DERIVED CATEGORIES OF TWISTED SHEAVES ON CALABI-YAU MANIFOLDS

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This dissertation is primarily concerned with the study of derived categories of twisted sheaves on Calabi-Yau manifolds. Twisted sheaves occur naturally in a variety of problems, but the most important situation where they are relevant is in the study of moduli problems of semistable sheaves on varieties. Although universal sheaves may not exist as such, in many cases one can construct them as twisted universal sheaves. In fact, the twisting is an intrinsic property of the moduli problem under consideration.

A fundamental construction due to Mukai associates to a universal sheaf a transform between the derived category of the original space and the derived category of the moduli space, which often turns out to be an equivalence. In the present work we study what happens when the universal sheaf is replaced by a twisted one. Under these circumstances we obtain a transform between the derived category of sheaves on the original space and the derived category of twisted sheaves on the moduli space.

The dissertation is divided into two parts. The first part presents the main technical tools: the Brauer group, twisted sheaves and their derived category, as well as a criterion for checking whether an integral transform is an equivalence (a so-called Fourier-Mukai transform). When this is the case we also obtain results regarding the cohomological transforms associated to the ones on the level of derived categories.

In the second part we apply the theoretical results of the first part to a large set of relevant examples. We study smooth elliptic fibrations and the relationship between the theory of twisted sheaves and Ogg-Shafarevich theory, K3 surfaces, and elliptic Calabi-Yau threefolds. In particular, the study of elliptic Calabi-Yau threefolds leads us to an example which is likely to provide a counterexample to the generalization of the Torelli theorem from K3 surfaces to threefolds. A similarity between the examples we study and certain examples considered by Vafa-Witten and Aspinwall-Morrison shows up, although we can only guess the relationship between these two situations at the moment.

Biographical Sketch

Andrei Căldăraru was born in Bucharest, Romania, in 1971. After attending the "Sf. Sava" high-school and one year at the Bucharest University, he moved to Jerusalem, Israel. He graduated *summa cum laude* from the Hebrew University in Jerusalem in 1993, with a double degree in Mathematics and Computer Science. After working for a year in the computer industry, he started his work towards a Ph.D. in Mathematics at Cornell University. Andrei is married to Daniela, and they currently live in New York City.

Tuturor celor din care am fost făcut, în spirit şi în lut.

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Convention

Throughout this work we make the assumption that the ground field is **C**. Most of the results hold over an arbitrary field as well, with only minor modifications.

It is also important to be precise about the topology being used. Most of the time we use the étale topology when working in the algebraic setting, and the Euclidean (analytic) topology in the analytic setting. For details regarding the étale topology the reader should consult [30]; the analytic case is studied in detail in [2].

Introduction and Overview

Twisted sheaves were introduced by Giraud ([16]) as part of his study of non-commutative cohomology, but passed relatively unnoticed in that context and were not studied as objects of intrinsic interest. The purpose of this dissertation is to present their theory, with an emphasis on their derived category, and to show that twisted sheaves provide a powerful and useful tool in the study of algebraic varieties and complex manifolds.

Informally speaking, a twisted sheaf consists of a collection of sheaves and gluing functions, with the apparent defect that these gluing functions "don't quite match up." In this respect they have a strong similarity with vector bundles on a space: these are vector spaces and gluing functions that also "don't quite match up." When the gluings match up, for a vector bundle, the result is a trivial vector bundle (i.e. a vector space). The similar situation for twisted sheaves gives a trivially twisted sheaf, which is a *sheaf*. In a certain sense, twisted sheaves are the next level of generalization, up from vector bundles.

Let's be a little more precise (the full definition of the category of twisted sheaves is given in Section 1.2). Consider a scheme or analytic space X, and an element α of the Brauer group of X (roughly, this is the same as $H^2(X, \mathscr{O}_X^*)$). Represent α by a Čech 2-cocycle on some open cover $\{U_i\}_{i\in I}$, i.e., find sections

$$\alpha_{ijk} \in \Gamma(U_i \cap U_j \cap U_k, \mathscr{O}_X^*)$$

that satisfy the coboundary condition and whose image in $H^2(X, \mathscr{O}_X^*)$ is α. An α-twisted sheaf is then a collection

$$(\{\mathscr{F}_i\}_{i\in I}, \{\varphi_{ij}\}_{i,j\in I})$$

of sheaves and isomorphisms such that $\mathscr{F}_i$ is a sheaf on U_i,

$$\varphi_{ij} : \mathscr{F}_j|_{U_i\cap U_j} \to \mathscr{F}_i|_{U_i\cap U_j}$$

is an isomorphism, and these isomorphisms satisfy

$$\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ki} = \alpha_{ijk} \cdot \mathrm{id}_{\mathscr{F}_i}$$

along $U_i \cap U_j \cap U_k$, for any $i, j, k \in I$.

It is easy to see that under the natural definitions of homomorphisms, kernel, cokernel, etc., the α-twisted sheaves form an abelian category. The question is, then, what information about X can we deduce from this category?

First, this category is invariant of the choices made: it does not depend on the choice of the open cover, or of the particular Čech cocycle. Moreover, in most interesting cases there is another, more natural description of this category, as the category of sheaves of modules over a sheaf of non-commutative algebras. In this sense, the category of twisted sheaves should be viewed as a non-commutative analogue of the category of coherent sheaves on the underlying scheme or analytic space X. However, this non-commutative situation is just a very mild generalization of the commutative setting, because the non-commutative algebras that appear in this way are the simplest possible ones, Azumaya algebras. (For a description of Azumaya algebras, the reader should consult Section 1.1; for the equivalence between twisted sheaves and sheaves of modules over an Azumaya algebra, see Section 1.3.)

The question that arises at this point is, why are twisted sheaves interesting, and where do they occur naturally? A starting point would be the study of Brauer-Severi varieties ($\mathbf{P}^n$-bundles which are *not* locally trivial in the Zariski topology, but are locally trivial in finer topologies, like the étale or the analytic topology). But the most important occurrence of twisted sheaves is in the study of moduli spaces of semistable sheaves on projective varieties.

The difficulty with most moduli problems is that they are not *fine*: although a moduli space exists (which parametrizes in a nice, algebraic way the objects under consideration), a universal object fails to exist. In the case of semistable sheaves on a projective variety X, this means that a moduli space M exists (whose points $[\mathscr{F}]$ correspond to semistable sheaves $\mathscr{F}$ on X), but there does not exist a universal sheaf on $X \times M$, i.e. a sheaf $\mathscr{U}$ such that $\mathscr{U}|_{X\times[\mathscr{F}]} \cong \mathscr{F}$. One cause for this problem is unsolvable: some points in M (the so-called properly semistable points) represent more than one semistable sheaf on X (they actually represent a whole S-equivalence class of sheaves). However, even when there are no properly semistable points in M, there may not exist a universal sheaf on $X \times M$. The reason is that although universal sheaves exist locally on M, they may not glue well along all of M. (Don't be fooled by the word *universal*: although the universal sheaves do represent a functor, they are not unique, and it is precisely this lack of uniqueness that causes them to fail to glue.)

This suggests that there may be a hope in gluing these local sheaves into a twisted universal sheaf: this is indeed the case, and for any moduli problem of semistable sheaves on a space X, we can find a twisted universal sheaf on $X \times M^s$, where M^s denotes the stable part of M (the set of points which are not properly semistable). The twisting depends only on the moduli problem under consideration (and not on the particular choice of local universal sheaves), and therefore we can also view it as the obstruction to the existence of a universal sheaf on $X \times M^s$ (if we wish to take a negative point of view!).

Next we study *integral transforms* between the derived categories of X and of M, defined by means of this twisted universal sheaf $\mathscr{U}$. It turns out that if $\mathscr{U}$ is twisted by $\alpha \in \mathrm{Br}(M)$, then we get integral functors

$$\Phi^{\mathscr{U}}_{M\to X} : \mathbf{D}^b_{\mathrm{coh}}(M, \alpha^{-1}) \to \mathbf{D}^b_{\mathrm{coh}}(X)$$
$$\Phi^{\mathscr{U}^\vee}_{X\to M} : \mathbf{D}^b_{\mathrm{coh}}(X) \to \mathbf{D}^b_{\mathrm{coh}}(M, \alpha^{-1})$$

where $\mathscr{U}^\vee$ is the dual (in the derived category) of $\mathscr{U}$, and $\mathbf{D}^b_{\mathrm{coh}}(M, \alpha^{-1})$ is the derived category of the category of α^{-1}-twisted coherent sheaves. These functors are defined in a manner entirely similar to the way correspondences in cohomology (given by a cocycle in $H^*(X \times M)$) are defined, and in fact sometimes there are such correspondences associated to these integral functors. For more details, see Section 3.1.

The reason this is a natural thing to do is the fact (discovered by Mukai) that in many cases of interest these integral functors turn out to be equivalences, providing powerful tools for studying the geometry of M. As an example of an application of this philosophy, Mukai proved (in [32]) that a compact, 2-dimensional component of the moduli space of stable sheaves on a K3 surface is again a K3 surface (Theorem 5.1.6). To use this idea we need a criterion for checking when an integral functor is an equivalence of derived categories, and we provide one, very similar to the one for untwisted derived categories due to Mukai, Bondal-Orlov and Bridgeland (Theorem 3.2.1).

We study what happens on the level of cohomology, when such an equivalence exists (for untwisted sheaves). We extend Mukai's results for K3 surfaces to higher dimensions, with the aim of applying them to examples of Calabi-Yau threefolds (in Chapter 6). The main result in this direction is the proof of the fact that under a certain modification of the usual cup product on the total cohomology of a space (similar to the Mukai product on a K3), the cohomological Fourier-Mukai transforms give isometries between the total cohomology groups of X and of M. This is used later to construct counterexamples to the generalization of the Torelli theorem from K3 surfaces to Calabi-Yau threefolds.

These results form the first part of the dissertation. We also include general results regarding the Brauer group, the category of twisted sheaves and its derived category, Morita theory on a scheme.

The second part of the work is devoted to the study of examples: smooth elliptic fibrations and Ogg-Shafarevich theory, K3 surfaces, and elliptic Calabi-Yau threefolds.

For smooth elliptic fibrations (which will provide the typical examples of occurrences of twisted sheaves in moduli problems), the situation is well-understood. Assume given a smooth morphism $X \to S$, between smooth varieties or complex manifolds, such that all the fibers are elliptic curves. (This is what we call a *smooth elliptic fibration.*) Such a fibration may have no sections; however, there is a standard construction (the *relative Jacobian*) which provides us with another smooth elliptic fibration $J \to S$, fibered over the same base, such that all the elliptic fibers are the same (i.e. $J_s \cong X_s$ for all closed points $s \in S$). However, unlike the initial fibration, the relative Jacobian always has a section. The standard construction of J is to view it as a relative moduli space (over S) of semistable sheaves on the fibers of X, of rank one and degree zero. (This may seem like a contorted way to say "degree zero line bundles", but it will pay off when we'll consider non-smooth fibrations.) Since J is a moduli space of stable sheaves on X, we obtain a twisting $\alpha \in \mathrm{Br}(J)$ and a twisted universal sheaf $\mathscr{U}$ on $X \times_S J$ (the product is the fibered product, since everything we do is relative to the base S).

Recall that in the case of smooth elliptic fibrations there is a well-known identification of $\mathrm{Br}(J)/\mathrm{Br}(S)$ with the Tate-Shafarevich group $\text{Ш}_S(J)$. The main property of $\text{Ш}_S(J)$ is that it classifies elliptic fibrations (possibly without a section) whose relative Jacobian is J. Having started with an elliptic fibration $X \to S$ whose Jacobian is J, we have thus defined two different elements of $\text{Ш}_S(J)$: the image of α, that measures the obstruction to the existence of a universal sheaf on $X \times_S J$, and $\alpha_{X/S}$, which represents $X \to S$ in $\text{Ш}_S(J) = \mathrm{Br}(J)/\mathrm{Br}(S)$. A first result in the study of these objects is Theorem 4.4.1:

$$\alpha \bmod \mathrm{Br}(S) = \alpha_{X/S}.$$

In other words, we can recover Ogg-Shafarevich theory from the theory of the obstructions to the existence of a universal sheaf for a moduli problem.

Next we move on to K3 surfaces, and we study moduli spaces of sheaves on them. We are mainly interested in the case when the moduli space is two-dimensional, in which case we know that it is again a K3 surface. Building upon the work of Mukai, we are able to identify the obstruction to the existence of a universal sheaf, only in terms of the cohomological Fourier-Mukai transform. This provides an interesting interpretation of the twisted derived categories of a K3 X: if the usual derived category corresponds (by a result of Orlov, [36]) to the Hodge structure on the transcendental lattice T_X of X, then the derived category of α-twisted sheaves on X corresponds to the induced Hodge structure on the sublattice $\mathrm{Ker}(\alpha)$ of T_X. (This makes sense, because for a K3 surface we have an identification

$$\mathrm{Br}(X) \cong T_X^\vee \otimes \mathbf{Q}/\mathbf{Z},$$

where $T_X^\vee$ denotes the dual lattice to T_X.) In particular, this allows us to conclude that in a number of interesting cases we have

$$\mathbf{D}^b_{\mathrm{coh}}(X, \alpha) \cong \mathbf{D}^b_{\mathrm{coh}}(X, \alpha^k)$$

for a K3 X, $\alpha \in \mathrm{Br}(X)$, and $k \in \mathbf{Z}$ coprime to the order of α. This result is quite striking: although a similar statement can be made over any scheme, it could not hold over the spectrum of any local ring! This result therefore illustrates the *global* nature of derived categories.

The next chapter deals with elliptic Calabi-Yau threefolds. We only restrict our attention to the simplest examples, where the singularities of the fibers are easiest to understand (see Chapter 6 for details). Birationally, these are the same as the smooth elliptic threefolds studied previously, but the existence of singular fibers introduces interesting features. Although the starting space X is smooth, the relative Jacobian J has singularities, and these cannot be removed in the algebraic category without losing the Calabi-Yau property $K_J = 0$. The solution to this is to work in the analytic category; we prove that for any analytic small resolution $\bar{J} \to J$ of the singularities of J we can find a twisted pseudo-universal sheaf $\mathscr{U}$ on $X \times_S \bar{J}$. (S is the base for the fibrations under consideration.) The reason we call this twisted sheaf "pseudo-universal" is the fact that over the stable points of $\bar{J}$ (which lie over the smooth points of J) $\mathscr{U}$ is equal to the twisted universal sheaf whose existence was asserted before. However, over points of $\bar{J}$ that lie over a singular – and semistable – point P of J, the sheaf $\mathscr{U}$ parametrizes *some* of the semistable sheaves in the S-equivalence class whose image in J is P.

Using this twisted pseudo-universal sheaf $\mathscr{U}$, we can define a Fourier-Mukai equivalence

$$\mathbf{D}^b_{\mathrm{coh}}(X) \cong \mathbf{D}^b_{\mathrm{coh}}(\bar{J}, \alpha^{-1}),$$

where $\alpha \in \mathrm{Br}'(\bar{J}) \subseteq \mathrm{Br}'(J)$ unique extension to $\bar{J}$ of the obstruction to the existence of a universal sheaf on $X \times_S J^s$.

This allows us to again construct a number of examples in which we have

$$\mathbf{D}^b_{\mathrm{coh}}(\bar{J}, \alpha) \cong \mathbf{D}^b_{\mathrm{coh}}(\bar{J}, \alpha^k)$$

for k coprime to the order of α. This seems to be a rather general phenomenon for twisted sheaves on schemes or complex manifolds!

One section of the chapter on elliptic Calabi-Yau threefolds is dedicated to applying the previously obtained results on cohomological Fourier-Mukai transforms. We are able to produce two elliptic Calabi-Yau threefolds which are not birational, but whose $\mathbf{Z}$-Hodge structures and cup-product structures are identical. This provides a counterexample to the generalization of the Torelli theorem from K3 surfaces to Calabi-Yau threefolds. (For K3's, the corresponding statement is that if the Hodge structures on the cohomologies of two K3's are the same, respecting the cup product, then the K3's are isomorphic. It was known before that there can be non-isomorphic Calabi-Yau's with isomorphic $\mathbf{Z}$-Hodge structures, but it was believed that they should at least be birational to each other. We are unable to prove that the two Calabi-Yau's we produce are not birational to each other, but we give strong supporting evidence that they are not.)

This construction has another interesting side to it: the apparition of the singularities in J seems to be caused by the same phenomenon that causes singularities to appear in an example of Vafa-Witten ([43]) and Aspinwall-Morrison ([1]). We give a brief suggestion as to why these singularities are unavoidable in Section 6.8.

The dissertation closes with a chapter in which we state a number of open questions, that would deserve further investigation. Hopefully, these questions will attract the interest of someone with mightier mathematical powers than mine...

Part I

Theoretical Results

Chapter 1

Twisted Sheaves

In this chapter we introduce twisted sheaves, which are the main object of study in this work. We begin with the definition of the Brauer group through various descriptions of its elements (as Čech 2-cocycles, as Azumaya algebras, as gerbes), and we continue with the definition and basic properties of twisted sheaves. These are viewed as local sheaves and isomorphisms that don't quite "match up" in Section 1.2. Section 1.3 gives a more natural interpretation of twisted sheaves, as modules over an Azumaya algebra. The material in this chapter is fundamental in the understanding of the sequel.

1.1 $H^2_{\text{ét}}(X, \mathscr{O}_X^*)$, the Brauer Group and Gerbes

The Cohomological Brauer Group

Let us start with an example, which will serve as the main motivation for the introduction of the Brauer group.

Example 1.1.1. Let $f : Y \to X$ be a smooth morphism of smooth algebraic varieties or complex analytic spaces, such that all the fibers of f are isomorphic to $\mathbf{P}^{n-1}$ (this will be called a *projective bundle*). We are interested in answering the question: is there a vector bundle $\mathscr{E}$ on X, of rank n, such that $Y \to X$ is isomorphic to $\mathbf{P}(\mathscr{E}) \to X$?

A possible way to study this problem is the following: first, solve the problem locally on X, using a fine enough topology (étale, or analytic). We can do this because locally in these topologies any projective bundle is trivial, so projectivizing the trivial bundle of rank n gives a solution.

Second, cover X with open sets U_i such that over U_i we can solve the problem, let $Y_i = f^{-1}(U_i)$, and on each U_i choose a vector bundle $\mathscr{E}_i$ and an isomorphism $\varphi_i : \mathbf{P}(\mathscr{E}_i) \xrightarrow{\sim} Y_i$ over U_i. Let

$$\varphi_{ij} = \varphi_i^{-1} \circ \varphi_j : \mathbf{P}(\mathscr{E}_j|_{U_i \cap U_j}) \to \mathbf{P}(\mathscr{E}_i|_{U_i \cap U_j}).$$

In order to be able to patch together the $\mathscr{E}_i$'s, we would need to be able to find liftings $\bar{\varphi}_{ij} : \mathscr{E}_j|_{U_i \cap U_j} \to \mathscr{E}_i|_{U_i \cap U_j}$ of φ_{ij} to isomorphisms of the vector bundles, such that $\mathbf{P}(\bar{\varphi}_{ij}) = \varphi_{ij}$. Of course, this can always be done on small enough open sets, but these liftings are only unique up to the choice of an element of $\Gamma(U_i \cap U_j, \mathscr{O}_X^*)$. This gives rise to the following problem: although $\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ki}$ is the identity on $\mathbf{P}(\mathscr{E}_i|_{U_i \cap U_j \cap U_k})$, the corresponding equality for $\bar{\varphi}_{ij}$'s only holds true up to an element of $\alpha_{ijk} \in \Gamma(U_i \cap U_j \cap U_k, \mathscr{O}_X^*)$, so we may not be able to patch together the $\mathscr{E}_i$'s via the $\bar{\varphi}_{ij}$'s.

One can express this fact cohomologically using the exact sequence

$$0 \to \mathscr{O}_X^* \to \mathrm{GL}(n) \to \mathrm{PGL}(n) \to 0,$$

(recall that we are working in either the étale or analytic topologies, in which this sequence is exact, as opposed to the case of the Zariski topology) whose long exact cohomology sequence yields

$$H^1(X, \mathscr{O}_X^*) \to H^1(X, \mathrm{GL}(n)) \to H^1(X, \mathrm{PGL}(n)) \xrightarrow{\delta} H^2(X, \mathscr{O}_X^*).$$

The geometric interpretation of this exact sequence is as follows: the given projective bundle $Y \to X$ corresponds to an element $[Y]$ of $H^1(X, \mathrm{PGL}(n))$. It lifts to an element of $H^1(X, \mathrm{GL}(n))$ (i.e. to a rank n vector bundle) if and only if

$$\delta([Y]) = 0.$$

If it does lift, the ambiguity is an element of $H^1(X, \mathscr{O}_X^*)$ (a line bundle).

We are thus led to studying the group $H^2(X, \mathscr{O}_X^*)$, as the place where the obstruction $\delta([Y/X])$ naturally lives. This will be the cohomological Brauer group.

In algebraic geometry one often encounters problems of this type: a solution can easily be found locally, but the result is unique only up to the choice of a line bundle. Typical examples of such problems are the construction of a universal sheaf for a moduli problem, and the lifting of a projective bundle to a line bundle (see the previous example and Chapter 3). However, a global solution to the problem in question may not exist, because the local solutions do not patch up nicely. One is usually interested in understanding the cohomological obstruction to patching up, and this obstruction is naturally an element of $H^2(X, \mathscr{O}_X^*)$. This higher-dimensional analogue of the Picard group (which equals $H^1(X, \mathscr{O}_X^*)$) is quite different from it in many ways, and will be the object of study in this section.

One important issue we need to take into account is the topology we use. In most problems of interest, the existence of a solution can not be guaranteed, even locally, if one works in the Zariski topology; however, a solution can be found by passing to the étale or analytic topologies. Because of this, the cohomology groups in question are considered in these topologies.

Definition 1.1.2. The *cohomological Brauer group* of a scheme X, $\mathrm{Br}'_{\text{ét}}(X)$, is defined to be $H^2_{\text{ét}}(X, \mathscr{O}_X^*)$. Similarly, if X is an analytic space, we define $\mathrm{Br}'_{\text{an}}(X)$

to be $H^2_{\text{an}}(X, \mathscr{O}_X^*)$. If we do not specify the topology used, it is implicitly assumed to be the étale topology in the case of a scheme, and the analytic topology in the case of an analytic space.

The main references for the Brauer group are Grothendieck's series of papers ([18]) and Milne's book on étale cohomology ([30]). Most of the results that follow can be found there. We will use the notations of the analytic topology (with intersections of open sets, for example, instead of fibered products), because these make the exposition more fluid; it is an easy exercise to rewrite everything to make sense in the étale topology. Whenever things are different for the étale case, it will be mentioned explicitly.

The main representation we shall use for the elements of $\mathrm{Br}'(X)$ is the one obtained from Čech cohomology: an element $\alpha \in \mathrm{Br}'(X)$ is given by specifying an open cover $\{U_i\}$ of X and sections $\{\alpha_{ijk}\}$ of $\Gamma(U_i \cap U_j \cap U_k, \mathscr{O}_X^*)$ for each triple of indices (i, j, k), satisfying the cocycle condition.

