PBW for an inclusion of Lie algebras

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Abstract

Let $\mathfrak{h} \subset \mathfrak{g}$ be an inclusion of Lie algebras with quotient $\mathfrak{h}$-module $\mathfrak{n}$. There is a natural degree filtration on the $\mathfrak{h}$-module $\mathbf{U}(\mathfrak{g})/\mathbf{U}(\mathfrak{g})\mathfrak{h}$ whose associated graded $\mathfrak{h}$-module is isomorphic to $\mathbf{S}(\mathfrak{n})$. We give a necessary and sufficient condition for the existence of a splitting of this filtration. In turn such a splitting yields an isomorphism between the $\mathfrak{h}$-modules $\mathbf{U}(\mathfrak{g})/\mathbf{U}(\mathfrak{g})\mathfrak{h}$ and $\mathbf{S}(\mathfrak{n})$. For the diagonal embedding $\mathfrak{h} \subset \mathfrak{h} \oplus \mathfrak{h}$ the condition is automatically satisfied and we recover the classical Poincaré-Birkhoff-Witt theorem.

The main theorem and its proof are direct translations of results in algebraic geometry, obtained using an *ad hoc* dictionary. This suggests the existence of a unified framework allowing the simultaneous study of Lie algebras and of algebraic varieties, and a closely related work in this direction is on the way.

1. Introduction

1.1. The aim

Let $\mathfrak{h} \hookrightarrow \mathfrak{g}$ be an inclusion of Lie algebras. Denote by $\mathfrak{n}$ the quotient $\mathfrak{g}/\mathfrak{h}$. The quotient $\mathbf{U}(\mathfrak{g})/\mathbf{U}(\mathfrak{g})\mathfrak{h}$ of $\mathbf{U}(\mathfrak{g})$ by the left ideal generated by $\mathfrak{h}$ is naturally an $\mathfrak{h}$-representation. The main purpose of this paper is to answer the following question (the PBW problem):

When is $\mathbf{U}(\mathfrak{g})/\mathbf{U}(\mathfrak{g})\mathfrak{h}$ isomorphic to $\mathbf{S}(\mathfrak{n})$ as $\mathfrak{h}$-representations?

A more precise way of stating the above question is the following. It is easy to see that $\mathbf{U}(\mathfrak{g})/\mathbf{U}(\mathfrak{g})\mathfrak{h}$ admits a natural filtration by $\mathfrak{h}$-modules whose associated graded $\mathfrak{h}$-module is $\mathbf{S}(\mathfrak{n})$. We ask for a necessary and sufficient condition for this filtration to split.

This question is important in deformation quantization, as the space of $\mathfrak{h}$-invariants $(\mathbf{U}(\mathfrak{g})/\mathbf{U}(\mathfrak{g})\mathfrak{h})^{\mathfrak{h}}$ can be given a natural structure of algebra by identifying it with the space of invariant differential operators on a homogeneous space [5]. An open conjecture of Duflo is concerned with understanding the center of this algebra in terms of the Poisson center of $\mathbf{S}(\mathfrak{n})^{\mathfrak{h}}$, which is thought of as the algebra of functions on a Poisson manifold obtained via reduction through the moment map $\mathfrak{g}^\vee \to \mathfrak{h}^\vee$. In order for this conjecture to make sense one needs to be in a situation where the PBW isomorphism holds. Traditionally this is achieved by assuming that the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ splits as a map of $\mathfrak{h}$-modules. We will see that this condition is unnecessarily restrictive: there are many pairs of Lie algebras for which there is a PBW isomorphism (and hence it makes sense to study the Duflo problem), but for which the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ does not split.

1.2. An analogous problem in algebraic geometry

Kontsevich and Kapranov [3] had the insight that we can view the shifted tangent tangent sheaf $\mathsf{T}_\mathsf{Y}[-1]$ of a smooth algebraic variety Y as a Lie algebra object in the derived category $\mathbf{D}(\mathsf{Y})$ of coherent sheaves on Y, with bracket given by the Atiyah class of the tangent sheaf. Moreover, the Atiyah class of any object in $\mathbf{D}(\mathsf{Y})$ gives it the structure of module over this Lie algebra object (see for example [4]). Loosely speaking $\mathbf{D}(\mathsf{Y})$ can be regarded as the category of representations of the shifted tangent sheaf. The role of the trivial representation is played by the structure sheaf $\mathscr{O}_\mathsf{Y}$.

An embedding $\mathfrak{i} : \mathsf{X} \hookrightarrow \mathsf{Y}$ of smooth algebraic varieties can be thought of as giving rise to an inclusion of Lie algebra objects in $\mathbf{D}(\mathsf{X})$

$$\mathfrak{h} = \mathsf{T}_\mathsf{X}[-1] \hookrightarrow \mathfrak{i}^*\mathsf{T}_\mathsf{Y}[-1] = \mathfrak{g}.$$

If E is an object in $\mathbf{D}(\mathsf{Y})$ then the Atiyah class of the restriction $\mathfrak{i}^*\mathsf{E}$ of E to X is precisely the composite of the above inclusion of Lie algebras with the restriction to X of the Atiyah class of E. In other words the functor

$$\mathfrak{i}^* : \mathbf{D}(\mathsf{Y}) \to \mathbf{D}(\mathsf{X})$$

can be interpreted as the restriction functor

$$\mathsf{Res} : \mathfrak{g}\text{-}\mathfrak{Mod} \to \mathfrak{h}\text{-}\mathfrak{Mod}.$$

(We think of all our functors between derived categories as being implicitly derived, so we write $\mathfrak{i}^*$ instead of $\mathbf{L}\mathfrak{i}^*$, etc.)

