space* X.

Observe that this conjecture includes, as a particular case, Kontsevich's Theorem 5.1.

Remark 5.3. This conjecture can be broadly classified to be a result of the same type as Tsygan's formality conjecture [20]. In general, such results describe various structures (product, pairing, Lie bracket, etc.) that are matched by a specific isomorphism between the Hochschild side and the harmonic side.

5.5. We conclude with a remark on a possible approach to proving Conjecture 5.2. For simplicity we restrict our attention to a discussion of the isomorphism on cohomology (where we know the conjecture is true by Kontsevich's result). Consider the sequence of morphisms

$$\begin{array}{ccccc}
\mathrm{Hom}^*_X(\bigoplus \Omega^i_X[i], \bigoplus \Omega^i_X[i]) & \xrightarrow{I} & \mathrm{Hom}^*_X(\Delta^*\mathcal{O}_\Delta, \Delta^*\mathcal{O}_\Delta) & & \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle -\circ\eta} & \nwarrow^{\Delta^*} & \\
\mathrm{Hom}^*_X(\bigoplus \Omega^i_X[i], \mathcal{O}_X) & \xrightarrow{I} & \mathrm{Hom}^*_X(\Delta^*\mathcal{O}_\Delta, \mathcal{O}_X) & \xrightarrow{\Delta_*(-)\circ\eta} & \mathrm{Hom}^*_{X\times X}(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \\
\| & & & & \| \\
HT^*(X) & & \xrightarrow{\qquad I^{\mathrm{HKR}} \qquad} & & HH^*(X).
\end{array}$$

The maps labeled I are isomorphisms induced by I; the arrow $\Delta_*(-)\circ\eta$ is the adjunction isomorphism. The map p is the projection of a matrix in $\mathrm{Hom}^*_X(\bigoplus \Omega^i_X[i], \bigoplus \Omega^i_X[i])$ onto its last column $\mathrm{Hom}^*_X(\bigoplus \Omega^i_X[i], \mathcal{O}_X)$. (The convention that we use is that morphisms of small degree appear at the *bottom* or *right* of column vectors/matrices.)

Observe that all the vector spaces in the diagram have ring structures, but only the top two and rightmost two have the ring structure given by the Yoneda product. Also, note that the arrows between these rings are obviously ring homomorphisms.

We are interested in the map

$$e : \mathrm{Hom}^j_X(\bigoplus \Omega^i_X[i], \mathcal{O}_X) \to \mathrm{Hom}^j_X(\bigoplus \Omega^i_X[i], \bigoplus \Omega^i_X[i])$$

which takes a column vector to a matrix, by the formula

$$\begin{pmatrix} v_n \\ v_{n-1} \\ v_{n-2} \\ \vdots \\ v_0 \end{pmatrix} \overset{e}{\mapsto} \begin{pmatrix} v_0 & v_1 & v_2 & \cdots & v_n \\ 0 & v_0 & v_1 & \cdots & v_{n-1} \\ 0 & 0 & v_0 & \cdots & v_{n-2} \\ & & \vdots & & \\ 0 & 0 & 0 & \cdots & v_0 \end{pmatrix}.$$

(For simplicity, at this point assume that we are only dealing with *homogeneous* elements in $\mathrm{Hom}^*_X(\bigoplus \Omega^i_X[i], \mathcal{O}_X)$.) It is easy to check that what we think of as "multiplication" in $\mathrm{Hom}^*_X(\bigoplus \Omega^i_X[i], \mathcal{O}_X)$ is the product

$$v * v' = p(e(v) \circ e(v')).$$

There is another map e' which takes a column vector and fills it up to a square matrix $e'(v)$. It is the map obtained by starting with $v \in \mathrm{Hom}^j_X(\bigoplus \Omega^i_X[i], \mathcal{O}_X)$ and following the arrows around the diagram to get $e'(v) \in \mathrm{Hom}^j_X(\bigoplus \Omega^i_X[i], \bigoplus \Omega^i_X[i])$. The fact that $p \circ e'$ is the identity means that the last column of $e'(v)$ is precisely v.

To prove that I^{HKR} is a ring isomorphism, it would suffice to show that $e' = e$. Unfortunately, Kontsevich's argument shows that this is not the case. The same argument, however, shows that if we repeat the above analysis with I replaced by I' (and I^{HKR} replaced by I^K) we do get a ring homomorphism. This leads us to state the following conjecture:

Conjecture 5.4. *Replacing I by I' in the above analysis yields $e = e'$.*

A proof of this conjecture, apart from providing a different proof of Kontsevich's result, would likely generalize to a proof of Conjecture 5.2.

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