The first thing one notes in Example 1.1.1 is the fact that $\delta([Y])$ (the obstruction to lifting $Y \to X$ to a vector bundle) is n-torsion. Indeed, instead of using the exact sequence

$$0 \to \mathscr{O}_X^* \to \mathrm{GL}(n) \to \mathrm{PGL}(n) \to 0$$

that we used in order to define the obstruction, we could have used the exact sequence

$$0 \to \mathbf{Z}/n\mathbf{Z} \to \mathrm{SL}(n) \to \mathrm{PGL}(n) \to 0.$$

(If we work over an arbitrary field, $\mathbf{Z}/n\mathbf{Z}$ should be replaced by μ_n, the group of n-th order roots of unity in the field we're working.) This yields a different coboundary map, $\delta' : H^1(X, \mathrm{PGL}(n)) \to H^2(X, \mathbf{Z}/n\mathbf{Z})$, and it is easy to see that the square

$$\begin{array}{ccc} H^1(X, \mathrm{PGL}(n)) & \xrightarrow{\delta'} & H^2(X, \mathbf{Z}/n\mathbf{Z}) \\ \| & & \downarrow \\ H^1(X, \mathrm{PGL}(n)) & \xrightarrow{\delta} & H^2(X, \mathscr{O}_X^*) \end{array}$$

commutes, where the rightmost vertical arrow is deduced from the natural inclusion $\mathbf{Z}/n\mathbf{Z} \hookrightarrow \mathscr{O}_X^*$. But $H^2(X, \mathbf{Z}/n\mathbf{Z})$ is obviously n-torsion, so $\delta([f])$ is n-torsion as well.

Hence it becomes relevant to study the torsion part of $\mathrm{Br}'(X)$. One gets easily the following description of it:

Theorem 1.1.3. *On a scheme or analytic space X there exists the following exact sequence:*

$$0 \to \mathrm{Pic}(X) \otimes \mathbf{Q}/\mathbf{Z} \to H^2(X, \mathbf{Q}/\mathbf{Z}) \to \mathrm{Br}'(X)_{\text{tors}} \to 0.$$

Proof. Start with the exact sequence

$$0 \to \mathbf{Z}/n\mathbf{Z} \to \mathscr{O}_X^* \xrightarrow{.n} \mathscr{O}_X^* \to 0,$$

which is exact in both the étale and analytic topologies. The long exact cohomology sequence gives

$$\operatorname{Pic}(X) \xrightarrow{\cdot n} \operatorname{Pic}(X) \xrightarrow{c_1} H^2(X, \mathbf{Z}/n\mathbf{Z}) \to \operatorname{Br}'(X) \xrightarrow{\cdot n} \operatorname{Br}'(X)$$

which implies that the n-torsion part of $\operatorname{Br}'(X)$, $\operatorname{Br}'(X)_n$, fits in the exact sequence

$$0 \to \operatorname{Pic}(X) \otimes \mathbf{Z}/n\mathbf{Z} \to H^2(X, \mathbf{Z}/n\mathbf{Z}) \to \operatorname{Br}'(X)_n \to 0.$$

Taking the direct limit over all n, we get the result. □

We also have:

Theorem 1.1.4. *If X is a smooth scheme then, in the étale topology,* $\operatorname{Br}'(X)$ *is torsion. For the associated analytic space, X^h, we have* $\operatorname{Br}'_{\text{ét}}(X) = \operatorname{Br}'_{\text{an}}(X^h)_{\text{tors}}$.

Proof. The first statement is just [18, II, 1.4]. For the second statement note that $H^2(X, \mathbf{Q}/\mathbf{Z})$ and $\operatorname{Pic}(X)$ are the same in the étale and analytic topologies, and use the previous theorem. □

There are two particular cases where we want to specialize these results further. One is the case of a smooth, simply connected surface. In this case we have $H^3(X, \mathbf{Z}) = 0$ and $H^2(X, \mathbf{Z})$ is torsion free and isomorphic to $H_2(X, \mathbf{Z})$ by Poincaré duality. From the universal coefficient theorem we get $H^2(X, \mathbf{Q}/\mathbf{Z}) \cong H^2(X, \mathbf{Z}) \otimes \mathbf{Q}/\mathbf{Z}$, so we conclude that

$$\operatorname{Br}'(X) = (H^2(X, \mathbf{Z})/\operatorname{NS}(X)) \otimes \mathbf{Q}/\mathbf{Z}.$$

The other case of interest is when X is a Calabi-Yau manifold. In this case we have $\operatorname{Pic}(X) \cong H^2(X, \mathbf{Z})$ because $H^1(X, \mathscr{O}_X) = H^2(X, \mathscr{O}_X) = 0$, so we conclude from the above two theorems and from the universal coefficient theorem that $\operatorname{Br}'(X) \cong H^3(X, \mathbf{Z})_{\text{tors}}$.

Azumaya Algebras and the Brauer Group

Of particular importance in the study of twisted sheaves will be those elements of the cohomological Brauer group that arise as $\delta([Y])$ in Example 1.1.1, i.e. those that lie in the images of the maps $H^1(X, \operatorname{GL}(n)) \xrightarrow{\delta} \operatorname{Br}'(X)$ for various n. This is obviously a subgroup of $\operatorname{Br}'(X)$, which is called the Brauer group of X and denoted by $\operatorname{Br}(X)$. In what follows we'll give a more intrinsic description of it.

Definition 1.1.5. Let R be a commutative ring, and let A be a (non-commutative) R-algebra. Assume that A is finitely generated projective as an R-module, and that the canonical homomorphism

$$A \otimes_R A^\circ \to \operatorname{End}_R(A)$$

is bijective. Then A is called an *Azumaya algebra* over R. The sheaf of algebras $\mathscr{A} = \tilde{A}$ over $\operatorname{Spec} R$ is called a sheaf of Azumaya algebras over R. (We follow the notation of [22, Section II.5], using $\tilde{A}$ for the sheafification of A.) If X is a scheme, $\mathscr{A}$ a sheaf of algebras over X, then $\mathscr{A}$ is said to be a sheaf of Azumaya algebras over X if and only if over each affine open set of the form $\operatorname{Spec} R$, $\mathscr{A}$ is isomorphic to the sheafification of $\tilde{A}$ of an Azumaya algebra A over R. It is an easy exercise to prove that this definition is consistent. (See, for example, [30, IV, 2.1].)

The following theorem details the local structure of Azumaya algebras:

Theorem 1.1.6. *Let X be a scheme, $\mathscr{A}$ a sheaf of algebras on X which is locally free of finite rank as a sheaf of $\mathscr{O}_X$-modules. Then $\mathscr{A}$ is a sheaf of Azumaya algebras if and only if for every $x \in X$ there exists a neighborhood (étale or analytic) $U \to X$ of x and a locally free sheaf $\mathscr{E}$ on U such that $\mathscr{A}_U$ is isomorphic (as an $\mathscr{O}_U$-algebra) with $\underline{\operatorname{End}}_{\mathscr{O}_U}(\mathscr{E})$.*

Proof. See [18, II, 5.1] for the étale case; the analytic case follows immediately from the étale one. □

The set of isomorphism classes of Azumaya algebras has a natural group structure under the tensor product operation, using the easy fact that

$$\underline{\operatorname{End}}(\mathscr{E}) \otimes_{\mathscr{O}_U} \underline{\operatorname{End}}(\mathscr{E}') \cong \underline{\operatorname{End}}(\mathscr{E} \otimes_{\mathscr{O}_U} \mathscr{E}')$$

and the previous theorem. One can consider the equivalence relation given by setting two Azumaya algebras $\mathscr{A}$ and $\mathscr{A}'$ to be equivalent if and only if there exist global vector bundles $\mathscr{E}$ and $\mathscr{E}'$ on X such that

$$\mathscr{A} \otimes \underline{\operatorname{End}}(\mathscr{E}) \cong \mathscr{A}' \otimes \underline{\operatorname{End}}(\mathscr{E}').$$

This is easily seen to be well defined and to be compatible with the group operation, hence one can take the quotient to obtain a new group.

Definition 1.1.7. The group of isomorphism classes of Azumaya algebras on X, under the tensor product operation, modulo the equivalence relation defined above, is called the Brauer group of X and is denoted by $\operatorname{Br}(X)$.

The relevance of Azumaya algebras in the context of the cohomological Brauer group stems from the following result:

Theorem 1.1.8. *There exists a natural injective map $\operatorname{Br}(X) \to \operatorname{Br}'(X)$, whose image coincides with the collection of all obstructions of the form $\delta([Y])$ (for various n) under the maps $\delta : H^1(X, \operatorname{PGL}(n)) \to H^2(X, \mathscr{O}_X^*)$.*

Proof. We'll only sketch the proof, since it is well known (see the standard references quoted in the beginning of this section). The way one constructs the map $\operatorname{Br}(X) \to \operatorname{Br}'(X)$ is by noting that since an Azumaya algebra of rank n^2 can be viewed as a bundle whose fibers are of the form $\mathscr{M}_n(\mathscr{O}_U)$, and the automorphism group of this algebra is isomorphic to $\mathrm{PGL}(n)$ by the Skolem-Noether theorem, we get a map from the set of Azumaya algebras of rank n^2 to $H^1(X, \mathrm{PGL}(n))$. In fact, it is easy to see that the set of isomorphism classes of Azumaya algebras of rank n^2 coincides with the set $H^1(X, \mathrm{PGL}(n))$, and that the equivalence relation on Azumaya algebras that we introduced does nothing more than quotient out the image of the map $H^1(X, \mathrm{GL}(n)) \to H^1(X, \mathrm{PGL}(n))$. □

We quote here, without proof, some results about Brauer groups.

Theorem 1.1.9. $\mathrm{Br}(X)$ *is torsion.*

Proof. Obvious from the previous theorem. □

Theorem 1.1.10. *If* X *is a smooth curve,* $\mathrm{Br}(X) = \mathrm{Br}'(X) = 0$*. If* X *is a smooth surface, then* $\mathrm{Br}(X) = \mathrm{Br}'(X)$.

Proof. The first statement follows easily from the exponential exact sequence. For the second one, see [30, IV.2.16]. □

Theorem 1.1.11. *If* X *is a proper scheme over* $\mathbf{C}$*, and if* $h : X^h \to X$ *is the natural continuous map from the associated analytic space* X^h*, then the map* $\mathscr{A} \mapsto \mathscr{A}^h = h^*(\mathscr{A})$ *takes an Azumaya algebra* $\mathscr{A}$ *on* X *to an Azumaya algebra* $\mathscr{A}^h$ *on* X^h *and induces an isomorphism of* $\mathrm{Br}(X)$ *with* $\mathrm{Br}(X^h)$.

Proof. Use [20, Exposé XII, 4.4] and [30, IV.2.1c] to construct an inverse. □

Elements of $H^2(X, \mathscr{O}_X^*)$ as Gerbes

There is a third way to describe the elements of $H^2(X, \mathscr{O}_X^*)$, similar to the way one describes line bundles via transition functions. The ideas (and the term *gerbe*) come from Giraud's non-abelian cohomology (see [16]), of which we'll only use a tiny amount. I follow roughly the ideas in [24], where much more detail can be found.

The idea is that we are comfortable thinking of objects that are defined on intersections of two open sets in a cover (like the transition functions of a line bundle), but we feel uneasy thinking of things defined on triple intersections. Thus we would prefer a description of the objects in terms of twofold intersections.

Elements of $H^2(X, \mathscr{O}_X^*)$ will be called gerbes. Any gerbe will be locally trivial (although nobody tells us what a trivial gerbe is, unlike the case of a line bundle, for which we have a clear geometric description). For example, if $\alpha \in H^2(X, \mathscr{O}_X^*)$ is given by a Čech cocycle $\{\alpha_{ijk}\}$ on the threefold intersections of an open cover $\{U_i\}$, then the gerbe will be trivial along the open sets in the cover.

Now we would like to understand what the "transition functions" are, i.e. what is the difference between two locally isomorphic gerbes. One way to understand this is via Čech cohomology: let α and β be trivial cocycles on the open sets U and V of X, that agree on $U \cap V$. This means that there are 1-cochains a and b

such that $\alpha = \delta a$ and $\beta = \delta b$. Since $\alpha = \beta$ on $U \cap V$, this means that $\delta(a - b) = 0$ on $U \cap V$, and therefore $a - b$ is a 1-cocycle on $U \cap V$. It thus represents a line bundle $\mathscr{L}$ on $U \cap V$ (recall that all cochains, cocycles, etc. take values in $\mathscr{O}_X^*$).

We conclude that one can give the following description of gerbes: a gerbe is given by fixing an open cover $\{U_i\}$, and giving a collection $\{\mathscr{L}_{ij}\}$ on the twofold intersections $U_i \cap U_j$, under the requirement that this collection satisfies the following cocycle conditions:

1. $\mathscr{L}_{ii} = \mathscr{O}_{U_i}$;
2. $\mathscr{L}_{ij} = \mathscr{L}_{ji}^{-1}$;
3. $\mathscr{L}_{ij} \otimes \mathscr{L}_{jk} \otimes \mathscr{L}_{ki} =: \mathscr{L}_{ijk}$ is trivial (but a trivialization is not being fixed);
4. $\mathscr{L}_{ijk} \otimes \mathscr{L}_{jkl}^{-1} \otimes \mathscr{L}_{kli} \otimes \mathscr{L}_{lij}^{-1}$ is canonically trivial (i.e. we have chosen an isomorphism of it with $\mathscr{O}_{U_i \cap U_j \cap U_k \cap U_l}$).

One way to understand conditions 3 and 4 above is by considering what happens when one refines the cover: one can get to a situation where all the line bundles $\mathscr{L}_{ij}$ are trivial, hence all the information must lie in the actual bundles, and not in their isomorphism type. A choice of trivialization for $\mathscr{L}_{ijk}$ would correspond then to a choice of an element of $\Gamma(U_i \cap U_j \cap U_k, \mathscr{O}_X^*)$, and we return to the old description of gerbes via Čech cohomology, where condition 4 corresponds to the fact that we are dealing with a cocycle.

These conditions will automatically be satisfied in all the constructions we'll do. For an explicitly worked example, see Chapter 4.

We would also like to express what it means to modify a gerbe (given by the above collection of line bundles) by a coboundary. Let $\{\mathscr{F}_i\}$ be line bundles on $\{U_i\}$. Then, modifying $\{\mathscr{L}_{ij}\}$ by $\partial\{\mathscr{F}_i\}$ gives the collection $\{\mathscr{L}_{ij} \otimes \mathscr{F}_i \otimes \mathscr{F}_j^{-1}\}$ (and therefore $\{\mathscr{L}_{ij}\}$ represents the trivial gerbe if and only if one can find line bundles $\{\mathscr{F}_i\}$ such that $\mathscr{L}_{ij} = \mathscr{F}_i \otimes \mathscr{F}_j^{-1}$).

For more details about the gerbe representation, the reader should consult [10] or the standard reference on non-abelian cohomology, [16].

1.2 Twisted Sheaves

If one considers Azumaya algebras (or, more generally, gerbes) as replacements for the structure sheaf of a scheme, then twisted sheaves are the natural objects to replace sheaves of modules. In this section we give their definition and basic properties, and show a first example where they naturally appear. For an extended example, see Chapter 4.

Throughout this section, $(X, \mathscr{O}_X)$ will be a scheme (considered with the étale topology) or an analytic space (with either the étale or analytic topology).

Definition 1.2.1. Let $\alpha \in \check{C}^2(X, \mathscr{O}_X^*)$ be a Čech 2-cocycle (in the topology under consideration), given by means of an open cover $\mathfrak{U} = \{U_i\}_{i \in I}$ and sections $\alpha_{ijk} \in \Gamma(U_i \cap U_j \cap U_k, \mathscr{O}_X^*)$. We define an α-twisted sheaf on X (or α-sheaf, in short) to consist of a pair $(\{\mathscr{F}_i\}_{i \in I}, \{\varphi_{ij}\}_{i,j \in I})$, with $\mathscr{F}_i$ being a sheaf of $\mathscr{O}_X$-modules on U_i and $\varphi_{ij} : \mathscr{F}_j|_{U_i \cap U_j} \to \mathscr{F}_i|_{U_i \cap U_j}$ being isomorphisms such that

1. φ_{ii} is the identity for all $i \in I$;
2. $\varphi_{ij} = \varphi_{ji}^{-1}$ for all $i, j \in I$;
3. $\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ki}$ is multiplication by α_{ijk} on $\mathscr{F}_i|_{U_i \cap U_j \cap U_k}$ for all $i, j, k \in I$.

A homomorphism f between α-twisted sheaves $\mathscr{F}, \mathscr{G}$ consists of a collection of maps $f_i : \mathscr{F}_i \to \mathscr{G}_i$ for each $i \in I$ such that $f_i \circ \varphi_{\mathscr{F},ij} = \varphi_{\mathscr{G},ij} \circ f_j$ for all $i, j \in I$. Composition of homomorphisms is defined in the obvious way, and this makes the class of all α-sheaves and homomorphisms (given along the cover $\mathfrak{U}$) into a category $\mathfrak{Mod}(X, \alpha, \mathfrak{U})$. We'll see (Lemma 1.2.3) that for different open covers on which α can be represented we get equivalent categories, so we'll just denote any of these equivalent categories by $\mathfrak{Mod}(X, \alpha)$. Also, Lemma 1.2.8 shows that for α and α' in the same cohomology class the categories $\mathfrak{Mod}(X, \alpha)$ and $\mathfrak{Mod}(X, \alpha')$ are equivalent, so we will use the notation $\mathfrak{Mod}(X, \alpha)$ for $\alpha \in \check{H}^2(X, \mathscr{O}_X^*)$ whenever the choice of the specific cocycle is irrelevant. Often we'll even abuse the notation by saying "let $\mathscr{F} \in \mathfrak{Mod}(X, \alpha), \mathscr{G} \in \mathfrak{Mod}(X, \alpha')$, etc. for some $\alpha, \alpha', \ldots \in \check{H}^2(X, \mathscr{O}_X^*)$," and mean by this "let $\alpha, \alpha', \ldots$ be any cocycles in $\check{C}^2(X, \mathscr{O}_X^*)$, let $\mathfrak{U}$ be an open cover over which $\alpha, \alpha', \ldots$ can be represented, and let $\mathscr{F}$ be an α-sheaf, $\mathscr{G}$ an α'-sheaf, etc. given over $\mathfrak{U}$." We will never deal with an infinite number of α's at the same time, so there is no problem finding a cover that works for all simultaneously.

An α-sheaf $\mathscr{F}$ on X is called coherent if all the underlying sheaves $\mathscr{F}_i$ are coherent. The category of α-twisted coherent sheaves will be denoted by $\mathfrak{Coh}(X, \alpha)$. One defines similarly the category of quasi-coherent α-sheaves, $\mathfrak{Qcoh}(X, \alpha)$.

It is easy to see that $\mathfrak{Mod}(X, \alpha)$, $\mathfrak{Coh}(X, \alpha)$ and $\mathfrak{Qcoh}(X, \alpha)$ are abelian categories, under the natural definitions of kernel, cokernel, etc.

Before we start into the general theory of twisted sheaves, let's consider a typical example. It is a continuation of Example 1.1.1. For another, more extensive example, see Chapter 4.

Example 1.2.2. Consider the setup of Example 1.1.1. Recall that we covered X by open sets U_i, and on each U_i we found a vector bundle $\mathscr{E}_i$ of rank n and isomorphisms $\bar{\varphi}_{ij} : \mathscr{E}_j \to \mathscr{E}_i$ with

$$\bar{\varphi}_{ij} \circ \bar{\varphi}_{jk} \circ \bar{\varphi}_{ki} = \alpha_{ijk} \cdot \mathrm{id}\,.$$

This collection forms an α-twisted sheaf $\mathscr{E}$ for the α defined by the collection $\{\alpha_{ijk}\}$ (which is easily seen to be a cocycle). Making different choices for the isomorphisms $\bar{\varphi}$ only changes α by a coboundary. We'll see in Section 1.3 that $\underline{\mathrm{End}}(\mathscr{E})$ is naturally an Azumaya algebra, and that $\mathscr{E}$ is a twisted sheaf of modules over this algebra.

Lemma 1.2.3. *Let $\alpha \in \check{C}^2(X, \mathscr{O}_X^*)$, and let $\mathfrak{U}'$ be a refinement of an open cover $\mathfrak{U}$ on which α can be represented. Then we have an equivalence of categories*

$$\mathfrak{Mod}(X, \alpha, \mathfrak{U}) \cong \mathfrak{Mod}(X, \alpha, \mathfrak{U}').$$

Before we prove this result, it is useful to recall the following result on gluing sheaves in the étale topology.

Lemma 1.2.4 (Gluing Sheaves in the Étale Topology). *Let X be a scheme, endowed with the étale topology, let $\mathfrak{U} = \{\rho_i : U_i \to X\}$ be an open cover of X, and suppose we are given for each i a sheaf $\mathscr{F}_i$ on U_i, and for each i, j an isomorphism*

$$\varphi_{ij} : (p_j^{ij})^* \mathscr{F}_j|_{U_i \times_X U_j} \to (p_i^{ij})^* \mathscr{F}_i|_{U_i \times_X U_j}$$

such that for each i, j, k,

$$(p_{ij}^{ijk})^*(\varphi_{ij}) \circ (p_{jk}^{ijk})^*(\varphi_{jk}) \circ (p_{ki}^{ijk})^*(\varphi_{ki}) = \mathrm{id}_{(p_i^{ijk})^* \mathscr{F}_i},$$

where p_{ij}^{ijk} is the projection from $U_i \times_X U_J \times_X U_k$ to $U_i \times_X U_j$, and similarly for the other projections. Then there exists a unique sheaf $\mathscr{F}$ on X, together with isomorphisms $\psi_i : \rho_i^* \mathscr{F} \xrightarrow{\sim} \mathscr{F}_i$ such that for each i, j,

$$(p_i^{ij})^*(\psi_i) = \varphi_{ij} \circ (p_j^{ij})^*(\psi_j)$$

on $U_i \times_X U_j$. We say loosely that $\mathscr{F}$ is obtained by gluing the sheaves $\mathscr{F}_i$ via the isomorphisms φ_{ij}.

Remark 1.2.5. We have phrased this lemma using pull-backs instead of restriction maps in order to make apparent its relationship to the standard lemma in descent theory (see, for example, [30, I.2.22]). From here on, however, we'll revert to the more convenient notation where if $\mathscr{F}$ is a sheaf on a space X, and $\varphi : U \to X$ is an étale open set, we write $\mathscr{F}|_U$ for $\varphi^* \mathscr{F}$, and if $f : \mathscr{F} \to \mathscr{G}$ is a map of sheaves on X, we write $f|_U$ for $\varphi^*(f)$. See also the definition of a presheaf in [30, p.47].

Proof. (We only sketch a proof, since this is well-known. For a topological space, see [22, Ex. II.1.22].) Let $U \to X$ be an open set in the étale topology. Define

$$\mathscr{F}(U) = \{(s_i) \in \prod_i \mathscr{F}_i(U_i \times_X U) \mid \varphi_{ij}(s_j|_{U_i \times_X U_j \times_X U}) = s_i|_{U_i \times_X U_j \times_X U} \text{ for all } i, j\},$$

and define ψ_i by

$$\psi_i(U)((s_j)_j) = s_i \in \mathscr{F}_i(U_i \times_X U).$$

It is now only a tedious check to see that $\mathscr{F}$ is a sheaf and that the collection $\{\psi_i\}$ satisfies the required properties.

The important thing to note, however, is that we have not made any choices here, and therefore this construction is entirely functorial. $\square$

Proof of Lemma 1.2.3. We only give a proof of this result in the analytic category, for ease of notation. The corresponding proof in the étale category is entirely similar. Since $\mathfrak{U}' = \{U'_j\}_{j\in J}$ is a refinement of $\mathfrak{U} = \{U_i\}_{i\in I}$ we are given a map $\lambda : J \to I$ such that for each $j \in J$ we have $U'_j \subseteq U_{\lambda(j)}$. If $\mathscr{F}$ is an α-twisted sheaf given on $\mathfrak{U}$, then there is a natural notion of refinement of $\mathscr{F}$ to $\mathfrak{U}'$: this is the pair

$$(\{\mathscr{F}_{\lambda(j)}|_{U_j}\}_{j\in J}, \{\varphi_{\lambda(i)\lambda(j)}|_{U_i\cap U_j}\}_{j\in J}),$$

and this construction clearly gives a refinement functor

$$\mathfrak{Mod}(X, \alpha, \mathfrak{U}) \to \mathfrak{Mod}(X, \alpha, \mathfrak{U}').$$

In order to show that the refinement functor is an equivalence of categories, we would need to show that it is fully faithful and that every α-sheaf $\mathscr{G}$ given on $\mathfrak{U}'$ is isomorphic to the refinement of an α-sheaf $\mathscr{F}$ on $\mathfrak{U}$ (see [12, p.84] or [9, p.26]). The first part (fully faithful) is an easy consequence of the definition of the morphisms, and it only remains to construct $\mathscr{F}$.