We now see a dictionary emerging between the worlds of Lie theory and of algebraic geometry. We can use this dictionary to translate naively the PBW question into a problem in algebraic geometry. The following concepts are matched by this dictionary:

Lie theory	Algebraic geometry
Lie algebras $\mathfrak{h}$, $\mathfrak{g}$	varieties X, Y, $\mathfrak{h} = \mathsf{T}_\mathsf{X}[-1]$, $\mathfrak{g} = \mathsf{T}_\mathsf{Y}[-1]$
inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$	closed embedding $i : \mathsf{X} \hookrightarrow \mathsf{Y}$
$\mathfrak{h}$-$\mathfrak{Mod}$, $\mathfrak{g}$-$\mathfrak{Mod}$	$\mathbf{D}(\mathsf{X})$, $\mathbf{D}(\mathsf{Y})$
$1_\mathfrak{h} \in \mathfrak{h}$-$\mathfrak{Mod}$	$\mathscr{O}_\mathsf{X} \in \mathbf{D}(\mathsf{X})$
$\mathsf{Res} : \mathfrak{g}\text{-}\mathfrak{Mod} \to \mathfrak{h}\text{-}\mathfrak{Mod}$	$i^* : \mathbf{D}(\mathsf{Y}) \to \mathbf{D}(\mathsf{X})$
$\mathsf{Ind} : \mathfrak{h}\text{-}\mathfrak{Mod} \to \mathfrak{g}\text{-}\mathfrak{Mod}$	$i_! : \mathbf{D}(\mathsf{X}) \to \mathbf{D}(\mathsf{Y})$

The last line is motivated by the fact that the induction functor Ind is the left adjoint of the restriction functor, hence in the right column we take the left adjoint $i_!$ of the pull-back functor, which exists for a closed embedding i of smooth varieties.

In representation-theoretic language the $\mathfrak{h}$-representation $\mathbf{U}(\mathfrak{g})/\mathbf{U}(\mathfrak{g})\mathfrak{h}$ arises as

$$\mathbf{U}(\mathfrak{g})/\mathbf{U}(\mathfrak{g})\mathfrak{h} = \mathsf{Res}\,\mathsf{Ind}\,1_\mathfrak{h} \in \mathfrak{h}\text{-}\mathfrak{Mod}.$$

Using the dictionary the latter corresponds to the object $i^* i_! \mathscr{O}_\mathsf{X}$ of the derived category $\mathbf{D}(\mathsf{X})$. Any object E of $\mathbf{D}(\mathsf{X})$ admits a natural filtration by successive truncations $\tau^{\geq k}\mathsf{E}$ whose k-th "quotient" is the cohomology sheaf $\mathscr{H}^k(\mathsf{E})[-k]$. An easy local calculation shows that for $\mathsf{E} = i^* i_! \mathscr{O}_\mathsf{X}$ we have

$$\mathscr{H}^k(i^* i_! \mathscr{O}_\mathsf{X}) = \wedge^k \mathsf{N}$$

where N is the normal bundle of X in Y. Thus the associated graded object of $i^* i_! \mathscr{O}_\mathsf{X}$ is precisely

$$\mathsf{gr}(i^* i_! \mathscr{O}_\mathsf{X}) = \bigoplus_k \wedge^k \mathsf{N}[-k] = \mathbf{S}(\mathsf{N}[-1]).$$

Since $\mathsf{N}[-1] = \mathsf{T}_\mathsf{Y}[-1]|_\mathsf{X}/\mathsf{T}_\mathsf{X}[-1]$ corresponds via the dictionary to $\mathfrak{n} = \mathfrak{g}/\mathfrak{h}$, this is the precise analogue of the statement that $\mathbf{U}(\mathfrak{g})/\mathbf{U}(\mathfrak{g})\mathfrak{h}$ admits a filtration whose associated graded is

$$\mathsf{gr}\left(\mathbf{U}(\mathfrak{g})/\mathbf{U}(\mathfrak{g})\mathfrak{h}\right) = \mathbf{S}(\mathfrak{n}).$$

The PBW question translates into the following question about a closed embedding $i : \mathsf{X} \hookrightarrow \mathsf{Y}$

When is $i^* i_! \mathscr{O}_\mathsf{X}$ isomorphic to $\mathbf{S}(\mathsf{N}[-1])$ in $\mathbf{D}(\mathsf{X})$?

Just like in the usual PBW problem, this question is better phrased by asking when the above filtration on $i^* i_! \mathscr{O}_\mathsf{X}$ splits. This question was addressed and solved recently by D. Arinkin and the second author in [1], where they prove the following result.

Theorem 1.2. *The following are equivalent:*

1. *the truncation filtration on* $i^* i_! \mathscr{O}_X$ *splits, and therefore*
$$i^* i_! \mathscr{O}_X \cong \mathbf{S}(N[-1]);$$
2. *the class*
$$\alpha \in \mathrm{Ext}^1_X(N[-1]^{\otimes 2}, N[-1])$$
obtained by composing the class of the normal bundle exact sequence with the Atiyah class of the normal bundle N*, is trivial;*
3. *the vector bundle* $N[-1]$ *admits an extension to the first infinitesimal neighborhood* $X^{(1)}$ *of* X *into* Y*.*

It is worth noting that there are many cases where the short exact sequence

$$0 \to T_X \to T_Y|_X \to N \to 0$$

does not split but the obstruction α is nonetheless trivial. For example this is the case when X is any non-linear hypersurface in $Y = \mathbf{P}^n$.