Let $\mathscr{G} = (\{\mathscr{G}_j\}, \{\psi_{jk}\})$ be an α-sheaf given along $\mathfrak{U}'$, and let U_i be any open set in $\mathfrak{U}$ ($i \in I$). Then define the sheaf $\mathscr{F}_i$ on U_i as follows: if $V \subseteq U_i$ is an open set, then

$$\mathscr{F}_i(V) = \{(s_j)_{j\in J} \in \prod_{j\in J} \mathscr{G}(U'_j \cap V) \mid \psi_{jk}(s_k) = \alpha_{i\lambda(j)\lambda(k)} s_j \text{ for all } j, k \in J\}.$$

It is not hard to see that this definition, along with the obvious restriction maps, makes $\mathscr{F}_i$ into a sheaf on U_i.

(Note that what we are doing is in fact gluing the sheaves $\mathscr{G}_j|_{U_i\cap U_j}$ along the isomorphisms

$$\alpha^{-1}_{i\lambda(j)\lambda(k)}\psi_{jk}|_{U_i\cap U_j\cap U_k}.$$

Using Lemma 1.2.4 we can do the same construction in the étale setup.)

Define the isomorphisms $\varphi_{ii'} : \mathscr{F}_{i'} \to \mathscr{F}_i$ that will make the collection $(\{\mathscr{F}_i\}, \{\varphi_{ii'}\})$ into an α-sheaf as follows:

$$\varphi_{ii'}((s_j)_{j\in J}) = (t_j)_{j\in J} = (\alpha_{ii'\lambda(j)} s_j)_{j\in J}$$

over any open set $V \subseteq U_i \cap U_j$. One now easily verifies that (t_j) is indeed a section of $\mathscr{F}_i$ over V, and that $\varphi_{ii'}$ is indeed an isomorphism. Finally, one checks that

$$\varphi_{ii'} \circ \varphi_{i'i''} \circ \varphi_{i''i} = \alpha_{ii'i''},$$

and thus that the collection $(\{\mathscr{F}_i\}, \{\varphi_{ii'}\})$ is an α-sheaf whose refinement to $\mathfrak{U}'$ is isomorphic to $\mathscr{G}$. □

Corollary 1.2.6. *Let $\alpha \in \check{C}^2(X, \mathscr{O}^*_X)$ and let $\mathscr{F}$ be an α-sheaf given along an open cover $\mathfrak{U}$. Let $\mathfrak{U}'$ be any open cover over which α can be represented. Then $\mathscr{F}$ can be represented by an α-sheaf on $\mathfrak{U}'$. In particular, let $\mathscr{F}$ be a sheaf (untwisted) whose support is contained in an open set U over which α is trivial. Then $\mathscr{F}$ can be also given the structure of an α-sheaf.*

Proof. Let $\mathfrak{U}''$ be a refinement of both $\mathfrak{U}$ and $\mathfrak{U}'$. By first refining $\mathscr{F}$ to $\mathfrak{U}''$, and then doing the construction from the previous lemma we obtain a representation of $\mathscr{F}$ on $\mathfrak{U}'$. □

Remark 1.2.7. We've already started abusing the notation. By "representing an α-sheaf on an open cover $\mathfrak{U}'$" we mean using the two functors (refinement and its inverse) to pass from the category of α-sheaves on the open cover $\mathfrak{U}$ to the category of α-sheaves on the new open cover $\mathfrak{U}'$. In the sequel we'll use these conventions without making explicitly note of it.

Lemma 1.2.8. *If α and α' represent the same element of $\check{H}^2(X, \mathscr{O}_X^*)$ then the categories $\mathfrak{Mod}(X, \alpha)$ and $\mathfrak{Mod}(X, \alpha')$ are equivalent. In particular, for any α that is trivial in $\check{H}^2(X, \mathscr{O}_X^*)$ the category $\mathfrak{Mod}(X, \alpha)$ is equivalent to the category $\mathfrak{Mod}(X)$ of non-twisted sheaves on X, and hence any α-sheaf on X can be viewed as a sheaf on X.*

Proof. α and α' represent the same element in $\check{H}^2(X, \mathscr{O}_X^*)$ if and only if there exists a 1-cochain g with values in $\mathscr{O}_X^*$ such that $\alpha = \alpha' + \partial g$. But then any α'-twisted sheaf $(\{\mathscr{F}_i\}, \{\varphi_{ij}\})$ can be replaced by $(\{\mathscr{F}_i\}, \{g_{ij}\varphi_{ij}\})$ to give an α-twisted sheaf, and this mapping is easily seen to be an equivalence of categories. □

Remark 1.2.9. Note that the choice of the 1-cochain g matters: different choices of g give different equivalences between $\mathfrak{Mod}(X, \alpha)$ and $\mathfrak{Mod}(X, \alpha')$ (any two such equivalences differ by tensoring with a line bundle on X). In most cases it will not matter which particular choice of g we take, but when it matters we'll mention explicitly which cochain we use.

Proposition 1.2.10. *If $\mathscr{F}$ is an α-sheaf and $\mathscr{G}$ is an α'-sheaf, then $\mathscr{F} \otimes \mathscr{G}$ is an $\alpha\alpha'$-sheaf and $\underline{\mathrm{Hom}}(\mathscr{F}, \mathscr{G})$ is an $\alpha^{-1}\alpha'$-sheaf ($\alpha, \alpha' \in \check{C}^2(X, \mathscr{O}_X^*)$). In particular, if $\mathscr{F}$ and $\mathscr{G}$ are α-sheaves, then $\underline{\mathrm{Hom}}(\mathscr{F}, \mathscr{G})$ is a sheaf. If $f : Y \to X$ is a morphism of ringed spaces, and $\mathscr{F}$ is an α-sheaf on X, then $f^*\mathscr{F}$ is an $f^*\alpha$-sheaf on Y. If $\mathscr{F}$ is an $f^*\alpha$-sheaf on Y then $f_*\mathscr{F}$ is an α-sheaf on X. Finally, if f is an open immersion and if $\mathscr{F}$ is an $f^*\alpha$-sheaf on Y then $f_!\mathscr{F}$ is also an α-sheaf on X. (Recall the definition of the "extension by zero outside an open set" functor $f_!$ for sheaves on a topological space in [22, II, Ex. 1.19], as well as the corresponding one for the étale topology in [30, II.3.18], where we replace the open immersion by an étale open set $f : U \to X$.)*

Proof. Refine the open cover enough to work for both $\mathscr{F}$ and $\mathscr{G}$. Define $\mathscr{F} \otimes \mathscr{G}$ to be the gluing of $\mathscr{F}_i \otimes \mathscr{G}_i$ along $\varphi_i \otimes \psi_i$ (we take $\mathscr{F} = (\{\mathscr{F}_i\}, \{\varphi_i\})$ and $\mathscr{G} = (\{\mathscr{G}_i\}, \{\psi_i\})$). This is obviously functorial and independent of the choice of the open covers.

For $\underline{\mathrm{Hom}}(\mathscr{F}, \mathscr{G})$ glue $\underline{\mathrm{Hom}}(\mathscr{F}_i, \mathscr{G}_i)$ along the isomorphisms

$$\underline{\mathrm{Hom}}(\mathscr{F}_i, \mathscr{G}_i) \xrightarrow{\psi_i} \underline{\mathrm{Hom}}(\mathscr{F}_i, \mathscr{G}_j) \xrightarrow{(\varphi_i^{-1})^\vee} \underline{\mathrm{Hom}}(\mathscr{F}_j, \mathscr{G}_j).$$

(Here $(\varphi_i^{-1})^\vee$ is the transpose of φ_i^{-1}, and we have omitted the restrictions to $U_i \cap U_j$.)

For $f^*\mathscr{F}$ take $(\{f^*\mathscr{F}_i\}, \{f^*\varphi_{ij}\})$ on $\{f^{-1}(U_i)\}$.

Now consider the case of f_*. Choose an open cover $\{U_i\}$ of X such that α is trivial along U_i for all i, and thus $f^*\alpha$ is trivial on $f^{-1}(U_i)$ for all i. Using Corollary 1.2.6 write $\mathscr{F}$ as $(\{\mathscr{F}_i\}, \{\varphi_{ij}\})$ on $\{f^{-1}(U_i)\}$. Take $f_*\mathscr{F}$ to be given by $\{f_*\mathscr{F}_i\}, \{f_*\varphi_{ij}\})$ on $\{U_i\}$.

A similar construction works for $f_!$. □

Remark 1.2.11. Note that if $\mathscr{F}$ and $\mathscr{G}$ are α-sheaves, then $\underline{\mathrm{Hom}}(\mathscr{F}, \mathscr{G})$ is a sheaf *without* needing to choose a 1-cocycle g as in Lemma 1.2.8. Thus the following lemma makes sense (if we had to choose one, this choice of a line bundle would have influenced the space of global sections).

Proposition 1.2.12. *For $\mathscr{F}, \mathscr{G} \in \mathfrak{Mod}(X, \alpha)$ we have*

$$\Gamma(X, \underline{\mathrm{Hom}}(\mathscr{F}, \mathscr{G})) \cong \mathrm{Hom}(\mathscr{F}, \mathscr{G}).$$

(Since $\underline{\mathrm{Hom}}(\mathscr{F}, \mathscr{G})$ is a regular sheaf, it makes sense to consider its global sections.)

Proof. Trivial chase through the definitions. □

Proposition 1.2.13. *The functor f_* is a right adjoint to f^*, as functors between $\mathfrak{Mod}(X, \alpha)$ and $\mathfrak{Mod}(Y, f^*\alpha)$. If f is an open immersion, then $f_!$ is a left adjoint to f^*.*

Proof. On X consider the sheaves $\mathscr{H}_1 = f_*\underline{\mathrm{Hom}}_{\mathfrak{Mod}(Y, f^*\alpha)}(f^*\mathscr{F}, \mathscr{G})$ and $\mathscr{H}_2 = \underline{\mathrm{Hom}}_{\mathfrak{Mod}(X,\alpha)}(\mathscr{F}, f_*\mathscr{G})$. If U is a small enough open set to trivialize α then there are natural isomorphisms $\mathscr{H}_1|_U \to \mathscr{H}_2|_U$ which glue along intersections of such U's, to give a natural isomorphism $\mathscr{H}_1 \cong \mathscr{H}_2$ and hence (taking global sections) a natural isomorphism

$$\mathrm{Hom}_{\mathfrak{Mod}(Y, f^*\alpha)}(f^*\mathscr{F}, \mathscr{G}) \cong \mathrm{Hom}_{\mathfrak{Mod}(X,\alpha)}(\mathscr{F}, f_*\mathscr{G}).$$

The same proof works for $f_!$. □

1.3 Modules over an Azumaya Algebra

When dealing with twisting classes $\alpha \in \mathrm{Br}(X)$, there is a more natural description of $\mathfrak{Mod}(X, \alpha)$ in terms of sheaves of modules over an Azumaya algebra (thus avoiding any problems related to open covers, refinements, etc.) In this section we describe this correspondence.

For a thorough treatment of Azumaya algebras over a scheme and of the Brauer group, the reader should consult [30, Chapter IV].

Lemma 1.3.1 (Skolem-Noether). *Let R be a commutative ring, and let E and F be free R-modules of finite rank. Then every isomorphism* $\mathrm{End}(E) \to \mathrm{End}(F)$ *(as R-algebras) is induced by an isomorphism $E \to F$.*

Proof. Since E and F must have the same rank, we can choose an isomorphism $q : F \to E$. Compose the given isomorphism $\mathrm{End}(E) \to \mathrm{End}(F)$ with the isomorphism $\mathrm{End}(F) \to \mathrm{End}(E)$ induced by q, to get an automorphism of $\mathrm{End}(E)$. By the classical Skolem-Noether theorem ([30, IV.1.4]), we conclude that there is an automorphism $p : E \to E$ that induces this automorphism of $\mathrm{End}(E)$. $q^{-1} \circ p$ is the desired isomorphism $E \to F$. □

Lemma 1.3.2. *Let R and E be as in the previous lemma, and let p be an automorphism of E. Then p induces the identity on $\mathrm{End}(E)$ if and only if p is multiplication by an unit in R.*

Proof. Tracing through the definitions we see that the induced map on $\mathrm{End}(E)$ takes $h \in \mathrm{End}(E)$ to $p \circ h \circ p^{-1}$. Therefore p must be in the center of $\mathrm{End}(E)$, which is known to consist of multiplications by elements of R. Since p is also invertible, the result follows. □

Lemma 1.3.3. *Let R and E be as before, and let $A = \mathrm{End}_R(E)$. It is a non-commutative R-algebra. Then E is naturally a left A-module,*

$$E^\vee = \mathrm{Hom}_R(E, R)$$

is naturally a right A-module (note that $A = E \otimes_R E^\vee$), and the evaluation map $E^\vee \otimes_A E \to R$ is an isomorphism of R-modules.

Proof. The module structure on E is given by evaluation, and on $E^\vee$ by composition. It is obvious that the evaluation map

$$\sum f_i \otimes_A e_i \mapsto \sum f_i(e_i)$$

is well-defined (here $f_i \in E^\vee$ and $e_i \in E$).

Let $e_1, \ldots, e_n$ be a basis for E, and let $e_1^\vee, \ldots, e_n^\vee$ be the dual basis. Then we have

$$\begin{aligned} e_i^\vee \otimes_A e_j &= 0 \text{ for } i \neq j \\ e_i^\vee \otimes_A e_i &= e_j^\vee \otimes_A e_j \text{ for all } i, j. \end{aligned}$$

To see this, let a be $e_j \otimes_R e_j^\vee \in A$. We have $a \cdot e_j = e_j$ and $e_i^\vee \cdot a = 0$ for $i \neq j$. Therefore we have

$$e_i^\vee \otimes_A e_j = e_i^\vee \otimes_A a \cdot e_j = e_i^\vee \cdot a \otimes_A e_j = 0.$$

Also, let a be $e_j \otimes_R e_i^\vee + e_i \otimes_R e_j^\vee$; it has the property that $a \cdot e_j = e_i$ and $e_i^\vee \cdot a = e_j^\vee$. Therefore

$$e_i^\vee \otimes_A e_i = e_i^\vee \otimes_A (a \cdot e_j) = (e_i^\vee \cdot a) \otimes_A e_j = e_j^\vee \otimes_A e_j.$$

Every element of $E^\vee \otimes_A E$ can be written as $\sum r_{ij} e_i^\vee \otimes_A e_j$. By the formulas just proved, this equals $\sum r_{ii} e_1^\vee \otimes e_1$, which maps under the evaluation map to $\sum r_{ii}$. This map is clearly an R-module isomorphism. □

Lemma 1.3.4. *Again, let R be a commutative ring, E a free R-module, and $A = \mathrm{End}_R(E)$. Then for any R modules F and G we have*

$$\mathrm{Hom}_R(F, G) \cong \mathrm{Hom}_A(F \otimes_R E^\vee, G \otimes_R E^\vee),$$

where Hom_A denotes the group of right A-module homomorphisms.

Proof. The map in one direction is $\cdot \otimes_R \mathrm{id}_{E^\vee}$. In the other direction take $\cdot \otimes_A \mathrm{id}_E$, and use the previous lemma. □

Theorem 1.3.5. *Let $\mathscr{A}$ be an Azumaya algebra over X, and let $\alpha \in \mathrm{Br}'(X)$ be the element that $\mathscr{A}$ represents (we'll often denote α by $[\mathscr{A}]$). Then there exists a locally free α-twisted sheaf $\mathscr{E}$ of finite rank (not necessarily unique) such that $\mathscr{A} \cong \underline{\mathrm{End}}(\mathscr{E})$. Conversely, for any $\alpha \in \mathrm{Br}'(X)$ such that there exists a locally free α-sheaf $\mathscr{E}$ of finite rank, $\underline{\mathrm{End}}(\mathscr{E})$ is an Azumaya algebra whose class in $\mathrm{Br}'(X)$ is α.*

Thus we have yet another characterization of the Brauer subgroup of the cohomological Brauer group: it is the subgroup of those twistings α for which there exist locally free α-twisted sheaves of finite rank (α-lffr's, in short).

Proof. Find a cover $\{U_i\}$, $\mathscr{O}_{U_i}$-lffr's $\mathscr{E}_i$ and isomorphisms $\varphi_i : \underline{\mathrm{End}}(\mathscr{E}_i) \to \mathscr{A}|_{U_i}$ as given by Theorem 1.1.6. By Lemma 1.3.1, the isomorphisms

$$\varphi_j^{-1} \circ \varphi_i : \underline{\mathrm{End}}(\mathscr{E}_i|_{U_i \cap U_j}) \to \underline{\mathrm{End}}(\mathscr{E}_i|_{U_i \cap U_j})$$

induce isomorphisms

$$\bar{\varphi}_{ij} : \mathscr{E}_i|_{U_i \cap U_j} \to \mathscr{E}_j|_{U_i \cap U_j}.$$

The threefold compositions $\bar{\varphi}_{ijk}$ are automorphisms of $\mathscr{E}_i$ such that the corresponding automorphisms on $\underline{\mathrm{End}}(\mathscr{E}_i)$ are the identity, hence by Lemma 1.3.2 they must be multiplications by sections α_{ijk} of $\mathscr{O}_X^*$ over $U_i \cap U_j \cap U_k$. Therefore we have found the data for an α-lffr $\mathscr{E}$, where α is the element of $\check{H}^2(X, \mathscr{O}_X^*)$ determined by $\{\alpha_{ijk}\}$. It is easy to see that this is the same correspondence as described in Theorem 1.1.8.

The fact that $\underline{\mathrm{End}}(\mathscr{E})$ is an Azumaya algebra follows immediately from Theorem 1.1.6. □

Proposition 1.3.6. *Let $\mathscr{A}$ be an Azumaya algebra over X, let $\alpha = [\mathscr{A}]$, and let $\mathscr{E}$ be an α-lffr such that $\mathscr{A} \cong \underline{\mathrm{End}}(\mathscr{E})$. Note that $\mathscr{E}$ is naturally a left $\mathscr{A}$-module. Define a functor F between the category $\mathfrak{Mod}(X, \alpha)$ of α-twisted sheaves and the category $\mathfrak{Mod}$-$\mathscr{A}$ of sheaves of right $\mathscr{A}$-modules on X by the formula*

$$F(\,\cdot\,) = \,\cdot\, \otimes_{\mathscr{O}_X} \mathscr{E}^\vee$$

(the right $\mathscr{A}$-module structure on $F(\,\cdot\,)$ is given by using the right $\mathscr{A}$-module structure of $\mathscr{E}^\vee$).

Then, for any pair of α-sheaves $\mathscr{F}$ and $\mathscr{G}$, F induces a functorial isomorphism

$$\underline{\mathrm{Hom}}_{\mathfrak{Mod}(X,\alpha)}(\mathscr{F}, \mathscr{G}) \cong \underline{\mathrm{Hom}}_{\mathfrak{Mod}\text{-}\mathscr{A}}(F(\mathscr{F}), F(\mathscr{G})).$$

Proof. Follows from Lemma 1.3.4 and the structure theorem for Azumaya algebras (1.1.6). □

Theorem 1.3.7. *Let $\mathscr{A}$ be an Azumaya algebra over X. Then the functor F defined above is an equivalence of categories between $\mathfrak{Mod}(X,\alpha)$ and $\mathfrak{Mod}$-$\mathscr{A}$.*

Proof. Let $\mathscr{E}$ be as before, and define $G : \mathfrak{Mod}\text{-}\mathscr{A} \to \mathfrak{Mod}(X,\alpha)$ by the formula

$$G(\,\cdot\,) = \,\cdot\, \otimes_{\mathscr{A}} \mathscr{E}.$$

(The tensor product over $\mathscr{A}$ is defined locally, as usual.)

From the formulas

$$\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{E}^\vee \cong \underline{\mathrm{End}}(\mathscr{E}) \cong \mathscr{A}$$

and

$$\mathscr{E}^\vee \otimes_{\mathscr{A}} \mathscr{E} \cong \mathscr{O}_X$$

(see lemma 1.3.3) it follows that F and G are inverse to one another. Taking global sections in Proposition 1.3.6 and using Proposition 1.2.12 we see that F is full and faithful, so it is an equivalence of categories. □

Remark 1.3.8. It is not hard to see that in fact all the functors we have defined in Proposition 1.2.10 ($\otimes$, $\underline{\mathrm{Hom}}$, f^*, f_*, $f_!$) are compatible with this equivalence. So from now on we'll just freely switch between the viewpoint that twisted sheaves are "local sheaves that don't quite match up" and the viewpoint that twisted sheaves are "modules over an Azumaya algebra". As an application of this, we prove the following theorem, which shows that on a proper scheme over $\mathbf{C}$, twisted coherent sheaves (in the étale topology) are the same as analytic twisted coherent sheaves (in the Euclidean topology) on the associated analytic space.

Theorem 1.3.9. *Let X be a proper scheme over $\mathbf{C}$, let $h : X^h \to X$ be the natural continuous map from the associated analytic space, and let α be an element of $\mathrm{Br}(X)$, represented by an Azumaya algebra $\mathscr{A}$. Let $\mathscr{A}^h = h^*\mathscr{A}$, and let $\alpha^h = a(\mathscr{A}^h)$. Fix an α-lffr $\mathscr{E}$ such that $\underline{\mathrm{End}}(\mathscr{E}) \cong \mathscr{A}$ and an α^h-lffr $\mathscr{E}^h$ such that $\underline{\mathrm{End}}(\mathscr{E}^h) \cong \mathscr{A}^h$. Let F be the functor*

$$F : \mathfrak{Mod}(X,\alpha) \to \mathfrak{Mod}(X^h,\alpha^h)$$

defined by

$$F(\,\cdot\,) = h^*(\,\cdot\, \otimes_{\mathscr{O}_X} \mathscr{E}^\vee) \otimes_{\mathscr{A}^h} \mathscr{E}^h.$$

Then F is exact, its restriction

$$F|_{\mathfrak{Coh}} : \mathfrak{Coh}(X,\alpha) \to \mathfrak{Coh}(X^h,\alpha^h)$$

is an equivalence of categories, and we have, for any $\mathscr{F},\mathscr{G} \in \mathfrak{Coh}(X,\alpha)$

$$h^*\underline{\mathrm{Hom}}_{\mathfrak{Mod}(X,\alpha)}(\mathscr{F},\mathscr{G}) = \underline{\mathrm{Hom}}_{\mathfrak{Mod}(X^h,\alpha^h)}(F(\mathscr{F}), G(\mathscr{G})).$$

Proof. F is exact because h is flat ([20, Exposé XII, 1.1]).

Define

$$G : \mathfrak{Coh}(X^h, \alpha^h) \to \mathfrak{Coh}(X, \alpha)$$

by the formula

$$G(\,\cdot\,) = H(\,\cdot\, \otimes_{\mathscr{O}_{X^h}} (\mathscr{E}^h)^\vee) \otimes_{\mathscr{A}} \mathscr{E}$$

where $H : \mathfrak{Coh}(X^h) \to \mathfrak{Coh}(X)$ is an inverse to $h^* : \mathfrak{Coh}(X) \to \mathfrak{Coh}(X^h)$ (see [20, Exposé XII, 4.4]). Then a local computation shows that $F|_{\mathfrak{Coh}}$ and G are inverse to one another, thus proving that $F|_{\mathfrak{Coh}}$ is an equivalence of categories. The last formula follows from Proposition 1.3.6 applied twice. □

Morita Theory for Azumaya Algebras

We conclude with a few comments regarding Morita theory for Azumaya algebras or sheaves of Azumaya algebras. This is only included here to give a flavor of the subject; for more details the reader should consult a standard book on Morita theory, for example [27].

Morita theory deals with the study of pairs of rings (possibly non-commutative) A and B, such that the categories $\mathfrak{Mod}$-A and $\mathfrak{Mod}$-B of right modules over A and B are equivalent. We make this into a definition:

Definition 1.3.10. Two rings A and B (possibly non-commutative, with unit) are said to be *Morita equivalent*, written $A \sim_M B$, if and only if the categories $\mathfrak{Mod}$-A and $\mathfrak{Mod}$-B of right modules over A and B are equivalent.

A typical result exemplifying Morita theory is the following theorem:

Proposition 1.3.11. *Let R be a (possibly non-commutative) ring. Then, if F is a free R-module of finite rank, we have*

$$R \sim_M \operatorname{End}_R(F).$$

Proof. Let $F^\vee = \operatorname{Hom}_R(F, R)$, and consider the two functors

$$\begin{aligned} \mathfrak{Mod}\text{-}R &\to \mathfrak{Mod}\text{-}\operatorname{End}_R(F) \qquad && M \mapsto M \otimes_R F^\vee \\ \mathfrak{Mod}\text{-}\operatorname{End}_R(F) &\to \mathfrak{Mod}\text{-}R && N \mapsto N \otimes_{\operatorname{End}_R(F)} F. \end{aligned}$$

Note that $F^\vee$ is a right $\operatorname{End}_R(F)$-module in a natural way, and that F is a left $\operatorname{End}_R(F)$-module.

Now it is easy to see, using a proof entirely similar to that of Proposition 1.3.7, that these two functors are inverse to one another and fully-faithful, and thus they are equivalences of categories. □

In fact, this situation is typical of what happens in Morita theory, as shown by the following theorem.

Definition 1.3.12. Let A be a ring. A right A-module is said to be an A-*progenerator* if it satisfies the following two conditions:

1. F is finitely generated projective;

2. F is a generator, i.e. the functor $\mathrm{Hom}_R(F, \cdot) : \mathfrak{Mod}\text{-}A \to \mathfrak{Ab}$ is faithful.

Theorem 1.3.13 (Fundamental Theorem of Morita Theory). *Let A, B be rings. Then $A \sim_M B$ if and only if there exists an A-progenerator F such that $B \cong \mathrm{End}_A(F)$.*

In this case, the functors

$$\begin{aligned} \mathfrak{Mod}\text{-}A \to \mathfrak{Mod}\text{-}B \qquad & M \mapsto M \otimes_A F^\vee \\ \mathfrak{Mod}\text{-}B \to \mathfrak{Mod}\text{-}A \qquad & N \mapsto N \otimes_B F, \end{aligned}$$

are mutually inverse, where $F^\vee = \mathrm{Hom}_A(F, A)$, (note that $F^\vee$ is naturally a left A-module and F is naturally a right B-module)

Proof. [27, Chap. 18, especially 18.24]. □

Lemma 1.3.14. *Let A, B and C be R-algebras over a commutative ring R, with C being flat as an R-module. If $A \sim_M B$ then $A \otimes_R C \sim_M B \otimes_R C$.*

Proof. Write $B \cong \mathrm{End}_A(F)$ for an A-progenerator F. Then we have isomorphisms of R-algebras:

$$B \otimes_R C \cong \mathrm{End}_A(F) \otimes_R C \cong \mathrm{End}_{A\otimes_R C}(F \otimes_R C).$$

To prove the last isomorphism, use [14, 2.10], noting that $A \otimes_R C$ is indeed a flat A-module (by the assumption that C is a flat R-module), and F is A-finitely presented being a progenerator.

On the other hand, it is easy to see that $F \otimes_R C$ is a progenerator for $A \otimes_R C$, using the fact that an A-module F is a progenerator if and only if F is a direct summand of a finite direct sum of copies of A and A is a direct summand of a finite direct sum of copies of F ([27, 18.9]). □

Theorem 1.3.15. *Two Azumaya algebras A and B over a commutative ring R are Morita equivalent if and only if there exist finitely generated projective R-modules F and F' such that $A \otimes_R \mathrm{End}(F) \cong B \otimes_R \mathrm{End}(F')$. (We assume that A, B, F and F' have positive rank on each component of* $\mathrm{Spec}\, R$*, to avoid trivial counterexamples.)*

Proof. Using [27, 18.11 and Ex. 2.24], we conclude that any finitely generated projective R-module (with positive rank on each component of $\mathrm{Spec}\, R$) is an R-progenerator. Thus, one implication is easy using Lemma 1.3.14 and Theorem 1.3.13.

Now let's assume $A \sim_M B$. Then, by Lemma 1.3.14, we have $A \otimes_R A^\vee \sim_M B \otimes_R A^\vee$. Since $A \otimes_R A^\vee \cong \mathrm{End}_R(A)$ we conclude using Theorem 1.3.13 that $B \otimes_R A^\vee \sim_M R$. Now using again Theorem 1.3.13, we conclude that

$$B \otimes_R A^\vee \cong \mathrm{End}_R(F)$$

for some R-progenerator F. Tensoring this isomorphism over R with A, we get

$$B \otimes_R \operatorname{End}_R(A) \cong A \otimes_R \operatorname{End}_R(F),$$

and thus taking $F' = A$ we get the result. □

This shows that in the local situation (over an affine scheme) the Brauer group of a commutative ring R is precisely the group of isomorphism classes of Azumaya algebras under the tensor product operation, modulo Morita equivalence.

Unfortunately this does not generalize well to Azumaya algebras over a scheme, as the following example shows:

Example 1.3.16. Work over the ground field $\mathbf{C}$. Let X be a double cover of $\mathbf{P}^2$ branched over a smooth sextic curve. Then it is known that X is a smooth K3 surface (see Chapter 5 for more information on K3 surfaces and their Brauer groups), and in this case we have

$$\operatorname{Br}(X) = T_X^\vee \otimes \mathbf{Q}/\mathbf{Z}$$

(Lemma 5.4.1), where T_X is the transcendental lattice of X, and $T_X^\vee$ is the dual lattice to T_X. There is a natural involution ι on X which interchanges the two sheets. The $+1$-eigenspace of the induced action of ι on $H^2(X, \mathbf{Z})$ is precisely the algebraic part of $H^2(X, \mathbf{Z})$, and therefore ι acts by -1 on T_X, and from Lemma 5.4.1 it also acts by -1 on $\operatorname{Br}(X)$.

Let $\mathscr{A}$ be a sheaf of Azumaya algebras on X whose image $[\mathscr{A}]$ in $\operatorname{Br}(X)$ is non-zero and of order not equal to 2. Obviously ι induces an equivalence of categories

$$\iota^* : \mathfrak{Coh}(X, \mathscr{A}) \to \mathfrak{Coh}(X, \iota^*\mathscr{A}).$$

But we assumed that $[\iota^*\mathscr{A}] = -[\mathscr{A}] \neq [\mathscr{A}]$, so we have found Azumaya algebras $\mathscr{A}$ and $\iota^*\mathscr{A}$ that are Morita equivalent, but not equal in the Brauer group.

In view of this example it makes sense to state the following conjecture:

Conjecture 1.3.17. *On a projective scheme X we have $\mathscr{A} \sim_M \mathscr{B}$ for sheaves of Azumaya algebras $\mathscr{A}$ and $\mathscr{B}$ if and only if there exists an automorphism $\varphi : X \to X$ such that $[\mathscr{A}] = [\varphi^*\mathscr{B}]$ in* $\operatorname{Br}(X)$.

We'll be interested in studying a coarser equivalence relation on the group of isomorphism classes of Azumaya algebras, and that is the notion of *derived Morita equivalence* (see the definition below). In the course of this work we'll not go into the details of the theory of derived equivalences for non-commutative rings (and tilting modules, tilting complexes, etc.) which is a whole subject in itself (see [38] for an introduction to the problem and main results), but just point out a number of surprising results that are consequences of the geometric theory we study here.

Definition 1.3.18. Two R-algebras A and B (or sheaves of algebras on a scheme X, $\mathscr{A}$ and $\mathscr{B}$) are said to be *derived Morita equivalent* if and only if the bounded derived categories $\mathbf{D}^b(\mathfrak{Mod}\text{-}A)$ and $\mathbf{D}^b(\mathfrak{Mod}\text{-}B)$ (or $\mathbf{D}^b_{\text{coh}}(\mathfrak{Mod}(X, \mathscr{A}))$ and $\mathbf{D}^b_{\text{coh}}(\mathfrak{Mod}(X, \mathscr{B}))$ in the case of a scheme) are equivalent as triangulated categories.

For interesting examples of derived Morita equivalence of sheaves of Azumaya algebras, see Section 5.5 and Section 6.6. (These examples do not come from Morita equivalences of the algebras involved, at least if one believes Conjecture 1.3.17.) These examples should be contrasted with the following result for Azumaya algebras over local rings:

Theorem 1.3.19. *Let R be a commutative local ring, A and B Azumaya algebras over R. Then the following are equivalent:*

1. $[A] = [B]$ *in* $\mathrm{Br}(R)$.
2. *A is Morita equivalent to B.*
3. *A is derived Morita equivalent to B.*

Proof. [45]. □

Chapter 2

Derived Categories of Twisted Sheaves

In this chapter we study the derived category of twisted sheaves on a scheme or analytic space, derived functors, and relationships among them. The theorems and proofs here are quite technical, and could be skipped on a first reading, the general idea being that all the results that hold in the untwisted case also hold in the twisted case, with minor modifications.

2.1 Preliminary Results

We start with a few remarks regarding injective and flat resolutions, and finiteness properties of these resolutions. Throughout this section X will denote a noetherian, separated scheme or analytic space, and α, α', etc. are elements of $H^2(X, \mathscr{O}_X^*)$.

Lemma 2.1.1. *$\mathfrak{Mod}(X, \alpha)$ has enough injectives for any $\alpha \in H^2(X, \mathscr{O}_X^*)$.*

Proof. The proof is the same as the one in [22, III, 2.2], using the correct f_*. □

Lemma 2.1.2. *Any α-sheaf is the quotient of an $\mathscr{O}_X$-flat α-sheaf.*

Proof. Let $\mathscr{F}$ be an α-sheaf on X. If U is any open set of X small enough to have $\alpha|_U$ trivial, then $\mathscr{O}_{X,U}$ (defined to be $j_!(\mathscr{O}_X|_U)$, where $j : U \to X$ is the inclusion and $j_!$ is the one defined in Proposition 1.2.10) is an $\mathscr{O}_X$-flat α-sheaf, and any direct sum of such is flat. For every pair (U, f) where U is a small enough open set to have $\alpha|_U$ trivial, and f is a section of $\mathscr{F}$ over U, consider the map $\mathscr{O}_{X,U} \to \mathscr{F}$ that over U takes the constant section "one" to f (such a map can be found using the adjunction property of $j_!$ and j^* from Proposition 1.2.13), and take the direct sum of all these maps over all pairs (U, f). This is the desired surjection. □

Remark 2.1.3. The α-sheaf that surjects onto $\mathscr{F}$, as constructed in the proof, has the property that the stalk at each point of each underlying local sheaf is a free module over the local ring of the structure sheaf at that point. We'll call this property “free on stalks” for the purpose of this chapter. Note that these sheaves are *not* locally free, as they are not quasi-coherent!

Lemma 2.1.4. *If $\alpha \in \mathrm{Br}(X)$, and if on X every coherent sheaf is the quotient of a locally free sheaf of finite rank (lffr), then the same holds for coherent α-sheaves.*

Proof. Let $\mathscr{G}$ be an α^{-1}-lffr, and let $\mathscr{F}$ be any coherent α-sheaf; then $\mathscr{F} \otimes \mathscr{G}$ is a coherent sheaf on X, hence we can find a lffr $\mathscr{L}$ that surjects onto $\mathscr{F} \otimes \mathscr{G}$. Tensor this surjection with $\mathscr{G}^\vee$, to get a surjective map $\mathscr{L} \otimes \mathscr{G}^\vee \to \mathscr{F} \otimes \mathscr{G} \otimes \mathscr{G}^\vee$. But there is a surjective trace map $\mathscr{G} \otimes \mathscr{G}^\vee \to \mathscr{O}_X$ given by the evaluation, and therefore a surjective composite map

$$\mathscr{L} \otimes \mathscr{G}^\vee \to \mathscr{F} \otimes \mathscr{G} \otimes \mathscr{G}^\vee \to \mathscr{F}$$

which is what we wanted. □

Lemma 2.1.5. *For any injective α-sheaf $\mathscr{I}$ and for any open set $U \subseteq X$ we have $\mathscr{I}|_U$ is an injective α-sheaf on U. In particular, if $\mathscr{I} = (\{\mathscr{I}_i\}, \{\varphi_{ij}\})$ along an open cover $\mathfrak{U} = \{U_i\}$, then the sheaf $\mathscr{I}_i$ is injective on U_i for all i.*

Proof. For the first part, use a proof identical to [22, III, 6.1] (note that, in fact, that proof can be written entirely in functorial terms, using the adjointness properties of $f_!$ and f^*, and using only the fact that $f_!$ takes injective maps to injective maps). For the last statement use Lemma 1.2.3 and the fact that the injectivity of an object is a categorical property. □

Proposition 2.1.6. *Assume that X is smooth, of dimension n. Then for any coherent α-sheaf $\mathscr{F}$ and any coherent α'-sheaf $\mathscr{G}$ we have*

$$\underline{\mathrm{Ext}}^i_X(\mathscr{F}, \mathscr{G}) = 0 \text{ for any } i > n.$$

(We define $\underline{\mathrm{Ext}}^i(\mathscr{F}, \cdot)$ as the right derived functors of $\underline{\mathrm{Hom}}(\mathscr{F}, \cdot)$.)

Remark 2.1.7. Note that we are, unfortunately, unable to conclude from this proposition that any coherent α-sheaf has injective dimension at most n on X. Indeed, to prove this, we would need to prove that the property

$$\text{“}\underline{\mathrm{Ext}}^i_X(\mathscr{F}, \mathscr{I}) = 0 \text{ for any coherent } \mathscr{F} \text{ and any } i > 0\text{”}$$

for an α-sheaf $\mathscr{I}$ implies that $\mathscr{I}$ is injective. Of course, if $\mathscr{I}$ were coherent, this property would say that $\mathscr{I}$ is indeed injective in $\mathfrak{Coh}(X, \alpha)$. But we can not hope, in general, to find resolutions of coherent α-sheaves by coherent injective α-sheaves, so we are forced to go to the larger category of non-coherent α-sheaves. On a scheme, a possible alternative would be to use the results in [23, II.7], but we do not know of similar results for analytic spaces. Fortunately, we can get by with the slightly weaker form of Proposition 2.1.6.

Proof. First, note that for any open set $U \subseteq X$ we have $\underline{\mathrm{Hom}}_U(\mathscr{F}|_U, \mathscr{G}|_U) = \underline{\mathrm{Hom}}_X(\mathscr{F}, \mathscr{G})|_U$ (where by restriction we mean the usual pull-back via the inclusion $U \hookrightarrow X$). Also, for any injective α-sheaf $\mathscr{I}$ on X we have $\mathscr{I}|_U$ injective (Lemma 2.1.5). So we can use the proof of [22, III, 6.2] to conclude that we have

$$\underline{\mathrm{Ext}}^i_U(\mathscr{F}|_U, \mathscr{G}|_U) \cong \underline{\mathrm{Ext}}^i_X(\mathscr{F}, \mathscr{G})|_U.$$

This allows us to reduce the problem to the case when X is affine (or Stein) and the cover $\mathfrak{U} = \{U_i\}$ on which we are working contains X, as the open set U_0. Since $\mathscr{F}$ and $\mathscr{G}$ are coherent, we conclude that $\underline{\mathrm{Ext}}^i_X(\mathscr{F}_0, \mathscr{G}_0) = 0$ for $i > n$, where $\mathscr{F}_0$ and $\mathscr{G}_0$ are the sheaves on U_0 in the representation of $\mathscr{F}$ and $\mathscr{G}$. (Use the fact that for an affine scheme $X = \operatorname{Spec} A$, and for coherent sheaves $\mathscr{F} = \tilde{M}$ and $\mathscr{G} = \tilde{N}$, we have $\underline{\mathrm{Ext}}^i_X(\mathscr{F}, \mathscr{G}) = \mathrm{Ext}^i_A(M, N)\tilde{}$, and that a regular ring of dimension n has global dimension n.) Since all the sheaves that make up the twisted sheaf $\underline{\mathrm{Ext}}^i_X(\mathscr{F}, \mathscr{G})$ are isomorphic to restrictions of $\underline{\mathrm{Ext}}^i_X(\mathscr{F}_0, \mathscr{G}_0)$ to smaller open sets, we conclude that they are all zero, hence $\underline{\mathrm{Ext}}^i_X(\mathscr{F}, \mathscr{G}) = 0$. □

Proposition 2.1.8. *If X is smooth, of dimension n, then any α-sheaf $\mathscr{F}$ has a finite flat resolution of length at most n.*

Proof. Using Lemma 2.1.2, construct a flat resolution

$$\mathscr{G}_n \xrightarrow{\varphi} \mathscr{G}_{n-1} \to \cdots \to \mathscr{G}_0 \to \mathscr{F} \to 0$$

by α-sheaves that are free on stalks (see Remark 2.1.3). Over each open set in the cover $\mathfrak{U} = \{U_i\}$ that we are working on we get a resolution of $\mathscr{F}_i$ by sheaves

$$\mathscr{G}_{n,i} \to \mathscr{G}_{n-1,i} \to \cdots \to \mathscr{G}_{0,i} \to \mathscr{F}_i \to 0,$$

where each $\mathscr{G}_{k,i}$ is free on stalks. Considering the corresponding exact sequence on stalks, and using Lemma 19.2, Theorem 19.2 and Theorem 2.5 in [29], we conclude that the kernel of the map $\mathscr{G}_{n,i} \to \mathscr{G}_{n-1,i}$ is free on stalks. Therefore replacing the initial resolution by

$$0 \to \operatorname{Ker} \varphi \to \mathscr{G}_{n-1} \to \cdots \to \mathscr{G}_0 \to \mathscr{F} \to 0$$

we obtain a resolution of $\mathscr{F}$ all of whose terms are α-sheaves that are free on stalks. This is the desired $\mathscr{O}_X$-flat resolution. □

Proposition 2.1.9. *Let $f : Y \to X$ be a proper morphism of schemes or analytic spaces, whose fibers have dimension at most n. Let $\alpha \in \check{H}^2(X, \mathscr{O}_X^*)$, and let $\mathscr{F}$ be a coherent $f^*\alpha$-twisted sheaf on Y. Then $R^i f_* \mathscr{F}$ is a coherent α-twisted sheaf for all i, and is zero for $i > n$. (We define $R^i f_*$ in the usual way, as the right derived functors of the left exact functor $f_* : \mathfrak{Mod}(Y, f^*\alpha) \to \mathfrak{Mod}(X, \alpha)$.)*

Proof. Using Corollary 1.2.6 represent $\mathscr{F}$ as $(\{\mathscr{F}_i\}, \{\varphi_{ij}\})$ along an open cover $\{f^{-1}(U_i)\}$, for some cover $\{U_i\}$ of X. Using Lemma 2.1.5, it is easy to see that computing $R^i f_* \mathscr{F}$ in the category of twisted sheaves (using twisted injective resolutions) gives the same result as gluing together $R^i f_* \mathscr{F}_i$ along the isomorphisms $R^i f_* \varphi_{ij}$. Now the result follows from the corresponding results for untwisted sheaves on schemes or analytic spaces (see [19, 3.2.1] and [2]). □

2.2 The Derived Category and Derived Functors

Definition 2.2.1. The α-twisted derived category of coherent sheaves, denoted by $\mathbf{D}^b_{\text{coh}}(X,\alpha)$, is the bounded derived category of the abelian category $\mathfrak{Mod}(X,\alpha)$, with all cohomology α-sheaves being coherent. In a similar fashion we define $\mathbf{D}^+_{\text{coh}}(X,\alpha)$, $\mathbf{K}^b_{\text{coh}}(X,\alpha)$, $\mathbf{K}^+_{\text{coh}}(X,\alpha)$, etc.

Remark 2.2.2. For all the notations pertaining to the derived category, we use the notations set up in [23, I.4]. For derived functors, our reference is [23, I.5].

Remark 2.2.3. Let $X^{\cdot}$ be a complex such that $H^i(X^{\cdot}) = 0$ for all $i > n_0$ for some n_0, and let

$$Y^{\cdot} = \quad \cdots \to X^{n_0-1} \to \operatorname{Ker} d^{n_0} \to 0 \to \cdots .$$

Then it is easy to see that there is a natural injective map $Y^{\cdot} \to X^{\cdot}$ which is a quasi-isomorphism. Similarly, if $H^i(X^{\cdot}) = 0$ for all $i < n_0$, then $X^{\cdot}$ there is a natural surjective quasi-isomorphism $X^{\cdot} \to Y^{\cdot}$, where

$$Y^{\cdot} = \quad \cdots \to 0 \to \operatorname{Coker} d^{n_0-1} \to X^{n_0+1} \to \cdots .$$

Let $\mathbf{D}$ be the full subcategory of $\mathbf{D}_{\text{coh}}(X,\alpha)$ consisting of complexes whose cohomology is zero except inside a bounded range. There is a natural functor $\mathbf{D}^b_{\text{coh}}(X,\alpha) \to \mathbf{D}$, and an easy application of [23, I.3.3] and of the previous comments shows that this functor is an equivalence of categories. Therefore we'll use the name "bounded complex" for either a complex which is zero outside a bounded range, or for one whose cohomology is zero outside a bounded range, and the context will make clear which one we mean (if it matters).

Theorem 2.2.4. *If X is a scheme or analytic space, $\alpha \in \check{H}^2(X, \mathscr{O}_X^*)$, and if $f : Y \to X$ is a morphism from another scheme or analytic space, then the following derived functors are defined:*

$$\begin{aligned}
\mathbf{R}\operatorname{Hom}^{\cdot} &: \mathbf{D}_{\text{coh}}(X,\alpha)^{\circ} \times \mathbf{D}^+_{\text{coh}}(X,\alpha) \to \mathbf{D}_{\text{coh}}(\mathfrak{Ab}),\\
\mathbf{R}\underline{\operatorname{Hom}}^{\cdot} &: \mathbf{D}_{\text{coh}}(X,\alpha)^{\circ} \times \mathbf{D}^+_{\text{coh}}(X,\alpha') \to \mathbf{D}_{\text{coh}}(X,\alpha^{-1}\alpha'),\\
\overset{\mathbf{L}}{\otimes} &: \mathbf{D}^-_{\text{coh}}(X,\alpha) \times \mathbf{D}^-_{\text{coh}}(X,\alpha') \to \mathbf{D}^-_{\text{coh}}(X,\alpha\alpha'),\\
\mathbf{L}f^* &: \mathbf{D}^-_{\text{coh}}(X,\alpha) \to \mathbf{D}^-_{\text{coh}}(Y,f^*\alpha).
\end{aligned}$$

If f is also proper then

$$\mathbf{R}f_* : \mathbf{D}_{\text{coh}}(Y,f^*\alpha) \to \mathbf{D}_{\text{coh}}(X,\alpha)$$

is also defined.