1.3. The result

Our main result is the following translation of the above theorem.

Theorem 1.3. *The following are equivalent:*

1. *the natural filtration on* $\mathbf{U}(\mathfrak{g})/\mathbf{U}(\mathfrak{g})\mathfrak{h}$ *splits, and therefore*
$$\mathbf{U}(\mathfrak{g})/\mathbf{U}(\mathfrak{g})\mathfrak{h} \cong \mathbf{S}(\mathfrak{n})$$
2. *the class* $\alpha \in \mathrm{Ext}^1_{\mathfrak{h}}(\mathfrak{n}^{\otimes 2}, \mathfrak{n})$*, obtained by composing the class of*
$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{n} \longrightarrow 0$$
with the $\mathfrak{h}$*-action, is trivial;*
3. *there exists a Lie algebra* $\mathfrak{h}^{(1)}$ *containing* $\mathfrak{h}$ *as a subalgebra with the property that* α *is trivial if and only if* $\mathfrak{n}$ *admits an extension to* $\mathfrak{h}^{(1)}$*.*

Observe that in the algebro-geometric context $X^{(1)}$ is singular even though X and Y are smooth. It turns out that the correct notion of tangent space for $X^{(1)}$ is that of the *tangent complex*, see [2]. Thus $\mathfrak{h}^{(1)}$ should be constructed by analogy to this tangent complex $\mathbb{L}_{X^{(1)}}$. Insight from Koszul duality tells us that $\mathfrak{h}^{(1)}$ should be a very natural quotient of the free Lie algebra generated by $\mathfrak{g}$. We will give its precise definition below, while the explanation for this definition will appear elsewhere.

The paper has two parts. The first part is concerned with defining the obstruction class α, the Lie algebra $\mathfrak{h}^{(1)}$, and proving Theorem 1.3. Almost all the results hold in arbitrary characteristic, except the final splitting argument for which we need to assume we work over a field of characteristic zero. We give a simple example of a pair of Lie algebras for which the class α is non-trivial in a very short and last section.

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2. A condition for the PBW isomorphism

Let $\mathfrak{h} \hookrightarrow \mathfrak{g}$ be an inclusion of Lie algebras and denote by $\mathfrak{n}$ the quotient $\mathfrak{h}$-module $\mathfrak{g}/\mathfrak{h}$. In this section we present a proof of Theorem 1.3 after defining the obstruction class α and the Lie algebra $\mathfrak{h}^{(1)}$ that appear in its statement.

The proof is motivated by Theorem 1.2 which is the analogous result in the setting of algebraic geometry. In fact we found it remarkable that the two proofs are almost identical after the appropriate translation between the two languages.

Unless otherwise stated we always consider modules over a Lie algebra as being acted on on the left. However, as the universal enveloping algebra admits an antipode, this left module structure induces a natural right module structure on any module. We will implicitly use this fact when forming tensor products over the universal enveloping algebra.

2.1. The extension class α. We begin with the definition of the extension class α that appears in the statement of Theorem 1.3. Consider the short exact sequence of $\mathfrak{h}$-modules

$$0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{n} \to 0. \tag{1}$$

Let E be an $\mathfrak{h}$-module. Tensoring (1) with E yields the sequence

$$0 \to \mathfrak{h} \otimes \mathsf{E} \to \mathfrak{g} \otimes \mathsf{E} \to \mathfrak{n} \otimes \mathsf{E} \to 0 \tag{2}$$

which remains exact because the tensor product of representations is the tensor product of vector spaces endowed with the $\mathfrak{h}$-module structure given by the Leibniz rule. The extension class of (2) is a map $\mathfrak{n} \otimes \mathsf{E} \to \mathfrak{h} \otimes \mathsf{E}[1]$ in the derived category of $\mathfrak{h}$-representations, which can be post-composed with the action map $\mathfrak{h} \otimes E \to E$ to give the map

$$\alpha_E : \mathfrak{n} \otimes E \to E[1].$$

Equivalently, we can define α_E as the class in $\mathrm{Ext}^1_{\mathfrak{h}}(\mathfrak{n} \otimes E, E)$ corresponding to the bottom extension in the diagram below:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{h} \otimes E & \longrightarrow & \mathfrak{g} \otimes E & \longrightarrow & \mathfrak{n} \otimes E & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & E & \longrightarrow & Q & \longrightarrow & \mathfrak{n} \otimes E & \longrightarrow & 0. \end{array} \tag{3}$$

Here the $\mathfrak{h}$-module Q is obtained by push-out in the first square of the above diagram. Explicitly, it is given by

$$Q = E \oplus (\mathfrak{g} \otimes E)/\langle (h(x), 0) - (0, h \otimes x) \rangle$$

where for $h \in \mathfrak{h}$ and $x \in E$ we have denoted by $h(x)$ the action of h on x and $h \otimes x$ is viewed as an element of $\mathfrak{g} \otimes E$ via the inclusion of $\mathfrak{h}$ into $\mathfrak{g}$.

We will be particularly interested in the class $\alpha_{\mathfrak{n}}$ of the $\mathfrak{h}$-module $\mathfrak{n}$. This special class will be denoted simply by α.

2.2. The first infinitesimal neighborhood algebra $\mathfrak{h}^{(1)}$. Consider the Lie algebra $\mathfrak{h}^{(1)}$ defined by

$$\mathfrak{h}^{(1)} := \mathbf{L}(\mathfrak{g})/\langle [h, g] - [h, g]_{\mathfrak{g}} \mid h \in \mathfrak{h}, g \in \mathfrak{g} \rangle$$

where $\mathbf{L}(\mathfrak{g})$ denotes the free Lie algebra generated by the vector space $\mathfrak{g}$. More precisely $\mathfrak{h}^{(1)}$ is the quotient of $\mathbf{L}(\mathfrak{g})$ in which the bracket between elements of $\mathfrak{h}$ and $\mathfrak{g}$ has been identified with the original one in $\mathfrak{g}$. Note that to define the Lie algebra $\mathfrak{h}^{(1)}$ we do not need $\mathfrak{g}$ to be a Lie algebra. The precise weaker condition for which this construction makes sense is given in Lemma 2.3 below.

There are obvious maps of Lie algebras

$$\mathfrak{h} \hookrightarrow \mathfrak{h}^{(1)} \text{ and } \mathfrak{h}^{(1)} \to \mathfrak{g}$$

which factor the original inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$. Given an $\mathfrak{h}$-representation E we can ask whether E extends to a representation of $\mathfrak{h}^{(1)}$. In other words we ask if on the vector space E we can find an $\mathfrak{h}^{(1)}$-module structure whose restriction to $\mathfrak{h}$ via the map $\mathfrak{h} \to \mathfrak{h}^{(1)}$ is the original one. The following lemma shows that this is the case if and only if $\alpha_E = 0$. We state the lemma in a slightly greater generality.

2.3. Lemma. *Let $\mathfrak{h}$ be a Lie algebra and let $\mathfrak{g}$ be an $\mathfrak{h}$-module that contains $\mathfrak{h}$ as an $\mathfrak{h}$-submodule. An $\mathfrak{h}$-module E is the restriction of an $\mathfrak{h}^{(1)}$-module if and only if its class α_E is trivial.*

Proof. We begin with the if part. Assume that the class α_E is trivial. This implies that the sequence (3) splits in the category of $\mathfrak{h}$-modules. Thus we get a map $j : Q \to E$ of $\mathfrak{h}$-modules that splits the canonical map $E \to Q$. Precomposing j with the middle vertical map in (3) yields a map of $\mathfrak{h}$-modules

$$\rho : \mathfrak{g} \otimes E \to E.$$

This map does not define a representation of $\mathfrak{g}$ on E, but it certainly defines a representation of $\mathbf{L}(\mathfrak{g})$ by the universal property of $\mathbf{L}(\mathfrak{g})$. The fact that ρ respects the $\mathfrak{h}$ structure translates into the fact that $\langle [h, g] - [h, g]_{\mathfrak{g}} \rangle$ is in the kernel of this representation. Thus ρ gives an $\mathfrak{h}^{(1)}$-module structure on E which lifts the original $\mathfrak{h}$-module structure because the first square in (3) commutes.

For the only if part assume we have an $\mathfrak{h}^{(1)}$-module structure on E that lifts the $\mathfrak{h}$ structure. Again denote this action by ρ. We can use the explicit description of Q above to define a splitting

$$(x, g \otimes y) \mapsto (x + \rho(g)(y)).$$

This map is obviously a splitting and it respects the $\mathfrak{h}$-module structure because $\langle [h, g] - [h, g]_{\mathfrak{g}} \rangle$ is in the kernel of the representation ρ. $\square$

2.4. Preparations for the relative PBW. Some notation is in order. Denote the Lie algebra inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ by i and denote the maps $\mathfrak{h} \to \mathfrak{h}^{(1)}$ and $\mathfrak{h}^{(1)} \to \mathfrak{g}$ by j and k respectively so that $i = k \circ j$. Denote by i^* the restriction functor from $\mathfrak{g}$-modules to $\mathfrak{h}$-modules and by $i_!$ the induction functor in the reverse direction. Thus we have $i_! \dashv i^*$. We also have similar functors for the maps j and k. Finally, we denote the 1-dimensional trivial representation of a Lie algebra $\mathfrak{g}$ by $\mathbf{1}_{\mathfrak{g}}$.

We want to understand the object

$$i^* i_!(\mathbf{1}_{\mathfrak{h}}) = \mathbf{U}(\mathfrak{g}) \otimes_{\mathbf{U}(\mathfrak{h})} \mathbf{1}_{\mathfrak{h}} = \mathbf{U}(\mathfrak{g})/\mathbf{U}(\mathfrak{g})\mathfrak{h}.$$

We begin by understanding the easier object $j^* j_!(\mathbf{1}_{\mathfrak{h}})$. It can be realized as the following quotient of the tensor algebra $\mathbf{T}(\mathfrak{g})$ on $\mathfrak{g}$:

$$j^* j_!(\mathbf{1}_{\mathfrak{h}}) = \mathbf{U}(\mathfrak{h}^{(1)}) \otimes_{\mathbf{U}(\mathfrak{h})} \mathbf{1}_{\mathfrak{h}} = \mathbf{T}(\mathfrak{g})/\langle J + \mathfrak{h} \rangle.$$

Here J is the two sided ideal generated by $[h, g] - [h, g]_{\mathfrak{g}}$ for $h \in \mathfrak{h}$, $g \in \mathfrak{g}$, and $\mathfrak{h}$ denotes the left ideal generated by $\mathfrak{h}$.