Proof. All these functors are defined exactly as in [23, II.2-II.4], using Propositions 2.1.1 and 2.1.2 to ensure the existence of the respective derived functors. Note that we need properness of f for $\mathbf{R}f_*$ in order to insure that the cohomology sheaves are coherent. $\square$

Proposition 2.2.5 (The Hypercohomology Spectral Sequence). *Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories, and let $F : \mathcal{A} \to \mathcal{B}$ be an additive, left exact functor. Assume that $\mathcal{A}$ has enough injectives, so that the derived functor*

$$\mathbf{R}F : \mathbf{D}^+(\mathcal{A}) \to \mathbf{D}^+(\mathcal{B})$$

exists. In particular the functors

$$R^iF : \mathcal{A} \to \mathcal{B}$$

are defined. Let $X^{\cdot}$ be a complex in $\mathbf{D}^+(\mathcal{A})$. Then there exists a spectral sequence $E_k^{i,j}$, such that

$$E_2^{i,j} = R^iF(H^j(X^{\cdot})),$$

and such that $E_k^{i,j} \Rightarrow H^{i+j}(\mathbf{R}F(X^{\cdot}))$.

If the functor F has finite cohomological dimension on $\mathcal{A}$ (i.e. there exists a fixed n such that we have $R^iF(A) = 0$ for all $A \in \mathcal{A}$ and all $i > n$) then the above statement holds with $\mathbf{D}^+$ replaced by $\mathbf{D}$.

Proof. See [30, Appendix C]. □

Theorem 2.2.6. *Under the assumptions of Theorem 2.2.4, assume furthermore that X and Y are smooth of finite dimension, and that the morphism f is proper. Then the following derived functors are defined:*

$$\begin{aligned}
\mathbf{R}\underline{\mathrm{Hom}}^{\cdot} &: \mathbf{D}^b_{\mathrm{coh}}(X,\alpha)^{\circ} \times \mathbf{D}^b_{\mathrm{coh}}(X,\alpha') \to \mathbf{D}^b_{\mathrm{coh}}(X,\alpha^{-1}\alpha'),\\
\overset{\mathbf{L}}{\otimes} &: \mathbf{D}^b_{\mathrm{coh}}(X,\alpha) \times \mathbf{D}^b_{\mathrm{coh}}(X,\alpha') \to \mathbf{D}^b_{\mathrm{coh}}(X,\alpha\alpha'),\\
\mathbf{L}f^* &: \mathbf{D}^b_{\mathrm{coh}}(X,\alpha) \to \mathbf{D}^b_{\mathrm{coh}}(Y,f^*\alpha),\\
\mathbf{R}f_* &: \mathbf{D}^b_{\mathrm{coh}}(Y,f^*\alpha) \to \mathbf{D}^b_{\mathrm{coh}}(X,\alpha).
\end{aligned}$$

If X is any scheme, or is a compact complex analytic space, then

$$\mathbf{R}\,\mathrm{Hom}^{\cdot} : \mathbf{D}^b_{\mathrm{coh}}(X,\alpha)^{\circ} \times \mathbf{D}^b_{\mathrm{coh}}(X,\alpha) \to \mathbf{D}^b_{\mathrm{coh}}(\mathfrak{Ab})$$

is also defined.

Proof. The only thing we need to do is prove that all the functors defined previously take complexes with bounded cohomology to complexes with bounded cohomology.

For $\mathbf{R}f_*$, $\mathbf{L}f^*$ and $\overset{\mathbf{L}}{\otimes}$ the result follows immediately from Proposition 2.2.5. Indeed, using Proposition 2.1.8 one finds that the $E_2^{i,j}$ terms of the hypercohomology spectral sequence are contained in a bounded rectangle of the (i,j)-plane, and therefore, after converging, the cohomology of the total complex is bounded.

Now consider the case of $\mathbf{R}\underline{\mathrm{Hom}}^{\cdot}(F^{\cdot}, G^{\cdot})$ for two bounded complexes $F^{\cdot} \in \mathbf{D}^b_{\mathrm{coh}}(X,\alpha)$, $G^{\cdot} \in \mathbf{D}^b_{\mathrm{coh}}(X,\alpha')$. Using again Proposition 2.2.5 we reduce to the case when $G^{\cdot}$ consists of a single sheaf. We'll use a technique known as *dévissage* to reduce to the case when $F^\cdot$ also consists of a single sheaf, which is just Proposition 2.1.6.

The dévissage technique is just induction on the number $n(F^\cdot)$ defined as

$$n(F^\cdot) = \max\{j - i \mid H^j(F^\cdot) \neq 0,\ H^i(F^\cdot) \neq 0\}.$$

The basis of the induction is the case $n(F^\cdot) = 0$, when $F^\cdot$ is quasi-isomorphic to a single sheaf, so that Proposition 2.1.6 gives the boundedness of $\mathbf{R}\underline{\mathrm{Hom}}^\cdot(F^\cdot, G^\cdot)$.

Now assume proven that $\mathbf{R}\underline{\mathrm{Hom}}^\cdot(F^\cdot, G^\cdot)$ is bounded for $n(F^\cdot) \leq n_0$ (for some $n_0 \geq 0$). Phrased differently, this means that we have

$$\underline{\mathrm{Ext}}^i(F^\cdot, G^\cdot) = 0 \text{ for any } F^\cdot \text{ with } n(F^\cdot) \leq n_0 \text{ and for all } |i| \gg 0.$$

Assume that $F^\cdot$ has $n(F^\cdot) = n_0 + 1$. Translating $F^\cdot$, we can assume that $H^i(F^\cdot) = 0$ for $i < 0$, and $H^0(F^\cdot) \neq 0$. Using Remark 2.2.3 we can assume that in fact $F^i = 0$ for $i < 0$. Let $F'^\cdot$ be the complex that consists of only one sheaf equal to $H^0(F^\cdot)$, in degree 0, and consider the natural map of complexes $F'^\cdot \to F^\cdot$ which is an isomorphism on H^0, and zero on the other cohomologies. Fit this morphism into a triangle

$$\begin{array}{ccc} F'^\cdot & \longrightarrow & F^\cdot \\ & \nwarrow \quad \swarrow & \\ & F''^\cdot, & \end{array}$$

and write down the long exact $\underline{\mathrm{Ext}}(\,\cdot\,, G^\cdot)$-sequence for this triangle, obtained as in [23, I.6.1]. Note that we do not need locally free resolutions for that, the proof being entirely similar to that in [22, III.6.4]. Since we have $n(F''^\cdot) < n(F^\cdot)$ (by the long exact cohomology sequence for the above triangle), we can use the induction hypothesis that $\underline{\mathrm{Ext}}^i(F''^\cdot, G^\cdot) = 0$ for $|i| \gg 0$ as well as the fact that $\underline{\mathrm{Ext}}^i(F'^\cdot, G^\cdot) = 0$ for $|i| \gg 0$, to conclude that $\underline{\mathrm{Ext}}^i(F^\cdot, G^\cdot) = 0$ for $|i| \gg 0$, which is what we wanted.

The case of $\mathbf{R}\operatorname{Hom}^\cdot$ will follow from the fact that we have

$$\mathbf{R}\operatorname{Hom}^\cdot(F^\cdot, G^\cdot) = \mathbf{R}\Gamma(\mathbf{R}\underline{\mathrm{Hom}}^\cdot(F^\cdot, G^\cdot)),$$

(Proposition 2.3.2) and the fact that on a scheme or on a compact analytic space $\mathbf{R}\Gamma$ takes bounded complexes to bounded complexes. □

Remark 2.2.7. Note that we have to go through all this tortuous process just because we do not know if locally free resolutions exist. If they did, a criterion entirely similar to that in Proposition 2.1.8, combined with a hypercohomology spectral sequence, would allow us to solve the problem as we did for $\overset{\mathbf{L}}{\otimes}$.

As another application of the dévissage technique, we prove the following theorem, which will be used to prove an analogue of GAGA (2.2.10).

Theorem 2.2.8. *Let $\mathcal{A}'$ and $\mathcal{B}'$ be thick subcategories of abelian categories $\mathcal{A}$ and $\mathcal{B}$, respectively, and let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor that takes $\mathcal{A}'$ to $\mathcal{B}'$. Assume furthermore that the following properties are satisfied:*

1. *$\mathcal{A}$ and $\mathcal{B}$ have enough injectives;*

2. *F is an equivalence of categories when restricted to $\mathcal{A}' \to \mathcal{B}'$;*

3. *F induces a natural isomorphism*

$$\mathrm{Ext}^i_{\mathcal{A}}(X, Y) \cong \mathrm{Ext}^i_{\mathcal{B}}(F(X), F(Y))$$

for any $X, Y \in \mathcal{A}'$ and any i.

Then the natural functor $\tilde{F} : \mathbf{D}^b_{\mathcal{A}'}(\mathcal{A}) \to \mathbf{D}^b_{\mathcal{B}'}(\mathcal{B})$ induced by F is an equivalence of categories.

Proof. As a first step we want to prove that $\tilde{F}$ is full and faithful, i.e. that for any $X^\cdot, Y^\cdot \in \mathbf{D}^b_{\mathcal{A}'}(\mathcal{A})$, $\tilde{F}$ induces an isomorphism

$$\mathrm{Hom}_{\mathbf{D}^b_{\mathcal{A}'}(\mathcal{A})}(X^\cdot, Y^\cdot) \cong \mathrm{Hom}_{\mathbf{D}^b_{\mathcal{B}'}(\mathcal{B})}(\tilde{F}(X^\cdot), \tilde{F}(Y^\cdot)).$$

We shall prove this statement by induction on $n = n(X^\cdot) + n(Y^\cdot)$. If $n = -\infty$, then one of $X^\cdot$ or $Y^\cdot$ is the zero complex, so there is nothing to prove. If $n = 0$, then both $X^\cdot$ and $Y^\cdot$ consist of a single object of $\mathcal{A}'$, hence the statement follows from [23, I.6.4] and property 3 above. This is the basis for the induction process.

Assume that $\tilde{F}$ induces an isomorphism

$$\mathrm{Hom}_{\mathbf{D}^b_{\mathcal{A}'}(\mathcal{A})}(X^\cdot, Y^\cdot) \cong \mathrm{Hom}_{\mathbf{D}^b_{\mathcal{B}'}(\mathcal{B})}(\tilde{F}(X^\cdot), \tilde{F}(Y^\cdot))$$

for all $X^\cdot, Y^\cdot \in \mathbf{D}^b_{\mathcal{A}'}(\mathcal{A})$ with $n(X^\cdot) + n(Y^\cdot) < n$, and let $X^\cdot, Y^\cdot$ be objects of $\mathbf{D}^b_{\mathcal{A}'}(\mathcal{A})$ with $n(X^\cdot) + n(Y^\cdot) = n > 0$. Without loss of generality assume $n(Y^\cdot) > 0$, and that $H^i(Y^\cdot) = 0$ for $i < 0$, and $H^0(Y^\cdot) \neq 0$. Using Remark 2.2.3 we can assume that $Y^i = 0$ for $i < 0$, and $H^0(Y^\cdot) \neq 0$.

Just as before there is a map of complexes between $Y'^\cdot = H^0(Y^\cdot)$ (the complex whose single nonzero object is $H^0(Y^\cdot)$ in degree 0) and $Y^\cdot$, that induces an isomorphism in cohomology in all degrees ≤ 0. Let $Y''^\cdot$ be the third vertex of a triangle built on this morphism. Again, we have $n(Y''^\cdot) < n(Y^\cdot)$; also, from the assumption, $n(Y'^\cdot) = 0 < n(Y^\cdot)$. From the long exact sequence of Hom's ([23, I.1.1b]), the five-lemma and the induction hypothesis we conclude that

$$\mathrm{Hom}^i_{\mathbf{D}^b_{\mathcal{A}'}(\mathcal{A})}(X^\cdot, Y^\cdot) \cong \mathrm{Hom}^i_{\mathbf{D}^b_{\mathcal{B}'}(\mathcal{B})}(\tilde{F}(X^\cdot), \tilde{F}(Y^\cdot)),$$

which is what we needed to prove that $\tilde{F}$ is full and faithful. (The case when $n(Y^\cdot) = 0$ but $n(X^\cdot) > 0$ follows in a similar way.)

We are left with proving that any object $Y^\cdot$ of $\mathbf{D}^b_{\mathcal{B}'}(\mathcal{B})$ is isomorphic (in $\mathbf{D}^b_{\mathcal{B}'}(\mathcal{B})$) to an object of the form $\tilde{F}(X^\cdot)$ for some $X^\cdot \in \mathbf{D}^b_{\mathcal{A}'}(\mathcal{A})$. We prove this by induction on $n = n(Y^\cdot)$: the case $n = -\infty$ is trivial, and $n = 0$ follows from property 2.

So assume $n > 0$, and as before construct an exact triangle $Y''^{\cdot} \to Y'^{\cdot} \to Y^{\cdot} \to Y''^{\cdot}[1]$, where again $Y'^{\cdot} = H^0(Y^{\cdot}) \neq 0$ and we assume $Y^{\cdot}$ is zero in degrees < 0. Since F is an equivalence of categories between $\mathcal{A}'$ and $\mathcal{B}'$, we can find an $X'^{\cdot} \in \mathcal{A}'$ such that $F(X'^{\cdot}) \cong Y'^{\cdot}$. Also, by the induction hypothesis, we can find an $X''^{\cdot} \in \mathbf{D}^b_{\mathcal{A}'}(\mathcal{A})$ such that $\tilde{F}(X''^{\cdot}) \cong Y''^{\cdot}$. Since we proved that $\tilde{F}$ is full and faithful, we can find a map $X''^{\cdot} \to X'^{\cdot}$ whose image by $\tilde{F}$ is just the side of the exact triangle constructed before; take $X^{\cdot}$ to be the third vertex of a triangle constructed on this map. Then, since $\tilde{F}$ is a ∂-functor (because F is exact), we see that $\tilde{F}(X^{\cdot})$ is isomorphic to $Y^{\cdot}$, as required. □

Proposition 2.2.9. *Let X be a proper, locally factorial scheme over $\mathbf{C}$, and let X^h be the associated analytic space. For $\alpha \in \mathrm{Br}(X)$, let $\alpha^h \in \mathrm{Br}(X^h)$ be the associated twisting class (see Theorem 1.1.11). Let $F : \mathfrak{Mod}(X, \alpha) \to \mathfrak{Mod}(X^h, \alpha^h)$ be the functor defined in Theorem 1.3.9. Then for any $\mathscr{F}, \mathscr{G} \in \mathfrak{Coh}(X, \alpha)$, and for any i, F induces isomorphisms*

$$\mathrm{Ext}^i_{\mathfrak{Coh}(X,\alpha)}(\mathscr{F}, \mathscr{G}) \cong \mathrm{Ext}^i_{\mathfrak{Coh}(X^h,\alpha^h)}(F(\mathscr{F}), F(\mathscr{G})).$$

Proof. First, F takes α-lffr's to α^h-lffr's. Also, on X there are enough α-lffr's by [22, Ex. III.6.8] and Proposition 2.1.4. Since $\underline{\mathrm{Ext}}$'s can be computed by means of locally free resolutions and because h is flat, we conclude using Theorem 1.3.9 that there is a functorial isomorphism

$$h^* \underline{\mathrm{Ext}}^i_{\mathfrak{Coh}(X,\alpha)}(\mathscr{F}, \mathscr{G}) \cong \underline{\mathrm{Ext}}^i_{\mathfrak{Coh}(X^h,\alpha^h)}(F(\mathscr{F}), F(\mathscr{G})).$$

Thus we obtain a morphism of spectral sequences

$$\begin{array}{ccc} H^j(X, \underline{\mathrm{Ext}}^i_{\mathfrak{Coh}(X,\alpha)}(\mathscr{F}, \mathscr{G})) & \Longrightarrow & \mathrm{Ext}^{i+j}_{\mathfrak{Coh}(X,\alpha)}(\mathscr{F}, \mathscr{G}) \\ \downarrow & & \downarrow \\ H^j(X^h, \underline{\mathrm{Ext}}^i_{\mathfrak{Coh}(X^h,\alpha^h)}(F(\mathscr{F}), F(\mathscr{G}))) & \Longrightarrow & \mathrm{Ext}^{i+j}_{\mathfrak{Coh}(X^h,\alpha^h)}(F(\mathscr{F}), F(\mathscr{G})). \end{array}$$

which is already an isomorphism at the $E_2^{i,j}$ level, because $\underline{\mathrm{Ext}}^i_{\mathfrak{Coh}(X,\alpha)}(\mathscr{F}, \mathscr{G})$ and $\underline{\mathrm{Ext}}^i_{\mathfrak{Coh}(X^h,\alpha^h)}(F(\mathscr{F}), F(\mathscr{G}))$ are coherent (use [20, Exposé XII, 4.3]). This finishes the proof. □

Theorem 2.2.10. *Let X be a proper, locally factorial scheme over $\mathbf{C}$, let $\alpha \in \mathrm{Br}(X)$, and let α^h be the associated analytic twisting class. Then the functor F of Theorem 1.3.9 induces an equivalence of categories $\mathbf{D}^b_{\mathrm{coh}}(X, \alpha) \cong \mathbf{D}^b_{\mathrm{coh}}(X^h, \alpha^h)$. F also induces an isomorphism*

$$h^* \underline{\mathrm{Ext}}^i_{\mathbf{D}^b_{\mathrm{coh}}(X,\alpha)}(F^{\cdot}, G^{\cdot}) \cong \underline{\mathrm{Ext}}^i_{\mathbf{D}^b_{\mathrm{coh}}(X^h,\alpha^h)}(F(F^{\cdot}), F(G^{\cdot})),$$

for $F^{\cdot} \in \mathbf{D}^b_{\mathrm{coh}}(X, \alpha)$, $G^{\cdot} \in \mathbf{D}^b_{\mathrm{coh}}(X, \alpha')$.

Proof. All the conditions of Theorem 2.2.8 are satisfied, for $\mathcal{A} = \mathfrak{Mod}(X, \alpha)$, $\mathcal{A}' = \mathfrak{Coh}(X, \alpha)$, $\mathcal{B} = \mathfrak{Mod}(X^h, \alpha^h)$, $\mathcal{B}' = \mathfrak{Coh}(X^h, \alpha^h)$ and F. (Use Theorem 1.3.9 and Proposition 2.2.9.) For the last statement, use again dévissage; the basis of the induction is provided by the fact that

$$h^* \underline{\mathrm{Ext}}^i_{\mathfrak{Coh}(X,\alpha)}(\mathscr{F}, \mathscr{G}) \cong \underline{\mathrm{Ext}}^i_{\mathfrak{Coh}(X^h,\alpha^h)}(F(\mathscr{F}), F(\mathscr{G}))$$

for any $\mathscr{F} \in \mathfrak{Coh}(X, \alpha), \mathscr{G} \in \mathfrak{Coh}(X, \alpha')$. □

2.3 Relations Among Derived Functors

The proofs of the following statements are entirely similar to those of the corresponding ones in [23, II.5], so they will be omitted. Note that the results are given in their full generality, but we will only need them in a much more restricted situation: all the spaces involved are smooth and compact, all morphisms are proper and smooth, all twistings are in the Brauer group and all complexes are bounded. These results hold then with no extra restrictions.

Proposition 2.3.1. *Let $f : X \to Y$ and $g : Y \to Z$ be two proper morphisms of schemes or analytic spaces, and let $\alpha \in \check{H}^2(Z, \mathscr{O}_Z^*)$. Then there is a natural isomorphism*

$$\mathbf{R}(g_* \circ f_*) \xrightarrow{\sim} \mathbf{R}g_* \circ \mathbf{R}f_*$$

of functors from $\mathbf{D}_{\mathrm{coh}}(X, f^*g^*\alpha)$ to $\mathbf{D}_{\mathrm{coh}}(Z, \alpha)$.

Proof. To prove this, all we need to do is show that if $\mathscr{I}$ is an injective $f^*g^*\alpha$-sheaf, then $f_*\mathscr{I}$ is a g_*-acyclic $g^*\alpha$-sheaf. But this statement is local on Z, and therefore it follows from the corresponding one for sheaves by Lemma 1.2.6. □

Proposition 2.3.2. *Let X be a scheme or analytic space, and let $\alpha \in \check{H}^2(X, \mathscr{O}_X^*)$. Then there is a natural isomorphism*

$$\mathbf{R}\operatorname{Hom}^{\cdot}(F^{\cdot}, G^{\cdot}) \xrightarrow{\sim} \mathbf{R}\Gamma(X, \mathbf{R}\underline{\operatorname{Hom}}^{\cdot}(F^{\cdot}, G^{\cdot}))$$

of bi-functors from $\mathbf{D}^-_{\mathrm{coh}}(X, \alpha)^\circ \times \mathbf{D}^+_{\mathrm{coh}}(X, \alpha)$ to $\mathbf{D}(\mathfrak{Ab})$.

Proof. The proof follows as in [23, I.5.3], using the easy fact that if $\mathscr{I}$ is an injective α-sheaf, $\underline{\operatorname{Hom}}(\mathscr{F}, \mathscr{I})$ is flasque for any α-sheaf $\mathscr{F}$ (easy). □

Proposition 2.3.3. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes or analytic spaces, and let $\alpha \in \check{H}^2(Z, \mathscr{O}_Z^*)$. Then there is a natural isomorphism*

$$\mathbf{L}(f^* \circ g^*) \xrightarrow{\sim} \mathbf{L}f^* \circ \mathbf{L}g^*$$

of functors from $\mathbf{D}^-_{\mathrm{coh}}(Z, \alpha)$ to $\mathbf{D}^-_{\mathrm{coh}}(X, f^*g^*\alpha)$.

Proposition 2.3.4. *Let $f : X \to Y$ be a proper morphism of schemes or analytic spaces, and let $\alpha, \alpha' \in \check{H}^2(Y, \mathscr{O}_Y^*)$. Then there is a natural functorial homomorphism*

$$\mathbf{R}f_*\mathbf{R}\underline{\mathrm{Hom}}^{\cdot}_X(F^{\cdot}, G^{\cdot}) \longrightarrow \mathbf{R}\underline{\mathrm{Hom}}^{\cdot}_Y(\mathbf{R}f_*F^{\cdot}, \mathbf{R}f_*G^{\cdot})$$

for $F^{\cdot} \in \mathbf{D}^-_{\mathrm{coh}}(X, f^*\alpha)$, $G^{\cdot} \in \mathbf{D}^+_{\mathrm{coh}}(X, f^*\alpha')$.

Proposition 2.3.5 (Projection Formula). *Let $f : X \to Y$ be a proper morphism of schemes or analytic spaces, and let $\alpha, \alpha' \in \check{H}^2(Y, \mathscr{O}_Y^*)$. Then there is a natural functorial isomorphism*

$$\mathbf{R}f_*(F^{\cdot}) \overset{\mathbf{L}}{\otimes}_Y G^{\cdot} \xrightarrow{\sim} \mathbf{R}f_*(F^{\cdot} \overset{\mathbf{L}}{\otimes}_X \mathbf{L}f^*G^{\cdot})$$

for $F^{\cdot} \in \mathbf{D}^-_{\mathrm{coh}}(X, f^*\alpha)$ and $G^{\cdot} \in \mathbf{D}^-_{\mathrm{coh}}(Y, \alpha')$.

Proposition 2.3.6. *Let $f : X \to Y$ be a flat morphism of schemes or analytic spaces, and let $\alpha, \alpha' \in \check{H}^2(Y, \mathscr{O}_Y^*)$. Then there is a natural functorial isomorphism*

$$f^*\mathbf{R}\underline{\mathrm{Hom}}^{\cdot}_Y(F^{\cdot}, G^{\cdot}) \xrightarrow{\sim} \mathbf{R}\underline{\mathrm{Hom}}^{\cdot}_X(f^*F^{\cdot}, f^*G^{\cdot})$$

for $F^{\cdot} \in \mathbf{D}^-_{\mathrm{coh}}(Y, \alpha)$, $G^{\cdot} \in \mathbf{D}^+_{\mathrm{coh}}(Y, \alpha')$. (We write f^* instead of $\mathbf{L}f^*$ because it is an exact functor.)