Elements of $\mathsf{T}^k(\mathfrak{g})$ will be said to have degree k. For a general (possibly inhomogeneous) element of $\mathsf{T}(\mathfrak{g})$ we shall define its degree to be the maximal degree of a monomial that appears in it. The elements of $\mathsf{T}(\mathfrak{g})$ of degree $\leq d$ form the d-th piece of an increasing filtration on $\mathsf{T}(\mathfrak{g})$ which we call the degree filtration.

The ideal $\langle J + \mathfrak{h} \rangle$ respects this filtration, and hence $j^* j_! (\mathbf{1}_{\mathfrak{h}})$ inherits a natural increasing filtration $F^0 \subset F^1 \subset F^2 \cdots \subset F^k \cdots$ whose k-th piece F^k consists of those elements of $j^* j_! \mathbf{1}_{\mathfrak{h}}$ that have a lift to $\mathsf{T}(\mathfrak{g})$ of degree $\leq k$. It is easy to see that the terms of this filtration are all $\mathfrak{h}$-submodules of $j^* j_! \mathbf{1}_{\mathfrak{h}}$.

2.5. Lemma. *The associated graded* $\mathfrak{h}$*-module* $\mathsf{gr}(F^{\cdot})$ *of the above filtration is precisely* $\mathsf{T}(\mathfrak{n})$*. In other words the successive quotients* F^k/F^{k-1} *are isomorphic, as* $\mathfrak{h}$*-modules, to* $\mathfrak{n}^{\otimes k}$*.*

Proof. Define a map $\sigma : F^k \to \mathfrak{n}^{\otimes k}$ as follows. For $x \in F^k$, pick a lift of it $\bar{x}$ in $\mathsf{T}(\mathfrak{g})$ of degree $\leq k$. (Such a lift exists by the assumption that $x \in F^k$.) Let $\bar{x}^k$ be the homogeneous degree k part of $\bar{x}$ (which may be zero). We define $\sigma(x)$ to be the image of $\bar{x}^d$ under the natural projection $\pi : \mathsf{T}^k(\mathfrak{g}) \to \mathsf{T}^k(\mathfrak{n})$.

We have to show that this map is well-defined. To do this let y and y' be two lifts of x to $\mathsf{T}(\mathfrak{g})$, both of degree k or less. We want to show that $\pi(y^k) = \pi(y'^k)$. The difference $y - y'$ is in $J + \mathfrak{h}$, i.e., it is a sum of terms of the form

$$a \otimes h_1 + b \otimes (h_2 \otimes g - g \otimes h_2 - [h_2, g]) \otimes c$$

for a, b, and c in $\mathsf{T}(\mathfrak{g})$. Moreover, these terms can all be taken to have degree $\leq k$ since the degree of $y - y'$ is $\leq k$. With the exception of terms of the form $b \otimes [h_2, g] \otimes c$ all the other terms are mapped to zero by the projection π. On the other hand $\deg b + \deg c$ can not exceed $k - 2$, so $\deg(b \otimes [h_2, g] \otimes c) < k$, hence the part of degree k of such terms is also zero. Thus σ is well-defined.

The subspace F^{k-1} is clearly in the kernel of σ so we get a map $F^k/F^{k-1} \to \mathfrak{n}^{\otimes k}$ which we shall denote by σ as well. It is obvious that this map is surjective. To check it is also injective let $x \in F^k$ be such that $\sigma(x) = 0$. Let $\bar{x}$ be a lift of x to $\mathsf{T}(\mathfrak{g})$ whose degree is minimal among all possible such lifts. We want to show that $\bar{x}$ has degree $\leq k - 1$, and hence x is in F^{k-1}.

Since $\sigma(x) = 0$, the homogeneous degree k part $\bar{x}^k$ of $\bar{x}$ is in the two-sided ideal generated by $\mathfrak{h}$. If $\bar{x}^k$ is non-zero we can use relations in $J + h$ to reduce the maximal tensor degree of elements in the two sided ideal $\langle \mathfrak{h} \rangle$. In other words we can still reduce the degree of $\bar{x}$, contradicting the assumption on the minimality of the degree of $\bar{x}$. Hence $\bar{x}^k = 0$, so $x \in F^{k-1}$, so the map $F^k/F^{k-1} \to \mathfrak{n}^{\otimes k}$ is injective. $\square$

2.6. It is easy to see that the inclusion $F^0 \hookrightarrow F^k$ always splits for any $k > 0$. We shall denote the reduced filtration by $\tilde{F}^1 \subset \cdots \tilde{F}^k \cdots$.

By the above lemma we have $\tilde{F}^1 \cong \mathfrak{n}$ and $\tilde{F}^2/\tilde{F}^1 \cong \mathfrak{n}^{\otimes 2}$. These $\mathfrak{h}$-modules fit into the short exact sequence

$$0 \to \mathfrak{n} \to \tilde{F}^2 \to \mathfrak{n}^{\otimes 2} \to 0. \tag{4}$$

The next lemma shows that the extension class of this sequence is precisely the class $\alpha := \alpha_{\mathfrak{n}} \in \mathrm{Ext}^1_{\mathfrak{h}}(\mathfrak{n} \otimes \mathfrak{n}, \mathfrak{n})$ defined in (2.1).

2.7. Lemma. *The short exact sequences* (3) *and* (4) *are isomorphic and hence both define the same obstruction class* α.

Proof. We construct a map between

$$Q := \mathfrak{n} \oplus (\mathfrak{g} \otimes \mathfrak{n})/\langle (h(x), 0) - (0, h \otimes x)\rangle$$

and $\tilde{F}^2$ which makes all the obvious squares commute. The required map has two components: one from $\mathfrak{n}$ and the other from $\mathfrak{g} \otimes \mathfrak{n}$. The first component is the natural inclusion map $\mathfrak{n} = \tilde{F}^1 \subset \tilde{F}^2$. The second one is given by

$$g \otimes x \mapsto [g \otimes \bar{x}]$$

where we first choose a lift $\bar{x}$ of $x \in \mathfrak{n}$ to $\mathfrak{g}$ and then take the class of $g \otimes x \in \mathbf{T}^2\mathfrak{g}$ in $\tilde{F}^2$. It is easy to see that this does not depend on the choice of lifting and that the resulting map factors through Q. A quick diagram chasing shows that all squares commute. □

2.8. Corollary. *The following two statements are equivalent:*

(a) The filtration $F^0 \subset \cdots \subset F^k \subset \cdots$ *splits.*

(b) The extension class α *is trivial.*

Proof. The implication (a)⇒(b) is trivial in view of Lemma 2.7. Now assume that $\alpha = 0$. Then the first and second inclusions are split (the first one is always split, and the second one splits by Lemma 2.7).

We have a short exact sequence for any $k \geq 2$

$$0 \to F^{k-1}/F^{k-2} \to F^k/F^{k-2} \to F^k/F^{k-1} \to 0. \tag{5}$$

In view of Lemma 2.5 this is the same as

$$0 \to \mathfrak{n}^{\otimes(k-1)} \to F^k/F^{k-2} \to \mathfrak{n}^{\otimes k} \to 0.$$

We will argue that the class of this extension is always

$$\rho \circ (\alpha \otimes \mathrm{id}_{\mathfrak{n}^{\otimes(k-1)}})$$

where

$$\rho : \mathfrak{h} \otimes \mathfrak{n}^{\otimes(k-1)} \to \mathfrak{n}^{\otimes(k-1)}$$

is the action of $\mathfrak{h}$ on the $\mathfrak{h}$-module $\mathfrak{n}^{\otimes(k-1)}$. The assumption that $\alpha = 0$ implies then that the extension (5) splits for all $k \geq 2$, and thus the filtration $F^{\cdot}$ is itself split.

To obtain the class $\rho \circ (\alpha \otimes \mathrm{id}_{\mathfrak{n}^{\otimes(n-1)}})$ we tensor the short exact sequence 1 with $\mathfrak{n}^{\otimes(k-1)}$ and post-compose with ρ. This is illustrated by the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{h} \otimes \mathfrak{n}^{\otimes(k-1)} \arrow[r] \arrow[d] & \mathfrak{g} \otimes \mathfrak{n}^{\otimes(k-1)} \arrow[r] \arrow[d] & \mathfrak{n} \otimes \mathfrak{n}^{\otimes(k-1)} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathfrak{n}^{\otimes(k-1)} \arrow[r] & Q^k \arrow[r] & \mathfrak{n} \otimes \mathfrak{n}^{\otimes(k-1)} \arrow[r] & 0.
\end{tikzcd}$$

The module Q^k is the push-out

$$\mathfrak{n}^{\otimes(k-1)} \oplus (\mathfrak{g} \otimes \mathfrak{n}^{\otimes(k-1)}) / \langle (h(x), 0) - (0, h \otimes x) \rangle.$$

We construct a map $Q^k \to F^k/F^{k-2}$ given by two components, one on $\mathfrak{n}^{\otimes(k-1)}$ and the other on $\mathfrak{g} \otimes \mathfrak{n}^{\otimes(k-1)}$. By Lemma 2.7 we identify $\mathfrak{n}^{\otimes(k-1)}$ with the space F^{k-1}/F^{k-2}. For the first component we take the natural inclusion map

$$\mathfrak{n}^{\otimes(k-1)} \cong F^{k-1}/F^{k-2} \hookrightarrow F^k/F^{k-2}.$$

On the other component, we define the map

$$(g \otimes [x]) \mapsto [g \otimes x]$$

where again we first lift $[x]$ to an element $x \in \mathbf{T}\mathfrak{g}$ and then take the class of $g \otimes x$. This map factor through Q^k as

$$\rho(h)([x]) = [h \otimes x]$$

in $\mathbf{T}(\mathfrak{g})/\langle J + \mathfrak{h} \rangle$. Moreover, chasing the diagrams yields the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{n}^{\otimes(k-1)} \arrow[r] \arrow[d] & Q^k \arrow[r] \arrow[d] & \mathfrak{n} \otimes \mathfrak{n}^{\otimes(k-1)} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & F^{\otimes(k-1)}/F^{k-2} \arrow[r] & F^k/F^{k-2} \arrow[r] & F^k/F^{\otimes(k-1)} \arrow[r] & 0.
\end{tikzcd} \tag{6}$$

Thus the splitting follows and the lemma is proved. $\square$

2.9. The relative PBW isomorphism. We now concentrate our attention on the $\mathfrak{h}$-representation

$$\mathbf{i}^*\mathbf{i}_!(\mathbf{1}_\mathfrak{h}) = \mathbf{U}(\mathfrak{g})/\mathbf{U}(\mathfrak{g})\mathfrak{h}.$$

This module can be realized as the quotient $\mathbf{T}(\mathfrak{g})/(I + \langle\mathfrak{h}\rangle)$ where I is the two-sided ideal generated by

$$\{g_1 \otimes g_2 - g_2 \otimes g_1 - [g_1, g_2] \mid g_1, g_2 \in \mathfrak{g}\}$$

and $\langle\mathfrak{h}\rangle$ is the right ideal generated by $\mathfrak{h}$. The ideal $I + \langle\mathfrak{h}\rangle$ is compatible with the degree filtration on $\mathbf{T}(\mathfrak{g})$ which descends to a filtration

$$R^0 \subset R^1 \subset \cdots \subset R^k \subset \cdots$$

by $\mathfrak{h}$-submodules of $\mathbf{i}^*\mathbf{i}_!(\mathbf{1}_\mathfrak{h})$. The corresponding successive quotients will be denoted by $G^k := R^k/R^{\otimes(k-1)}$.