Proposition 2.3.7. *Let $f : X \to Y$ be a morphism of schemes or analytic spaces, and let $\alpha, \alpha' \in \check{H}^2(Y, \mathscr{O}_Y^*)$. Then there is a natural functorial isomorphism*

$$\mathbf{L}f^*(F^{\cdot}) \overset{\mathbf{L}}{\otimes}_X \mathbf{L}f^*(G^{\cdot}) \xrightarrow{\sim} \mathbf{L}f^*(F^{\cdot} \overset{\mathbf{L}}{\otimes}_Y G^{\cdot})$$

for $F^{\cdot} \in \mathbf{D}^-_{\mathrm{coh}}(Y, \alpha)$, $G^{\cdot} \in \mathbf{D}^-_{\mathrm{coh}}(Y, \alpha')$.

Proposition 2.3.8. *Let $f : X \to Y$ be a proper morphism of schemes or analytic spaces, and let $\alpha, \alpha' \in \check{H}^2(Y, \mathscr{O}_Y^*)$. Then there is a natural functorial homomorphism*

$$F^{\cdot} \to \mathbf{R}f_*\mathbf{L}f^*F^{\cdot}$$

for $F^{\cdot} \in \mathbf{D}^-_{\mathrm{coh}}(Y, \alpha)$ which gives rise by proposition 2.3.4 to a natural functorial isomorphism

$$\mathbf{R}f_*\mathbf{R}\underline{\mathrm{Hom}}^{\cdot}_X(\mathbf{L}f^*F^{\cdot}, G^{\cdot}) \xrightarrow{\sim} \mathbf{R}\underline{\mathrm{Hom}}^{\cdot}_Y(F^{\cdot}, \mathbf{R}f_*G^{\cdot})$$

for $F^{\cdot} \in \mathbf{D}^-_{\mathrm{coh}}(Y, \alpha), G^{\cdot} \in \mathbf{D}^+_{\mathrm{coh}}(X, f^*\alpha')$.

Corollary 2.3.9 (Adjoint property of f_* and f^*). *If $f : X \to Y$ is a proper morphism of schemes or analytic spaces, and if $\alpha \in \check{H}^2(Y, \mathscr{O}_X^*)$ then we have*

$$\mathrm{Hom}_{\mathbf{D}_{\mathrm{coh}}(X, f^*\alpha)}(\mathbf{L}f^*F^{\cdot}, G^{\cdot}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}_{\mathrm{coh}}(Y,\alpha)}(F^{\cdot}, \mathbf{R}f_*G^{\cdot})$$

for $F^{\cdot} \in \mathbf{D}^-_{\mathrm{coh}}(Y, \alpha), G^{\cdot} \in \mathbf{D}^+_{\mathrm{coh}}(X, f^*\alpha')$. In other words $\mathbf{L}f^*$ and $\mathbf{R}f_*$ are adjoint functors from $\mathbf{D}^-_{\mathrm{coh}}(Y, \alpha)$ to $\mathbf{D}^-_{\mathrm{coh}}(X, f^*\alpha)$ and $\mathbf{D}^+_{\mathrm{coh}}(X, f^*\alpha)$ to $\mathbf{D}^+_{\mathrm{coh}}(Y, \alpha)$, respectively.

Proposition 2.3.10. *Let $f : X \to Y$ be a proper morphism of schemes or analytic spaces, and let $\alpha \in \check{H}^2(Y, \mathscr{O}_Y^*)$. Let $u : Y' \to Y$ be a flat morphism, let $X' = X \times_Y Y'$, and let v, g be the projections, as shown:*

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ {\scriptstyle g}\Big\downarrow & & \Big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow{u} & Y. \end{array}$$

Then there is a natural functorial isomorphism

$$u^* \mathbf{R} f_* F^{\cdot} \xrightarrow{\sim} \mathbf{R} g_* v^* F^{\cdot}$$

for $F^{\cdot} \in \mathbf{D}_{\mathrm{coh}}(X, f^*\alpha)$.

Proposition 2.3.11. *Let X be a scheme or analytic space, and let $\alpha, \alpha', \alpha'' \in \check{H}^2(X, \mathscr{O}_X^*)$. Then there are natural functorial isomorphisms*

$$F^{\cdot} \overset{\mathbf{L}}{\otimes} G^{\cdot} \xrightarrow{\sim} G^{\cdot} \overset{\mathbf{L}}{\otimes} F^{\cdot}$$

and

$$F^{\cdot} \overset{\mathbf{L}}{\otimes} (G^{\cdot} \overset{\mathbf{L}}{\otimes} H^{\cdot}) \xrightarrow{\sim} (F^{\cdot} \overset{\mathbf{L}}{\otimes} G^{\cdot}) \overset{\mathbf{L}}{\otimes} H^{\cdot}$$

for $F^{\cdot} \in \mathbf{D}^-_{\mathrm{coh}}(X, \alpha)$, $G^{\cdot} \in \mathbf{D}^-_{\mathrm{coh}}(X, \alpha')$, $H^{\cdot} \in \mathbf{D}^-_{\mathrm{coh}}(X, \alpha'')$.

Proposition 2.3.12. *Let X be a scheme or analytic space, and let $\alpha, \alpha', \alpha'' \in \check{H}^2(X, \mathscr{O}_X^*)$. Then there is a natural functorial isomorphism*

$$\mathbf{R}\underline{\mathrm{Hom}}^{\cdot}(F^{\cdot}, G^{\cdot}) \overset{\mathbf{L}}{\otimes} H^{\cdot} \xrightarrow{\sim} \mathbf{R}\underline{\mathrm{Hom}}^{\cdot}(F^{\cdot}, G^{\cdot} \overset{\mathbf{L}}{\otimes} H^{\cdot})$$

for $F^{\cdot} \in \mathbf{D}^-_{\mathrm{coh}}(X, \alpha), G^{\cdot} \in \mathbf{D}^+_{\mathrm{coh}}(X, \alpha')$ and $H^{\cdot} \in \mathbf{D}_{\mathrm{coh}}(X, \alpha'')_{\mathrm{fTd}}$.

Proposition 2.3.13. *Let X be a scheme or analytic space, let $\alpha, \alpha' \in \mathrm{Br}(X)$, $\alpha'' \in \check{H}^2(X, \mathscr{O}_X^*)$, and assume that every coherent sheaf on X is a quotient of a lffr. Then there is a natural functorial isomorphism*

$$\mathbf{R}\underline{\mathrm{Hom}}^{\cdot}(F^{\cdot}, \mathbf{R}\underline{\mathrm{Hom}}^{\cdot}(G^{\cdot}, H^{\cdot})) \xrightarrow{\sim} \mathbf{R}\underline{\mathrm{Hom}}^{\cdot}(F^{\cdot} \overset{\mathbf{L}}{\otimes} G^{\cdot}, H^{\cdot})$$

for $F^{\cdot} \in \mathbf{D}^-_{\mathrm{coh}}(X, \alpha), G^{\cdot} \in \mathbf{D}^-_{\mathrm{coh}}(X, \alpha')$ and $H^{\cdot} \in \mathbf{D}^+_{\mathrm{coh}}(X, \alpha'')$.

Proposition 2.3.14. *Let X be a scheme or analytic space, let $\alpha, \alpha', \alpha'' \in \check{H}^2(X, \mathscr{O}_X^*)$, and let $L^{\cdot}$ be a bounded complex of α-lffr's. Let $L^{\cdot\vee} = \underline{\mathrm{Hom}}^{\cdot}(L^{\cdot}, \mathscr{O}_X)$. Then there are natural functorial isomorphisms*

$$\mathbf{R}\underline{\mathrm{Hom}}^{\cdot}(F^{\cdot}, G^{\cdot}) \overset{\mathbf{L}}{\otimes} L^{\cdot} \xrightarrow{\sim} \mathbf{R}\underline{\mathrm{Hom}}^{\cdot}(F^{\cdot}, G^{\cdot} \overset{\mathbf{L}}{\otimes} L^{\cdot}) \xrightarrow{\sim} \mathbf{R}\underline{\mathrm{Hom}}^{\cdot}(F^{\cdot} \overset{\mathbf{L}}{\otimes} L^{\cdot\vee}, G^{\cdot})$$

for $F^{\cdot} \in \mathbf{D}^-_{\mathrm{coh}}(X, \alpha'), G^{\cdot} \in \mathbf{D}^+_{\mathrm{coh}}(X, \alpha'')$.

2.4 Duality for Proper Smooth Morphisms

Theorem 2.4.1. *Let $f : X \to Y$ be a proper smooth morphism of relative dimension n between smooth schemes or between smooth analytic spaces, and let $\alpha \in \mathrm{Br}(Y)$. Define $f^! : \mathbf{D}^b_{\mathrm{coh}}(Y, \alpha) \to \mathbf{D}^b_{\mathrm{coh}}(X, f^*\alpha)$ by*

$$f^!(\,\cdot\,) = \mathbf{L}f^*(\,\cdot\,) \otimes_X \omega_{X/Y}[n]$$

where $\omega_{X/Y} = \wedge^n \Omega_{X/Y}$ (here $\Omega_{X/Y}$ is the sheaf of relative differentials, which is locally free by [22, III.10.0.2]). Then for any $G^{\cdot} \in \mathbf{D}^b_{\mathrm{coh}}(Y, \alpha)$ there is a natural homomorphism

$$\mathbf{R}f_* f^! G^{\cdot} \to G^{\cdot},$$

which by Proposition 2.3.4 induces a natural homomorphism

$$\mathbf{R}f_*\mathbf{R}\underline{\mathrm{Hom}}^{\cdot}_X(F^{\cdot}, f^! G^{\cdot}) \longrightarrow \mathbf{R}\underline{\mathrm{Hom}}^{\cdot}_Y(\mathbf{R}f_* F^{\cdot}, G^{\cdot})$$

for every $F^{\cdot} \in \mathbf{D}^b_{\mathrm{coh}}(X, f^*\alpha)$, which is an isomorphism.

Proof. To define the trace morphism $\mathbf{R}f_* f^! G^{\cdot} \to G^{\cdot}$ use the projection formula

$$\mathbf{R}f_* f^! G^{\cdot} = \mathbf{R}f_*(\mathbf{L}f^* G^{\cdot} \otimes \omega_{X/Y}[n]) = G^{\cdot} \overset{\mathbf{L}}{\otimes} \mathbf{R}f_*\omega_{X/Y}[n]$$

along with the canonical trace map for untwisted sheaves

$$\mathbf{R}f_*\omega_{X/Y}[n] \to \mathscr{O}_Y$$

which comes from standard duality.

Once we have defined the homomorphism

$$\mathbf{R}f_*\mathbf{R}\underline{\mathrm{Hom}}^{\cdot}_X(F^{\cdot}, f^! G^{\cdot}) \longrightarrow \mathbf{R}\underline{\mathrm{Hom}}^{\cdot}_Y(\mathbf{R}f_* F^{\cdot}, G^{\cdot}),$$

to check that it is an isomorphism is a local statement, and therefore it follows from the standard duality theorem for smooth morphisms ([23, III.11.1] or [37]). □

Remark 2.4.2. We only prove this rather weak result here, because it is all that we need in our context. I believe though that the analogue of this theorem holds for arbitrary proper morphisms.

Corollary 2.4.3. *In the context of the previous theorem, $f^!$ is a right adjoint to $\mathbf{R}f_*$ as functors between $\mathbf{D}^b_{\mathrm{coh}}(X, f^*\alpha)$ and $\mathbf{D}^b_{\mathrm{coh}}(Y, \alpha)$.*

Proof. Apply $\mathbf{R}\Gamma$ to both sides of the isomorphism in the previous theorem, and use Proposition 2.3.2 and [23, II.5.2]. □

Chapter 3

Fourier-Mukai Transforms

This chapter collects various results regarding integral functors and Fourier-Mukai transforms, as well as general results on the existence of relative moduli spaces and quasi-universal sheaves.

The first section introduces integral functors between derived categories and their associated functors in cohomology, and examines the relationship between them. We draw our inspiration from [25, Chapter 6], although the case of odd dimensional cohomology classes is completely new.

In the second section we sketch a proof of the generalization (Theorem 3.2.1), to the case of twisted sheaves, of the criterion for when an integral functor is an equivalence. The original form of this criterion was obtained in a restricted form by Mukai, and in its general form by Bondal-Orlov and Bridgeland. We follow the general structure of Bridgeland's excellent paper [5] with modifications for twisted sheaves.

The chapter concludes with a section on the existence of relative moduli spaces and quasi-universal sheaves. Most of the statements here are known, and they are only included for future reference.

3.1 Integral Functors

A fundamentally important class of morphisms between derived categories is the class of *integral functors*, introduced by Mukai. In this section we define them for the case of twisted derived categories. When the twisting is trivial, we also define the associated maps on cohomology, which leads to the definition of the Mukai intersection pairing and to the study of the relationship between the transforms on derived categories and the ones on cohomology.

Definition 3.1.1. Let X and Y be proper and smooth schemes over $\mathbf{C}$ or compact complex manifolds, and let

$$\begin{array}{ccc} X \times Y & \xrightarrow{\pi_Y} & Y \\ \pi_X \downarrow & & \\ X & & \end{array}$$

be the two projections. Let $\alpha \in \check{H}^2(Y, \mathscr{O}_Y^*)$, let $\bar{\alpha} = \pi_Y^* \alpha$ and let $U^\cdot \in \mathbf{D}^b_{\mathrm{coh}}(X \times Y, \bar{\alpha}^{-1})$. Then a functor

$$\Phi^{U^\cdot}_{Y \to X} : \mathbf{D}^b_{\mathrm{coh}}(Y, \alpha) \to \mathbf{D}^b_{\mathrm{coh}}(X)$$

of the form

$$\Phi^{U^\cdot}_{Y \to X}(\,\cdot\,) = \mathbf{R}\pi_{X,*}(U^\cdot \overset{\mathbf{L}}{\otimes} \mathbf{L}\pi_Y^*(\,\cdot\,))$$

is called an *integral functor*. If it is an equivalence of categories, then it is called a *Fourier-Mukai transform*.

In section 3.2 we'll give a general criterion for when an integral functor is an equivalence (Theorem 3.2.1); however, for the rest of this section we'll only be concerned with the case when X and Y are complex projective manifolds, and $\alpha = 0$, so that the functor $\Phi^{U^\cdot}_{Y \to X}$ takes $\mathbf{D}^b_{\mathrm{coh}}(Y)$ to $\mathbf{D}^b_{\mathrm{coh}}(X)$. For the remainder of this section we assume this to be the case.

For the following definition it is necessary to observe that because we have $\mathrm{td}_0(X) = 1$ for any complex manifold X, we can express $\sqrt{\mathrm{td}(X)}$ and $(\sqrt{\mathrm{td}(X)})^{-1}$ by means of power series.

Definition 3.1.2. Let

$$H^*(X, \mathbf{C}) = \bigoplus_i H^i(X, \mathbf{C}),$$

and define

$$v : \mathbf{D}^b_{\mathrm{coh}}(X) \to H^*(X, \mathbf{C})$$

by the formula

$$v(E^\cdot) = \mathrm{ch}(E^\cdot).\sqrt{\mathrm{td}(X)}.$$

(The product here is the cup product in the cohomology of X. The exponential Chern character of a complex $E^\cdot$ is defined to be

$$\mathrm{ch}(E^\cdot) = \sum_i (-1)^i \,\mathrm{ch}(E^i).$$

It is easy to see that this definition descends to give a map

$$\mathrm{ch} : \mathbf{D}^b_{\mathrm{coh}}(X) \to H^*(X, \mathbf{C}).)$$

Let $\tau : H^*(X,\mathbf{C}) \to H^*(X,\mathbf{C})$ be the map given by

$$\tau(v_0, v_1, v_2, \ldots, v_{2n}) = (v_0, iv_1, -v_2, \ldots, i^{2n}v_{2n}),$$

where $i = \sqrt{-1} \in \mathbf{C}$ and $n = \dim_{\mathbf{C}} X$.

For any $v \in H^*(X,\mathbf{C})$ define $v^\vee$ by the formula

$$v^\vee = \tau(v).\sqrt{\operatorname{td}(X)}.\tau(\sqrt{\operatorname{td}(X)})^{-1}.$$

On $H^*(X,\mathbf{C})$ consider the bilinear pairing given by

$$(v,w) = \int_X v^\vee.w;$$

this pairing will be called the *generalized Mukai product* on $H^*(X,\mathbf{C})$.

Remark 3.1.3. Just for a reminder, we include here the first four terms in the usual expansion of the Todd class of a vector bundle $\mathscr{E}$ on a complex manifold X:

$$\begin{aligned}\operatorname{td}(\mathscr{E}) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 \\ - \frac{1}{720}(c_1^4 - 4c_1^2c_2 - 3c_2^2 - c_1c_3 + c_4) + \ldots,\end{aligned}$$

where we set $c_i = c_i(\mathscr{E})$. Whenever we refer to $\operatorname{td}(X)$, we actually refer to $\operatorname{td}(\mathscr{T}_X)$, the Todd class of the tangent bundle of X.

Remark 3.1.4. Note that if X satisfies the following condition:

(TD) $\operatorname{td}(X)$ consists only of classes of dimension divisible by 4

(which happens, for example, if $c_1(X) = 0$ and $\dim X \leq 4$, which are all the cases we're interested in), the definition of $v^\vee$ simplifies to the easier form $v^\vee = \tau(v)$, which agrees with that of [25, Chapter 6] along the even dimensional part of the cohomology. Also, note that although the classes involved can have complex coefficients, if v, w are rational or real, the product (v,w) is again rational or real, because X has even (real) dimension.

Remark 3.1.5. If $u \in H^k(Y,\mathbf{C})$, $v \in H^{n-k}(Y,\mathbf{C})$, the product (u,v) only differs from the usual one (given by cupping and integrating along Y) only by a factor of i^k, which depends only on k. We'll use this fact later, when studying Calabi-Yau threefolds.

Lemma 3.1.6. *Let X, Y be complex manifolds, both satisfying (TD), let $\pi_X : X \times Y \to X$ be the projection onto the first factor, and let $u \in H^*(X \times Y, \mathbf{C})$, $v, v' \in H^*(X,\mathbf{C})$. The following formulas hold:*

$$\begin{aligned}(v.v')^\vee &= v^\vee.v'^\vee,\\ \pi_X^*(v^\vee) &= (\pi_X^* v)^\vee,\\ \pi_{X,*}(u^\vee) &= (-1)^{\dim_{\mathbf{C}} Y}\pi_{X,*}(u)^\vee;\end{aligned}$$

if v consists only of even dimensional classes we have

$$v^{\vee\vee} = v;$$

and, if $E^{\cdot} \in \mathbf{D}^b_{\mathrm{coh}}(X)$, we have

$$v(E^{\cdot\vee}) = v(E^{\cdot})^{\vee},$$

where $E^{\cdot\vee} = \mathbf{R}\underline{\mathrm{Hom}}(E^{\cdot}, \mathscr{O}_X)$. (This last formula holds without requiring that X satisfy (TD).)

Proof. All the results follow easily once one notices that $\tau(v.v') = \tau(v).\tau(v')$, $\mathrm{ch}(E^{\cdot\vee}) = \tau(\mathrm{ch}(E^{\cdot}))$ and $\mathrm{td}(X \times Y) = \pi_X^* \,\mathrm{td}(X).\pi_Y^* \,\mathrm{td}(Y)$. □

The reason to use this product on the complex cohomology of X is the following lemma:

Lemma 3.1.7. *For $E^{\cdot}, F^{\cdot} \in \mathbf{D}^b_{\mathrm{coh}}(X)$ we have*

$$\chi(E^{\cdot}, F^{\cdot}) = (v(E^{\cdot}), v(F^{\cdot})),$$

where

$$\chi(E^{\cdot}, F^{\cdot}) = \sum_i (-1)^i \dim \mathbf{R}^i \,\mathrm{Hom}(E^{\cdot}, F^{\cdot})$$

is the Euler characteristic *of $(E^{\cdot}, F^{\cdot})$.*

Proof. When $E^{\cdot}$ is a locally free sheaf and $F^{\cdot}$ is any single sheaf, we have

$$\begin{aligned}
\chi(E^{\cdot}, F^{\cdot}) &= \chi(E^{\cdot\vee} \otimes F^{\cdot}) \\
&= \int_X \mathrm{ch}(E^{\cdot\vee}).\,\mathrm{ch}(F^{\cdot}).\,\mathrm{td}(X) \\
&= \int_X \mathrm{ch}(E^{\cdot\vee}).\sqrt{\mathrm{td}(X)}.\,\mathrm{ch}(F^{\cdot}).\sqrt{\mathrm{td}(X)} \\
&= \int_X v(E^{\cdot\vee}).v(F^{\cdot}) \\
&= \int_X v(E^{\cdot})^{\vee}.v(F^{\cdot}) \\
&= (v(E^{\cdot}), v(F^{\cdot})),
\end{aligned}$$

where the second use of χ is the usual Euler characteristic, and the second equality is just Grothendieck-Riemann-Roch. Now use the fact that every bounded complex with coherent cohomology is quasi-isomorphic to one consisting of locally free sheaves to reduce the general case to this one. □

Definition 3.1.8. Let X and Y be complex projective manifolds, and let $U^\cdot \in \mathbf{D}^b_{\mathrm{coh}}(X \times Y)$. Define the *cohomological integral transform* associated to $U^\cdot$, $\varphi^{U^\cdot}_{Y \to X}$, by

$$\varphi^{U^\cdot}_{Y \to X} : H^*(Y, \mathbf{C}) \to H^*(X, \mathbf{C}),$$
$$\varphi^{U^\cdot}_{Y \to X}(v) = \pi_{X,*}(v(U^\cdot).\pi_Y^*(v)).$$

Proposition 3.1.9. *The diagram*

$$\begin{array}{ccc} \mathbf{D}^b_{\mathrm{coh}}(Y) & \xrightarrow{\Phi^{U^\cdot}_{Y\to X}} & \mathbf{D}^b_{\mathrm{coh}}(X) \\ \downarrow v & & \downarrow v \\ H^*(Y,\mathbf{C}) & \xrightarrow{\varphi^{U^\cdot}_{Y\to X}} & H^*(X,\mathbf{C}) \end{array}$$

commutes.