Consider the map

$$j^*j_!(\mathbf{1}_\mathfrak{h}) \to j^*k^*k_!j_!(\mathbf{1}_\mathfrak{h}) = i^*i_!(\mathbf{1}_\mathfrak{h})$$

constructed using the unit map of the adjunction $k_! \dashv k^*$. This map preserves the filtrations and descends to maps between associated graded $\mathfrak{h}$-modules

$$\tau : \mathbf{T}(\mathfrak{n}) = \mathrm{gr}(j^*j_!(\mathbf{1}_\mathfrak{h})) \to \mathrm{gr}(i^*i_!(\mathbf{1}_\mathfrak{h})).$$

It is easy to see that τ is surjective.

2.10. Lemma. *The kernel of the map τ is generated by the commutators $x \otimes y - y \otimes x$ for $x, y \in \mathfrak{n}$.*

Proof. The idea is to use the standard PBW theorem. Consider the increasing filtration $E^0 \subset \cdots \subset E^k \subset \cdots$ on the universal enveloping algebra $\mathbf{U}(\mathfrak{g})$. The standard PBW theorem asserts that the kernel of the canonical quotient map $\mathfrak{g}^{\otimes k} \to E^k/E^{k-1}$ is generated by the commutators of elements in $\mathfrak{g}$, thus yielding an isomorphism between the k-th symmetric tensors on $\mathfrak{g}$ and E^k/E^{k-1}.

As all these filtration are compatible (they all arise from the degree filtration on $\mathbf{T}(\mathfrak{g})$) we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I_1 & \longrightarrow & \mathfrak{g}^{\otimes k} & \longrightarrow & E^k/E^{k-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I_2 & \longrightarrow & \mathfrak{n}^{\otimes k} & \longrightarrow & G^k & \longrightarrow & 0
\end{array}$$

where I_1 is the degree k part of the commutator ideal in $\mathbf{T}(\mathfrak{g})$ by the PBW theorem and I_2 is the kernel of the quotient map $\mathfrak{n}^{\otimes k} \to G^k$.

We want to show that I_2 is the k-th commutator in $\mathfrak{n}$. It suffices to show that the map $I_1 \to I_2$ is surjective. By the snake lemma this is equivalent to showing that the map from the kernel of $\mathfrak{g}^{\otimes k} \to \mathfrak{n}^{\otimes k}$ to the kernel of $E^k/E^{k-1} \to G^k$ is surjective.

For that we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E^{k-1} & \longrightarrow & E^k & \longrightarrow & E^k/E^{k-1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & R^{k-1} & \longrightarrow & R^k & \longrightarrow & G^k & \longrightarrow & 0. \end{array}$$

Again by the snake lemma we conclude that the right ideal generated by $\mathfrak{h}$ in $\mathbf{U}(\mathfrak{g})$ surjects onto the kernel of the map $E^k/E^{k-1} \to G^k$. But the kernel of the map $\mathfrak{g}^{\otimes k} \to \mathfrak{n}^{\otimes k}$ is the two-sided ideal in $\mathbf{T}(\mathfrak{g})$ generated by $\mathfrak{h}$ which certainly surjects onto the right sided one. Thus the lemma is proved. $\square$

To state an if and only if condition for the relative PBW isomorphism, we need the following lemma concerning the obstruction class α.

2.11. Lemma. *The obstruction class* $\alpha \in \mathrm{Ext}^1(\mathfrak{n} \otimes \mathfrak{n}, \mathfrak{n})$ *factors through* $\mathbf{S}^2(\mathfrak{n})$.

Proof. The lemma is an easy corollary of Lemma 2.7 and Lemma 2.10. Indeed, by Lemma 2.7, we can consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{n} & \longrightarrow & \tilde{F}^2 & \longrightarrow & \mathfrak{n} \otimes \mathfrak{n} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathfrak{n} & \longrightarrow & \tilde{R}^2 & \longrightarrow & G^2 & \longrightarrow & 0 \end{array}$$

where the vertical maps are all defined via the adjunction $j^* j_!(\mathbf{1}_{\mathfrak{h}}) \to i^* i_!(\mathbf{1}_{\mathfrak{h}})$. By Lemma 2.10 $G^2 = \mathbf{S}^2\mathfrak{n}$ and the last vertical map is the canonical quotient from the tensor product to the symmetric product. It is easy to see that the second square in the above diagram is cartesian. Thus the lemma is proved. $\square$

We can summarize our main result in the following theorem which is what we mean by an if and only if condition for the relative PBW theorem.

2.12. Theorem. *Let* $\mathbf{k}$ *be a field and let* $\mathfrak{h} \hookrightarrow \mathfrak{g}$ *be an inclusion of Lie algebras over* $\mathbf{k}$. *Consider the two filtrations* $R^0 \subset R^1 \subset \cdots \subset R^k \subset \cdots$ *and* $F^0 \subset F^1 \subset \cdots \subset F^k \subset \cdots$ *defined above. We have:*

(a) $\mathsf{gr}(F^{\cdot}) = \mathbf{T}(\mathfrak{n})$;

(b) $\mathsf{gr}(R^{\cdot}) = \mathbf{S}(\mathfrak{n})$.