Proof. We have

$$\begin{aligned} \varphi^{U^\cdot}_{Y \to X}(v(E^\cdot)) &= \pi_{X,*}(\pi_Y^*(v(E^\cdot)).v(U^\cdot)) \\ &= \pi_{X,*}(\pi_Y^*(\mathrm{ch}(E^\cdot).\sqrt{\mathrm{td}(Y)}).\,\mathrm{ch}(U^\cdot).\sqrt{\mathrm{td}(X \times Y)}) \\ &= \pi_{X,*}(\mathrm{ch}(\mathbf{L}\pi_Y^* E^\cdot).\,\mathrm{ch}(U^\cdot).\pi_Y^*\,\mathrm{td}(Y).\pi_X^*\sqrt{\mathrm{td}(X)}) \\ &= \pi_{X,*}(\mathrm{ch}(\mathbf{L}\pi_Y^* E^\cdot \overset{\mathbf{L}}{\otimes} U^\cdot).\,\mathrm{td}(\mathscr{T}_{\pi_X})).\sqrt{\mathrm{td}(X)} \\ &= \mathrm{ch}(\mathbf{R}\pi_{X,*}(\mathbf{L}\pi_Y^* E^\cdot \overset{\mathbf{L}}{\otimes} U^\cdot)).\sqrt{\mathrm{td}(X)} \\ &= \mathrm{ch}(\Phi^{U^\cdot}_{Y \to X}(E^\cdot)).\sqrt{\mathrm{td}(X)} \\ &= v(\Phi^{U^\cdot}_{Y \to X}(E^\cdot)), \end{aligned}$$

which is what we wanted. Here, $\mathscr{T}_{\pi_X}$ is the relative tangent bundle of $\pi_X : X \times Y \to X$, which is obviously isomorphic to $\pi_Y^* \mathscr{T}_Y$. ∎

Proposition 3.1.10. *Let X, Y and Z be complex projective manifolds, and let $E^\cdot \in \mathbf{D}^b_{\mathrm{coh}}(X \times Y)$, $F^\cdot \in \mathbf{D}^b_{\mathrm{coh}}(Y \times Z)$. Denote by p_{XY}, p_{YZ} and p_{XZ} the projections from $X \times Y \times Z$ to $X \times Y$, $Y \times Z$ and $X \times Z$, respectively, and define $E^\cdot \circ F^\cdot \in \mathbf{D}^b_{\mathrm{coh}}(X \times Z)$ by*

$$E^\cdot \circ F^\cdot = \mathbf{R}p_{XZ,*}(\mathbf{L}p_{XY}^* E^\cdot \overset{\mathbf{L}}{\otimes} \mathbf{L}p_{YZ}^* F^\cdot).$$

Similarly, for $e \in H^*(X \times Y, \mathbf{C})$ and $f \in H^*(Y \times Z, \mathbf{C})$ consider

$$e \circ f = p_{XZ,*}((p_{XY}^*\, e).(p_{YZ}^*\, f)).$$

Then we have

$$v(E^\cdot \circ F^\cdot) = v(E^\cdot) \circ v(F^\cdot).$$

The reason for the notation is that we have $\Phi^{E^\cdot}_{Y\to X} \circ \Phi^{F^\cdot}_{Z\to Y} = \Phi^{E^\cdot\circ F^\cdot}_{Z\to X}$, (see, for example, [4, 1.4]) and similarly for the transforms on the level of rational cohomology.

Proof. We have

$$\begin{aligned} v(E^\cdot \circ F^\cdot) &= \operatorname{ch}(E^\cdot \circ F^\cdot).\sqrt{\operatorname{td}(X\times Z)} \\ &= \operatorname{ch}(\mathbf{R}p_{XZ,*}(\mathbf{L}p^*_{XY}E^\cdot \overset{\mathbf{L}}{\otimes} \mathbf{L}p^*_{YZ}F^\cdot)).\sqrt{\operatorname{td}(X\times Z)} \\ &= p_{XZ,*}(\operatorname{ch}(\mathbf{L}p^*_{XY}E^\cdot \overset{\mathbf{L}}{\otimes} \mathbf{L}p^*_{YZ}F^\cdot).p^*_Y\operatorname{td}(Y)).\sqrt{\operatorname{td}(X\times Z)} \\ &= p_{XZ,*}(p^*_{XY}\operatorname{ch}(E^\cdot).p^*_{YZ}\operatorname{ch}(F^\cdot).p^*_Y\operatorname{td}(Y).p^*_X\sqrt{\operatorname{td}(X)}.p^*_Z\sqrt{\operatorname{td}(Z)}) \\ &= p_{XZ,*}(p^*_{XY}v(E^\cdot).p^*_{YZ}v(F^\cdot)) \\ &= v(E^\cdot)\circ v(F^\cdot) \end{aligned}$$

□

Proposition 3.1.11. *Let X and Y be complex projective manifolds, and assume that they satisfy condition (TD). For $U^\cdot \in \mathbf{D}^b_{\mathrm{coh}}(X\times Y)$, let*

$$U^{\cdot\vee} = \mathbf{R}\underline{\operatorname{Hom}}(U^\cdot, \mathscr{O}_{X\times Y}).$$

Then, for every $c \in H^*(X,\mathbf{C})$, $c' \in H^*(Y,\mathbf{C})$, we have

$$(c, \varphi^{U^\cdot}_{Y\to X}(c')) = (-1)^{\dim_{\mathbf{C}} X}(\varphi^{U^{\cdot\vee}}_{X\to Y}(c), c'),$$

where the pairings are the Mukai products on X and on Y, respectively.

Proof. Let $u = v(U^\cdot)$, and note that $u^\vee = v(U^{\cdot\vee})$. We have

$$\begin{aligned} (c, \varphi^{U^\cdot}_{Y\to X}(c')) &= \int_X c^\vee.\varphi^{U^\cdot}_{Y\to X}(c') \\ &= \int_X c^\vee.\pi_{X,*}(u.\pi^*_Y c') \\ &= \int_X \pi_{X,*}(\pi^*_X c^\vee.u.\pi^*_Y c') \\ &= \int_{X\times Y} \pi^*_X c^\vee.u.\pi^*_Y c' \\ &= \int_{X\times Y} ((\pi^*_X c).u^\vee)^\vee.\pi^*_Y c' \\ &= \int_Y \pi_{Y,*}(((\pi^*_X c).u^\vee)^\vee).c' \\ &= (-1)^{\dim_{\mathbf{C}} X}\int_Y \pi_{Y,*}((\pi^*_X c).u^\vee)^\vee.c' \\ &= (-1)^{\dim_{\mathbf{C}} X}\int_Y \varphi^{U^{\cdot\vee}}_{X\to Y}(c)^\vee.c' \\ &= (-1)^{\dim_{\mathbf{C}} X}(\varphi^{U^{\cdot\vee}}_{X\to Y}(c), c'). \end{aligned}$$

(We have used Lemma 3.1.6 at various stages of the computation.) □

Remark 3.1.12. This result is good enough for what we need, which will be the cases of K3 surfaces and Calabi-Yau three-folds. However, it looks like there should be a better definition of the Mukai vector for arbitrary complex projective manifolds, which would take into account the canonical class of the space, and which would remove the requirement that the spaces satisfy (TD).

Corollary 3.1.13. *Assume that X and Y satisfy (TD), and let $U^{\cdot} \in \mathbf{D}^b_{\mathrm{coh}}(X \times Y)$ be such that $\Phi^{U^{\cdot}}_{Y \to X}$ is a Fourier-Mukai transform, i.e. it is an equivalence $\mathbf{D}^b_{\mathrm{coh}}(Y) \to \mathbf{D}^b_{\mathrm{coh}}(X)$. Then $\varphi^{U^{\cdot}}_{Y \to X}$ is an isometry between $H^*(Y, \mathbf{C})$ and $H^*(X, \mathbf{C})$ (preserving the inner product defined in 3.1.2).*

Proof. First, note that condition (TD) immediately implies that X and Y have trivial canonical class. It is also well known that $\mathbf{D}^b_{\mathrm{coh}}(X) \cong \mathbf{D}^b_{\mathrm{coh}}(Y)$ implies that X and Y have the same dimension n. Let $U^{\cdot\vee} = \mathbf{R}\operatorname{Hom}(U^{\cdot}, \mathscr{O}_{X\times Y})$, so that the functor $\Phi^{U^{\cdot\vee}[n]}_{X \to Y}$ is a left and right adjoint functor to $\Phi^{U^{\cdot}}_{Y \to X}$, by [5, 4.5]. But since $\Phi^{U^{\cdot}}_{Y \to X}$ is an equivalence, $\Phi^{U^{\cdot\vee}[n]}_{X \to Y}$ must be an inverse to it, so that we must have $U^{\cdot} \circ U^{\cdot\vee}[n] = \mathscr{O}_{\Delta_X}$ and $U^{\cdot\vee}[n] \circ U^{\cdot} = \mathscr{O}_{\Delta_Y}$, where Δ_X and Δ_Y are the diagonals in $X \times X$ and $Y \times Y$, respectively. Therefore, using Proposition 3.1.10, we conclude that

$$\varphi^{U^{\cdot}}_{Y \to X} \circ \varphi^{U^{\cdot\vee}[n]}_{X \to Y} = \varphi^{\mathscr{O}_{\Delta_X}}_{X \to X},$$

and

$$\varphi^{U^{\cdot\vee}[n]}_{X \to Y} \circ \varphi^{U^{\cdot}}_{Y \to X} = \varphi^{\mathscr{O}_{\Delta_Y}}_{Y \to Y}.$$

But it is well known (see for example [25, Chapter 6]) that $\varphi^{\mathscr{O}_{\Delta_X}}_{X \to X}$ is the identity on $H^*(X, \mathbf{C})$, and similarly for Y, so that $\varphi^{U^{\cdot}}_{Y \to X}$ and $\varphi^{U^{\cdot\vee}[n]}_{X \to Y}$ are inverse to one another. Shifting by n in the derived category has the effect of introducing a $(-1)^n$ sign when taking the Chern character, so we conclude that $\varphi^{U^{\cdot}}_{Y \to X}$ and $\varphi^{U^{\cdot\vee}}_{X \to Y}$ are inverse to one another, up to a $(-1)^n$ sign. This fact, combined with Proposition 3.1.11 shows that $\varphi^{U^{\cdot}}_{Y \to X}$ is an isometry between $H^*(Y, \mathbf{C})$ and $H^*(X, \mathbf{C})$. □

Proposition 3.1.14. *Let $U^{\cdot} \in \mathbf{D}^b_{\mathrm{coh}}(X \times Y)$, with $\dim X = n, \dim Y = m$. If $c \in H^{p,q}(Y)$ then*

$$\varphi^{U^{\cdot}}_{Y \to X}(c) \in \bigoplus_{k=0}^{m+n} H^{p+k-m,q+k-m}(X).$$

In particular, if X and Y are K3 surfaces, $\varphi^{U^{\cdot}}_{Y \to X}$ maps $H^{2,0}(Y)$ to $H^{2,0}(X)$, and if X and Y are Calabi-Yau threefolds, $\varphi^{U^{\cdot}}_{Y \to X}$ maps $H^{p,q}(Y)$ to $H^{p,q}(X)$ for $p+q = 3$.

Remark 3.1.15. Throughout this entire work, our definition for a Calabi-Yau threefold is that it is a complex, projective manifold X of dimension 3, simply connected (equivalent, through Hodge theory, to $H^1(X, \mathscr{O}_X) = 0$), and with trivial canonical class ($K_X = 0$).

Proof. The first statement follows immediately from the fact that

$$v(U^{\cdot}) \in \bigoplus_{k=0}^{m+n} H^{k,k}(X \times Y),$$

and the other two are immediate (use the fact that a Calabi-Yau has $H^{1,0}$ and $H^{0,1}$ zero). □

We include here, for future reference, the following result due to Orlov ([36]):

Theorem 3.1.16. *Let X and Y be complex projective manifolds, and let $F : \mathbf{D}^b_{\mathrm{coh}}(Y) \to \mathbf{D}^b_{\mathrm{coh}}(X)$ be an equivalence of categories. Then F is a Fourier-Mukai transform, i.e. it is of the form $\Phi^{U^{\cdot}}_{Y\to X}$ for some object $U^{\cdot} \in \mathbf{D}^b_{\mathrm{coh}}(X \times Y)$. Furthermore, $U^{\cdot}$ is unique up to isomorphism.*

(In fact this theorem holds in the more general setting when F is full and faithful, and has a left adjoint, but we won't use this stronger form of the theorem.)

3.2 Equivalences of Twisted Derived Categories

In this section we give a criterion, very similar to one due to Mukai ([32]), Bondal-Orlov ([4]) and Bridgeland ([5]), for an integral functor between twisted derived categories to be an equivalence. We return to the situation of Definition 3.1.1, i.e. we no longer assume X and Y to be necessarily projective, nor do we assume α to be 0. The spaces in question are still assumed to be smooth and compact.

Theorem 3.2.1. *The functor $F = \Phi^{U^{\cdot}}_{Y\to X}$ is fully faithful if, and only if, for each point $y \in Y$,*

$$\mathrm{Hom}_{\mathbf{D}^b_{\mathrm{coh}}(X)}(F\mathscr{O}_y, F\mathscr{O}_y) = \mathbf{C},$$

and for each pair of points $y_1, y_2 \in Y$, and each integer i,

$$\mathrm{Ext}^i_{\mathbf{D}^b_{\mathrm{coh}}(X)}(F\mathscr{O}_{y_1}, F\mathscr{O}_{y_2}) = 0$$

unless $y_1 = y_2$ and $0 \leq i \leq \dim Y$. (Here $\mathscr{O}_y$ is the skyscraper sheaf $\mathbf{C}$ on y, which is naturally an α-sheaf by 1.2.10.)

Assuming the above conditions satisfied, then F is an equivalence of categories if, and only if, for every point $y \in Y$,

$$F\mathscr{O}_y \overset{\mathbf{L}}{\otimes} \omega_X \cong F\mathscr{O}_y.$$

Proof. The proof would be nothing more than a rewriting of Bridgeland's excellent paper ([5]), so we'll not do it here. Here are some fine points about the proof (the numbers refer to sections or theorems in [5]):

2.2. Note that since X is smooth, we have Serre duality by Theorem 2.4.3.

3. We don't need to prove anything here.

4.3. If Y and S are schemes or analytic spaces, and $\alpha \in \mathrm{Br}(Y \times S)$, then it makes sense to say that an α-sheaf $\mathscr{F}$ is flat over S: demand that the stalks of the local sheaves be flat modules over the corresponding stalks of $\mathscr{O}_S$.

4.4. We'll only use the statement in the same setting as in the original paper (no α-twisted version.)

4.5. Follows from the results in Sections 2.3 and 2.4.

5.1. The criterion for a complex to consist of just one sheaf supported at a point and the fact that a composite of two Fourier-Mukai transforms is again a Fourier-Mukai transform is proved like [4, 1.5] and [4, 1.4]. Note that $\mathscr{Q}$ is now a complex of $(p_1^*\alpha) \cdot (p_2^*\alpha^{-1})$-twisted sheaves on $Y \times Y$ (with p_1, p_2 the projections).

5.2. Local statement on Y.

5.3. By restricting ourselves to a small enough S, we can assume that $\mathscr{Q}$ is not twisted over S. Furthermore, since in the case where we apply this lemma we actually have an open immersion $i : S \to Y$, and for a point $s \in S$, $\mathscr{Q}_s$ is supported at the point $i(s)$ of Y, we see that $\mathscr{Q}$ can actually be taken to be non-twisted. It is important to note here that if an α-sheaf $\mathscr{F}$ is supported inside an open set U such that $\alpha|_U$ is trivial, then

$$\mathrm{Ext}^i_{\alpha\text{-}\mathfrak{Mod}}(\mathscr{F}, \mathscr{F}) = \mathrm{Ext}^i_{\mathfrak{Mod}}(\mathscr{F}, \mathscr{F}),$$

because an α-injective resolution of $\mathscr{F}$ can be restricted to U to give a (non-twisted) injective resolution of $\mathscr{F}$ (viewed as a non-twisted sheaf now).

5.4. There is nothing special about α-sheaves here.

□

3.3 Moduli Spaces and Universal Sheaves

We collect in this section various results regarding the existence and construction of absolute/relative moduli spaces, and of (possibly twisted) universal sheaves.

Theorem 3.3.1 (Existence of Relative Moduli Spaces). *Let $f : X \to S$ be a projective morphism of $\mathbf{C}$-schemes of finite type with connected fibers, and let $\mathscr{O}_X(1)$ be a line bundle on X very ample relative to S. Then for a given polynomial P there is a projective morphism $M_{X/S}(P) \to S$ which universally corepresents the functor*

$$\mathscr{M}_{X/S} : (\mathfrak{Sch}/S)^\circ \to \mathfrak{Ens},$$

which by definition associates to an S-scheme T of finite type the set of isomorphism classes of T-flat families of semistable sheaves on the fibers of the morphism $X_T := T \times_S X \to T$ with Hilbert polynomial P. In particular, for any closed point $s \in S$ one has $M_{X/S}(P)_s \cong M_{X_s}(P)$. Moreover there is an open subscheme $M^s_{X/S}(P) \subseteq M_{X/S}(P)$ that universally corepresents the subfunctor $\mathscr{M}^s_{X/S} \subseteq \mathscr{M}_{X/S}$ of families of stable sheaves.

Proof. See [25, 4.3.7]. □

Proposition 3.3.2. *Let X/S be a flat, projective morphism, and let $\mathscr{O}(1)$ be a relatively ample sheaf on X/S. For a polynomial P, consider the relative moduli space M^s/S of stable sheaves with Hilbert polynomial P on the fibers of X/S. Then there exists a covering $\{U_i\}$ of M^s (by analytic open sets in the analytic setting, and by étale open sets in the algebraic setting) such that on each $X \times_S U_i$ there exists a local universal sheaf $\mathscr{U}_i$. Furthermore, there exists an $\alpha \in \check{H}^2(M^s, \mathscr{O}^*_{M^s})$ (that only depends on X/S, $\mathscr{O}(1)$ and P) and isomorphisms $\varphi_{ij} : \mathscr{U}_j|_{U_i \cap U_j} \to \mathscr{U}_i|_{U_i \cap U_j}$ that make $(\{\mathscr{U}_i\}, \{\varphi_{ij}\})$ into an α-sheaf (which we will call a* universal α-sheaf*).*

Definition 3.3.3. The element $\alpha \in \check{H}^2(M^s, \mathscr{O}^*_{M^s})$ described above is called the *obstruction to the existence of a universal sheaf on $X \times_S M$*, and is denoted by $\mathrm{Obs}(X/S, P)$, with $\mathscr{O}(1)$ being understood.

Proof. For simplicity, we prove the statement in the absolute setting as the relative case is entirely similar; also, we work in the analytic category. We use the notations of [25, Section 4.6]. Note that $R^s \to M^s$ is a principal $\mathrm{PGL}(V)$-bundle ([25, 4.3.5]) and hence R^s is isomorphic, locally on M^s, to the product of M^s and $\mathrm{PGL}(V)$. If U is any open set in M^s over which R^s is trivial, say isomorphic to $\mathrm{PGL}(V) \times U \to U$, we can find over it $\mathrm{GL}(V)$-linearized line bundles of Z-weight 1: for example, $\mathrm{GL}(V) \times U \to \mathrm{PGL}(V) \times U$ is such a line bundle. Now apply a local version of [25, 4.6.2] to conclude that for every open set U of M^s such that $R^s \times_{M^s} U$ is trivial (as a $\mathrm{PGL}(V)$-bundle), there exists a local universal sheaf $\mathscr{U}$ on $X \times U$.

The existence of $\alpha \in \check{H}^2(M^s, \mathscr{O}^*_{M^s})$ and of the isomorphisms φ_{ij} that make $(\{\mathscr{U}_i\}, \{\varphi_{ij}\})$ into a universal α-sheaf now follows in exactly the same way as in the proof of [32, A.6], and the uniqueness of α is routine checking. □

Proposition 3.3.4 (Mukai, [32, A.6]). *Let X, M, S be proper schemes or analytic spaces over* **C***, with M integral, and assume given morphisms $X \to S$ and $M \to S$, such that $X \to S$ is projective. Let p_1 and p_2 be the projections to X and M from $X \times_S M$. For $\alpha \in \check{H}^2(M, \mathscr{O}^*_M)$ assume that there exists a coherent $p_2^*\alpha$-twisted sheaf $\mathscr{F}$ on $X \times_S M$ that is flat over M. Then α is in fact in* $\mathrm{Br}(M)$.

Proof. Using Lemma 1.2.6, represent $\mathscr{F}$ as $(\{\mathscr{F}_i\}, \{\varphi_{ij}\})$ on a cover $\{p_2^{-1}(U_i)\}$, where $\{U_i\}$ is an open cover of M. Pull back a relatively ample sheaf on X/S via p_1 to get a relatively ample sheaf $\mathscr{O}(1)$ on $X \times_S M \to M$, flat over M. Now using semicontinuity, for each i we can find n_0 such that for $n \geq n_0$ we have $p_{2*}(\mathscr{F}_i \otimes \mathscr{O}(n))$ locally free on U_i. Since M is proper over **C**, we can choose n_0

large enough to work for all U_i at once, so we conclude that $p_{2*}(\mathscr{F} \otimes \mathscr{O}(n))$ is an α-lffr, which implies that $\alpha \in \mathrm{Br}(M)$. □

The next theorem allows us to compare the absolute and the relative settings for generating equivalences, when dealing with moduli problems:

Proposition 3.3.5. *Let $f : X \to S$ be a morphism of schemes or analytic spaces, with S of the form* $\operatorname{Spec} R$ *for a regular local ring R. If s is the closed point of S, let $i : X_s \to X$ be the inclusion into X of the fiber X_s over s, and let $\mathscr{F}, \mathscr{G}$ be sheaves on X_s. Then*

1. *if* $\operatorname{Ext}^j_{X_s}(\mathscr{F}, \mathscr{G}) = 0$ *for all j then* $\operatorname{Ext}^j_X(i_*\mathscr{F}, i_*\mathscr{G}) = 0$ *for all j;*
2. *if* $\operatorname{Ext}^j_{X_s}(\mathscr{F}, \mathscr{G}) = 0$ *for all $j > n_0$ then* $\operatorname{Ext}^j_X(i_*\mathscr{F}, i_*\mathscr{G}) = 0$ *for all $j > n_0 + \dim S$.*

Proof. (An adaptation of the proof of [7, 3.1].) We have

$$\mathbf{R}\operatorname{Hom}^{\cdot}_X(i_*\mathscr{F}, i_*\mathscr{G}) = \mathbf{R}\operatorname{Hom}^{\cdot}_{X_s}(\mathbf{L}i^* i_*\mathscr{F}, \mathscr{G})$$

by the adjunction of $\mathbf{L}i^*$ and i_*. Furthermore,

$$\mathbf{L}i^* i_*\mathscr{F} = \mathscr{F} \overset{\mathbf{L}}{\otimes}_{X_s} \mathbf{L}i^* i_*\mathscr{O}_{X_s}$$

by the projection formula. Since S is smooth at s, writing down the Koszul resolution for $\mathscr{O}_s$ and pulling back via f we get a free resolution of $\mathscr{O}_{X_s}$ on X which can be used to compute $\mathbf{L}i^* i_*\mathscr{F}$. This gives

$$H^q(\mathbf{L}i^* i_*\mathscr{F}) = \mathscr{F} \otimes \bigwedge^q \mathscr{O}_{X_s}^{\oplus m}$$

where $m = \dim S$. Now the hypercohomology spectral sequence

$$\begin{aligned} E_2^{p,q} = \operatorname{Ext}^p_{X_s}(H^q(\mathbf{L}i^* i_*\mathscr{F}), \mathscr{G}) \quad &\Longrightarrow \quad H^{p+q}(\mathbf{R}\operatorname{Hom}^{\cdot}_{X_s}(\mathbf{L}i^* i_*\mathscr{F}, \mathscr{G})) \\ &= \quad H^{p+q}(\mathbf{R}\operatorname{Hom}^{\cdot}_X(i_*\mathscr{F}, i_*\mathscr{G})) \\ &= \quad \operatorname{Ext}^{p+q}_X(i_*\mathscr{F}, i_*\mathscr{G}) \end{aligned}$$

gives the results. □

Part II

Applications

Chapter 4

Smooth Elliptic Fibrations

In this chapter we consider the first non-trivial example of an occurrence of twisted sheaves, in the construction of the twisted Poincaré bundle for an elliptic fibration without a section. Here we only analyze the case of fibrations without singular fibers, and investigate the relationship of our approach to Ogg-Shafarevich theory. Most of the interesting phenomena will only occur when we'll take into account fibrations with singular fibers, which we'll do in Chapter 6, but this chapter should be viewed as a "birational analysis" of what comes later. Also, the results in this chapter serve to illustrate some of the concepts introduced in the previous chapters.