Moreover, if the field $\mathbf{k}$ *has characteristic zero, then the following are equivalent:*

(a) The extension class α *is trivial.*

(b) The filtration $F^0 \subset F^1 \subset \cdots \subset F^k \subset \cdots$ *splits;*

(c) The filtration $R^0 \subset R^1 \subset \cdots \subset R^k \subset \cdots$ *splits.*

In fact, if the extension class α *is trivial, we have the following explicit splitting of the filtration* $R^{\cdot}$ *that resembles the standard* PBW *isomorphism:*

$$I : \mathbf{S}(\mathfrak{n}) \to \mathbf{T}(\mathfrak{n}) \cong j^* j_!(\mathbf{1}_{\mathfrak{h}}) \to i^* i_!(\mathbf{1}_{\mathfrak{h}}) \cong \mathbf{U}(\mathfrak{g})/\mathbf{U}(\mathfrak{g})\mathfrak{h}.$$

Here the first arrow is given by

$$x_1 x_2 \cdots x_k \mapsto \frac{1}{k!} \sum_{\sigma \in S_k} x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes \cdots \otimes x_{\sigma(k)}.$$

Proof. With Lemmas 2.5, 2.7, 2.10, 2.11, we have proved most of the theorem. The only thing left is to show that when the field has characteristic zero the map I constructed above gives an explicit splitting of the filtration R. The diagram below commutes

$$\begin{tikzcd} \mathbf{S}^k(\mathfrak{n}) \arrow[r, "\text{symmetrization}"] & \mathbf{T}^k(\mathfrak{n}) = F^k/F^{k-1} \arrow[r, "\text{splitting of } F"] \arrow[d] & F^k/F^{k-2} \arrow[d] \\ & \mathbf{S}^k(\mathfrak{n}) = R^k/R^{k-1} & R^k/R^{k-2}. \arrow[l] \end{tikzcd}$$

Moreover, the composition of the symmetrization map $\mathbf{S}^k(\mathfrak{n}) \to \mathbf{T}^k(\mathfrak{n})$ with the projection map $\mathbf{T}^k(\mathfrak{n}) \to \mathbf{S}^k(\mathfrak{n})$ is the identity. Therefore the composition of the top map with the rightmost vertical one is a splitting of the bottom map $R^k/R^{k-2} \to R^k/R^{k-1}$. We conclude that all the maps $R^k/R^{k-2} \to R^k/R^{k-1}$ are split, hence the entire filtration $R^{\cdot}$ splits. $\square$

3. An example of a non trivial class

We now give an example of an inclusion of Lie algebras $\mathfrak{h} \hookrightarrow \mathfrak{g}$ for which the obstruction class is non trivial. Let $\mathfrak{g} = \mathfrak{sl}_2$; recall that it is generated by e, h and f, satisfying the relations

$$[e, f] = h\,, \quad [h, e] = 2e\,, \quad [h, f] = -2f\,.$$

Now let $\mathfrak{h}$ be the Lie subalgebra generated by e and h. Then $\mathfrak{n} = \mathfrak{g}/\mathfrak{h}$ is the one dimensional $\mathfrak{h}$-module generated as a vector space by f, with module structure defined by

$$e \cdot f = 0 \quad \text{and} \quad h \cdot f = -2f\,.$$

3.1. Proposition. *The obstruction class α is non-trivial.*

Proof. First observe that the Chevalley-Eilenberg 1-cocycle

$$c \in C^1\big(\mathfrak{h}, \mathsf{Hom}(\mathfrak{n}, \mathfrak{h})\big)$$

given by

$$c(e)(f) = e \cdot f - [e, f] = -h, \quad c(h)(f) = h \cdot f - [h, f] = 0$$

is a representative of the exact sequence

$$0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{n} \to 0\,.$$

Therefore the 1-cocycle $a \in C^1\big(\mathfrak{h}, \mathrm{Hom}(\mathfrak{n}^{\otimes 2}, \mathfrak{n})\big)$ given by

$$a(e)(f, f) = -h \cdot f = 2f, \quad a(h)(f, f) = 0$$

is a representative of the obstruction class α.

Finally, observe that since e acts trivially on $\mathfrak{n}$, then it acts trivially on $\mathrm{Hom}(\mathfrak{n}^{\otimes 2}, \mathfrak{n})$. Consequently, for any $b \in \mathrm{Hom}(\mathfrak{n}^{\otimes 2}, \mathfrak{n})$ we have $d(b)(e) = 0$, so that $a \neq d(b)$. Thus $\alpha \neq 0$. □

References

[1] D. Arinkin, A. Căldăraru, When is the self-intersection of a subvariety a fibration?, `arXiv:1007.1671`

[2] L. Illusie, *Complexe cotangent et déformations. I*, Lecture Notes in Mathematics, Vol. **239**. Springer-Verlag, Berlin-New York, 1971

[3] M. Kapranov, Rozansky-Witten invariants via Atiyah classes, Compositio Math. **115** (1999), no. 1, 71–113

[4] J. Roberts, S. Willerton, On the Rozansky-Witten weight system, `arXiv:math/0602653`

[5] C. Torossian, Applications de la bi-quantification à la théorie de Lie, `arXiv:0711.3553`

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