4.1 Elliptic Fibrations

Definition 4.1.1. A smooth, projective morphism of smooth schemes (or analytic spaces), $p_X : X \to S$, such that for all $s \in S$, X_s (the fiber of p_X over s) is a curve of genus one over $k(s)$, is called a *smooth elliptic fibration*.

Note that although X_s has lots of points defined over $k(s)$ when s is a closed point of S, X_s might not have any point defined over $k(s)$ when s is, for example, η, the generic point of S. This will be the main issue in the study of elliptic fibrations. In fact, most of the results that will be discussed in this chapter are concerned primarily with what happens around η, so that in many cases we'll be able to remove the loci in S where "bad" things happen, and still get our results.

Just in order to have a specific example in mind, we give here an example of a smooth elliptic fibration.

Example 4.1.2. Let X be a general bidegree $(3, 3)$ hypersurface in $\mathbf{P}^2 \times \mathbf{P}^2$, and consider the map $f : X \to \mathbf{P}^2$ given by projection on the first factor of the product $\mathbf{P}^2 \times \mathbf{P}^2$. Using Bertini's theorem X is smooth and the general fiber of f is smooth. If $\Delta \subseteq \mathbf{P}^2$ is the discriminant locus of this fibration (the locus over which the fiber is not smooth), and if $S = X \setminus \Delta$, then $X_S = X \times_{\mathbf{P}^2} S \to S$ is a smooth elliptic fibration.

In this example we can in fact see that f does not have a rational section. Indeed, the closure of a rational section over all of $\mathbf{P}^2$ would be a divisor D in

X which would intersect a general fiber of the map $X \to \mathbf{P}^2$ in one point. But Lefschetz's theorem tells us that the map $\mathrm{Pic}(\mathbf{P}^2 \times \mathbf{P}^2) \to \mathrm{Pic}(X)$ is surjective. Since all divisors in $\mathrm{Pic}(\mathbf{P}^2 \times \mathbf{P}^2)$ have intersection number with the general fiber of $X \to \mathbf{P}^2$ divisible by 3, we conclude that we cannot find a rational section in $X_S \to S$. On the other hand, the pull-back of the general hyperplane section of the second factor of $\mathbf{P}^2 \times \mathbf{P}^2$ provides an example of a 3-section (a flat, effective divisor that meets each fiber in 3 points).

Therefore we conclude that X_η has no rational points (a rational point would correspond to a rational section of $X_S \to S$), but that it has a rational divisor of degree 3 (corresponding to the 3-section).

4.2 The Relative Jacobian

Given a smooth elliptic fibration $p_X : X \to S$, there is a naturally constructed elliptic fibration $p_J : J \to S$ that has the same fibers as $X \to S$ over the closed points of S, but which has a section. This fibration, called the *relative Jacobian* of $X \to S$ is in a certain sense a universal object, and it will constitute the starting point of our study of smooth elliptic fibrations.

Definition 4.2.1. Fix a relatively ample sheaf $\mathscr{O}_X(1)$ for the morphism $p_X : X \to S$, and let $p_J : J \to S$ be the relative moduli space of semistable sheaves of rank 1, degree 0 on the fibers of p_X (see Theorem 3.3.1 for existence and properties). The fibration $J \to S$ (which is a smooth elliptic fibration) is called the *relative Jacobian* of $X \to S$.

Proposition 4.2.2. *The map p_J is a smooth elliptic fibration with a section. Over a closed point $s \in S$, we have $J_s \cong X_s$. If we set $\mathscr{L} = p_{J,*}\omega_{J/S}$, then any smooth elliptic fibration $p' : J' \to S$ possessing a section, having $J'_s \cong X_s$ for all closed points $s \in S$, and for which $p'_*\omega_{J'/S} \cong \mathscr{L}$, is isomorphic (over S) to $J \to S$.*

Proof. The map p_J is a flat, projective morphism by the very definition of relative moduli spaces. The flat family $\mathscr{O}_X$ of sheaves of rank 1, degree 0 on the fibers of p_X gives by the universal property of $J \to S$ a map $s_0 : S \to J$ which is a section of p_J. The fiber J_s of p_J over a closed point $s \in S$ is isomorphic to $\mathrm{Pic}^\circ(X_s)$, and since X_s is a smooth elliptic curve over an algebraically closed field, we conclude that $J_s \cong X_s$. The smoothness of J follows at once from local dimension estimates (see [25, Section 4.5]). Therefore p_J is a smooth elliptic fibration.

Now we can apply [13, 2.2 and 2.3] to $J' \to S$ to obtain the uniqueness property of p_J. Note that since all the fibers of p_X are connected, the same holds for p_J, and therefore the birational morphism g that is given to us by [loc.cit.] is in fact an isomorphism.

A final note: since locally (in the étale or analytic topologies) p_X has sections (because there are no multiple fibers), this proof actually shows that X is locally isomorphic to J, over S. We'll give a direct proof of this fact in the next proposition. □

Proposition 4.2.3. *Assume that p_X has a section $s : S \to X$. Then J and X are isomorphic as schemes or analytic spaces over S; an isomorphism $\varphi : X \to J$ can be chosen in such a way that the natural section s_0 of p_J (that corresponds to $\mathscr{O}_X$ on X) is mapped to s. The moduli problem is fine, under these circumstances, and a universal sheaf can be taken to be*

$$\mathscr{O}_{X\times_S J}(\Gamma) \otimes \pi_X^* \mathscr{O}_X(-s),$$

where Γ is the (scheme theoretic) graph of φ inside $X \times_S J$, $\pi_X : X \times_S J \to X$ is the projection, and $\mathscr{O}_X(-s)$ is the line bundle on X defined by the divisor $-s$, the image of the section $s : S \to X$.

Proof. This is nothing more than a relative version of the corresponding, well known statement for elliptic curves. This definition only works here for smooth elliptic fibrations (because Γ is a Cartier divisor). In the general case we'll have to replace $\mathscr{O}_{X\times_S J}(\Gamma)$ with $\mathscr{I}_\Gamma^\vee$, the dual of the ideal sheaf of Γ (which makes sense always). □

This identification allows us to give an explicit construction of the Jacobian of $X \to S$. For simplicity we'll work in the analytic category. Cover S with open sets U_i such that if we set $X_i = X \times_S U_i$, the projection $X_i \to U_i$ admits a section. For each i, fix once and for all a section $s_i : U_i \to X_i$. We identify s_i with its image in X_i, whenever there is no danger of confusion. It is a divisor in X_i, whose restriction to each fiber has degree 1.

We digress to talk about translations on elliptic curves. Let C be a smooth curve of genus 1 over an algebraically closed field (i.e., an elliptic curve, but without the choice of an origin). On C there exists a natural identification between line bundles of degree 0 and translations of C (under the group law on C that is obtained by fixing some origin), which is independent of the choice of an origin. This identification can be described as follows: let $\mathscr{L} \in \mathrm{Pic}^\circ(C)$; fixing an origin $s_0 \in C$, $\mathscr{L}$ can be written in a unique way as $\mathscr{L} \cong \mathscr{O}_C(s_0 - p)$ for some $p \in C$. The translation associated to $\mathscr{L}$, $\tau_{\mathscr{L}}$, is $x \mapsto x + p$ for $x \in C$, where the operation is the one given in the group law of C, with origin fixed at s_0. It is a trivial check to see that $\tau_{\mathscr{L}}$ is in fact independent of the choice of s_0.

Conversely, given a translation τ, we can associate to it an element $\mathscr{L}_\tau$ of $\mathrm{Pic}^\circ(C)$ by the formula

$$\mathscr{L}_\tau = \tau^*(\mathscr{F}) \otimes \mathscr{F}^{-1},$$

where $\mathscr{F}$ is some line bundle of degree 1 on C. This is independent of the choice of $\mathscr{F}$, and the operations $\mathscr{L} \mapsto \tau_{\mathscr{L}}$, $\tau \mapsto \mathscr{L}_\tau$ are inverse to one another. The important property of $\mathscr{L}_\tau$ is the following: if $\mathscr{F}$ is a line bundle on C (not necessarily of degree 0), then we have

$$\tau^*\mathscr{F} \cong \mathscr{F} \otimes \mathscr{L}_\tau^{\deg \mathscr{F}}.$$

A consequence of this formula is that if $\mathscr{F} \in \mathrm{Pic}^\circ(C)$, then $\tau^*\mathscr{F} \cong \mathscr{F}$, so if C' is isomorphic to C via a translation, there is a well defined notion of pull-back of $\mathscr{F}$

to C', independent of the choice of translation: all pull-backs will give the same line bundle (up to isomorphism).

We now return to the initial situation. Define $U_{ij} = U_i \cap U_j$, $X_{ij} = X_i \cap X_j$, and similarly U_{ijk}, X_{ijk}, etc. On each X_{ij} there are two sections, s_i and s_j, which define the line bundle

$$\mathscr{L}_{ij} = \mathscr{O}_{X_{ij}}(s_j - s_i),$$

whose degree along the fibers of $X_{ij} \to U_{ij}$ is zero. As such it defines a translation $\tau_{ij} : X_{ij} \to X_{ij}$ over U_{ij}. It is useful to note that $\tau_{ji} = \tau_{ij}^{-1}$, and $\tau_{ij} \circ \tau_{jk} \circ \tau_{ki} = \mathrm{id}$ on X_{ijk}.

We think of these translations as gluing functions for a new space J, obtained from the "slices" X_i, glued along

$$\tau_{ij} : X_j|_{X_i \cap X_j} \to X_i|_{X_i \cap X_j}.$$

Since the compatibility relations (checked above) hold, we can actually glue together this space, and it obviously comes with a natural map $p_J : J \to S$ which makes it into a smooth elliptic fibration. This construction naturally provides us with isomorphisms $\rho_i : J_i \to X_i$ over U_i, that satisfy

$$\rho_i \circ \rho_j^{-1} = \tau_{ij}$$

over X_{ij}. The images of the sections $s_i \subseteq X_i$ and $s_j \subseteq X_j$, under ρ_i^{-1} and ρ_j^{-1}, coincide in J_{ij}: indeed, this is because we have

$$\rho_i \circ \rho_j^{-1}(s_j) = \tau_{ij}(s_j) = s_j - (s_j - s_i) = s_i.$$

(We have written here $-(s_j - s_i)$ for $\tau_{ij} = \tau_{\mathscr{O}(s_j - s_i)}$, to make things more evident.) Therefore the images of all the local sections under the local isomorphisms glue together in J to give a global section t_0 of J.

Now we can see how a point $p \in J$ that lies over a point $s \in S$ corresponds to a line bundle on X_s: let U_i be an open set such that $s \in U_i$, and associate to p the line bundle $\rho_i^* \mathscr{O}_{J_s}(p - t_0(s))$. Since this line bundle has degree 0, the resulting line bundle on X_s does not change if we make another choice for the open set in which s lies, say U_j. In that case, we would obtain $\rho_j^* \mathscr{O}_{J_s}(p - t_0(s))$, and the two line bundles differ by the pull-back by a translation of X_s. Since we are working with line bundles of degree 0, the isomorphism type of the result is not affected by this operation.

4.3 The Twisted Poincaré Bundle

We consider the question of the existence of a universal sheaf on the product $X \times_S J$, since in this setting we can explicitly construct (as a gerbe) the twisting α of Definition 3.3.3.

We continue with the notations and assumptions from the previous section. Let $P_i = X_i \times_S J_i$, and similarly P_{ij}, etc. Note that over each P_i we have a universal sheaf given by

$$\mathscr{U}_i = \mathscr{O}_{X \times_S J}(\Gamma_i) \otimes \pi_X^* \mathscr{O}_{X_i}(-s_i),$$

as in Proposition 4.2.3. (We abuse the notation and keep using π_X for the projection $X_i \to U_i$.)

The natural question to ask at this point is why don't these local universal sheaves glue to a global one? We need to look over an intersection P_{ij}, and see what is the failure of $\mathscr{U}_i|_{P_{ij}}$ and $\mathscr{U}_j|_{P_{ij}}$ to be isomorphic.

Let

$$\mathscr{M}_{ij} = \mathscr{U}_j|_{P_{ij}} \otimes \mathscr{U}_i^{-1}|_{P_{ij}},$$

an operation which we can do in this case since we are dealing with line bundles; we have

$$\mathscr{M}_{ij} = \pi_X^* \mathscr{O}_{X_i}(s_i - s_j) \otimes \mathscr{O}(\Gamma_j) \otimes \mathscr{O}(\Gamma_i)^{-1}.$$

Let p be any point in J_{ij}, let s be its image in S, and consider the restriction of $\mathscr{M}_{ij}$ to X_s. It is isomorphic to

$$\mathscr{O}_{X_s}(s_i - s_j + \rho_j(p) - \rho_i(p)),$$

which is trivial along X_s (we could have seen this also from the fact that $\mathscr{U}_i$ and $\mathscr{U}_j$ are local universal sheaves, so over $p \in J_{ij}$ they must give the same line bundle on X_s, the one represented by p as a point of the moduli space J).

We conclude that $\mathscr{M}_{ij}$ is the pull-back of a line bundle $\mathscr{F}_{ij}$ from J_{ij}. The collection $\{\mathscr{F}_{ij}\}$, viewed as a gerbe $\alpha \in H^2(J, \mathscr{O}_J^*)$ will represent the twisting that is the obstruction to the existence of a universal sheaf on $X \times_S J$.

It is worthwhile to note that the twisting $\pi_J^*\alpha$ is in fact *trivial* on $X \times_S J$: indeed, the collection $\{\mathscr{U}_{ij}\}$ is a $\pi_J^*\alpha$-twisted line bundle on $X \times_S J$, so α must have order divisible by 1 (the rank of this vector bundle). In other words α must be trivial. The point here is that although we could modify this line bundle to make it glue well on $X \times_S J$, this operation would not preserve the fibers of this bundle on fibers of π_J, so the result would no longer be a universal sheaf.

It is not hard to see that the isomorphism type of $\mathscr{F}_{ij}$ (at least along the fibers of $J_{ij} \to S$) can be described as follows: if on X_{ij} we consider the sheaf

$$\mathscr{L}_{ij}^{-1} = \mathscr{O}_{X_{ij}}(s_i - s_j),$$

then $\mathscr{F}_{ij}$ is isomorphic to the sheaf $\rho_i^* \mathscr{L}_{ij}^{-1}$. The remarks in the previous section regarding the invariance of line bundles of degree zero under translations show that this gives a well-defined isomorphism type for $\mathscr{F}_{ij}$. An easy verification shows that the collection $\{\rho_i^* \mathscr{L}_{ij}^{-1}\}$ satisfies the conditions to form a gerbe.

However, this construction is not entirely natural: we have made the choice of using ρ_i as an isomorphism between J_{ij} and X_{ij}. In fact, the verification that we can do to ensure that $\{\mathscr{F}_{ij}\}$ is indeed a gerbe only tests this along the fibers of

$J_{ij} \to S$. This means that if $\mathscr{F}_{ij}$ were twisted by line bundles from U_{ij}, we would not be able to detect this. To make this construction independent of choices, one could consider the sheaves

$$\mathscr{F}_{ij} = \pi_{J,*}\mathscr{M}_{ij}.$$

These only differ from the $\mathscr{F}_{ij}$'s constructed before by the global line bundle

$$\pi_{J,*}\mathscr{O}_{X\times_S J},$$

and therefore the gerbe determined by $\{\mathscr{F}_{ij}\}$ is the one we want. Under this definition of $\mathscr{F}_{ij}$ it is obvious that the compatibility relations are satisfied.

This situation should be contrasted to the one in the next section, where Ogg-Shafarevich theory produces a natural element of $\mathrm{Br}(J)/\mathrm{Br}(S)$. This should be thought of as meaning that by just looking at the picture along the fibers (which is what we did in this section) we can only recover α up to possible twists from $\mathrm{Br}(S)$.

4.4 Ogg-Shafarevich Theory

As alluded to at the end of the previous section, this picture has a strong relationship to the classification theory for elliptic fibrations, as developed by Ogg and Shafarevich. For a reference to the technical aspects of this theory, see [13].

The setup is as before: $X \to S$ is a smooth elliptic fibration, and $J \to S$ is its relative Jacobian fibration. Since $X \to S$ is obtained from $J \to S$ by the "cut and reglue" procedure described before, one can try to understand this process from a cohomological point of view. One approach is to note that the regluing data consists of translations along the fibers of each J_i; the group of these translations can be identified with the group $J^\#(U_i)$ of sections of $J_i \to U_i$ (which can be naturally given the structure of a group because we have a fixed, natural section t_0 of $J \to S$). Therefore the data needed to reconstruct the fibration $X \to S$ out of $J \to S$ is an element α_X of the Tate-Shafarevich group of J, $Ш_S(J)$, defined as

$$Ш_S(J) = H^1(S, J^\#),$$

where $J^\#$ is the sheaf of abelian groups on S defined by

$$J^\#(U) = \text{the group of sections of } J_U \to U.$$

This makes sense in any topology, and if one considers everything in the analytic or étale topology, one obtains a 1-1 correspondence between elliptic fibrations $X \to S$ whose Jacobian is $J \to S$, and elements of $Ш_S(J)$. (It is important to note here that some fibrations that correspond to elements of $H^1_{\mathrm{an}}(S, J^\#)$ can only be constructed in the analytic category and have no algebraic counterpart; to remain inside the algebraic category one has to work with in the étale setup, and thus obtain fewer elliptic fibrations.)

The next step is to understand this picture further, and using the Leray spectral sequence one shows that if the fibrations in question have no singular fibers there is an exact sequence

$$0 \to \mathrm{Br}(S) \longrightarrow \mathrm{Br}(J) \xrightarrow{\pi} Ш_S(J) \to 0$$

which splits. (The first map is just the pull-back map on the Brauer groups.) The splitting is provided by, say, the pull-back from J to S via a section of $J \to S$.

Let's see how this relates to the situation described in the previous section: the construction of the gerbe α provides an element of $\mathrm{Br}(J)$, and projecting we obtain an element $\pi(\alpha)$ of $Ш_S(J)$. In fact, the line bundles $\rho_i^* \mathscr{O}_{X_{ij}}(s_i - s_j)$ can be written in a unique way as $\mathscr{O}_{J_{ij}}(t_{ij} - t_0)$ for some $t_{ij} \in J^{\#}(U_{ij})$, and it is clear from the construction that in order to reconstruct $X \to S$ one needs to reglue the J_i's via the translations

$$\tau_{ij}^{-1} = \tau_{\mathscr{O}_{J_{ij}}(t_{ij} - t_0)}$$

which are the inverses of the translations τ_{ij} used before to construct J from X.

But the translations $\{\tau_{ij}^{-1}\}$ give the element α_X of $Ш_S(J)$ that represents X (see the construction described above), and tracing through all the identifications it is easy (!) to see that α_X is precisely the projection $\pi(\alpha)$ of α.

Therefore we obtain:

Theorem 4.4.1. *Let $X \to S$ be a smooth elliptic fibration, and let $J \to S$ be its relative Jacobian. Let $\alpha_X \in Ш_S(J)$ be the element of the Tate-Shafarevich group that represents $X \to S$, and let $\alpha \in \mathrm{Br}(J)$ be the obstruction to the existence of a universal sheaf on $X \times_S J$ (for the moduli problem on X that gives J). Then $\alpha_X = \pi(\alpha)$, where π is the projection*

$$0 \to \mathrm{Br}(S) \longrightarrow \mathrm{Br}(J) \xrightarrow{\pi} Ш_S(J) \to 0.$$

Theorem 4.4.2. *Let n denote the smallest positive fiber degree of a divisor on X. (The fiber degree of a divisor D is defined as the degree of the restriction $D|_F$ of D to a fiber of $X \to S$.) Then the projection $\alpha_X = \pi(\alpha)$ is n-torsion in $Ш_S(J)$.*

Proof. Since we work in the Tate-Shafarevich group we can ignore contributions from the Brauer group of the base, and we only need to prove that there exist line bundles $\mathscr{L}_i$ on J_i such that

$$\mathscr{F}_{ij}^{\otimes n} \cong \mathscr{L}_i \otimes \mathscr{L}_j^{-1}$$

on J_{ij}. (This proves that, up to contributions from $\mathrm{Br}(S)$, the gerbe $\{\mathscr{F}_{ij}\}$ defined in the previous section is trivial.)

Let D be a divisor on X of fiber degree n, and let $\mathscr{L} = \mathscr{O}_X(D)$. Define

$\mathscr{L}_i = \rho_i^*\mathscr{L}^{-1}|_{X_i}$. Then we have

$$\begin{aligned}\mathscr{L}_i \otimes \mathscr{L}_j^{-1} &= \rho_i^*\mathscr{L}^{-1} \otimes \rho_j^*\mathscr{L} \\ &= \rho_i^*(\mathscr{L}^{-1} \otimes (\rho_i^{-1})^*\rho_j^*\mathscr{L}) \\ &= \rho_i^*(\mathscr{L}^{-1} \otimes \tau_{ji}^*\mathscr{L}) \\ &= \rho_i^*(\mathscr{L}^{-1} \otimes \mathscr{L} \otimes \mathscr{L}_{\tau_{ji}}^{\otimes n}) \\ &= \rho_i^*(\mathscr{L}^{-1} \otimes \mathscr{L} \otimes \mathscr{L}_{ji}^{\otimes n}) \\ &= \rho_i^*\mathscr{O}_X(s_i - s_j)^{\otimes n} \\ &= \mathscr{F}_{ij}^{\otimes n},\end{aligned}$$

where we have used the comments in Section 4.2 regarding translations of line bundles of various degrees. (Note that we have used the fact that $\deg \mathscr{L} = n$.) $\square$

Remark 4.4.3. In fact, using some slightly more detailed information that one can get from Ogg-Shafarevich theory it can be proven that the order of α_X in $Ш_S(J)$ is precisely n.

4.5 Other Fibrations

The Jacobian fibration that we considered in the previous sections is not the only one we can naturally construct from the original fibration $X \to S$. In a certain sense the Jacobian is the common denominator of the other ones, but these present some interest as well, and we study them briefly here.

We keep the notation from the previous sections.

Definition 4.5.1. For any integer k, let $Y \to S$ be the relative moduli space of semistable sheaves of rank 1, degree k on the fibers of $X \to S$. We call $Y \to S$ the k-th twisted power of $X \to S$, and denote it by $X^k \to S$.

Theorem 4.5.2. *For any integer k, $X^k \to S$ is a smooth elliptic fibration whose relative Jacobian is isomorphic in a natural way to the relative Jacobian $J \to S$ of the initial fibration $X \to S$. If $\alpha \in Ш_S(J)$ is the element of the Tate-Shafarevich group representing $X \to S$, then $X^k \to S$ is represented by α^k. Consequently the fibration $(X^k)^{k'} \to S$ is isomorphic to $X^{kk'} \to S$. Let n denote the order of α in $Ш_S(J)$. A universal sheaf exists on $X \times_S X^k$ if and only if*

$$(k, n) = 1,$$

and in this case X can be viewed as a moduli space on X^k (we have

$$X \cong (X^k)^{k'}$$

as fibrations over S, where k' is any integer such that $kk' = 1 \mod n$).

Proof. Note that $X^k \to S$ is naturally an elliptic fibration, because the moduli space of rank 1, degree k semistable sheaves on an elliptic curve is isomorphic to the curve itself. This also proves that over each closed point $s \in S$ we have $X^k_s \cong X_s$. In fact, the same construction as in Proposition 4.2.3 shows that X^k and X are locally isomorphic over S. (The universal sheaf in this case is given by

$$\mathscr{U} = \mathscr{O}_{X \times_S X^k}(\Gamma) \otimes \pi_X^* \mathscr{O}_X((k-1)s),$$

where s is the local section.)

Recall that we have a cover of S by open sets $\{U_i\}$, such that the restrictions $X_i \to U_i$ have sections $s_i : U_i \to X_i$. In order to construct J we reglued these pieces via translations

$$\tau_{ij} : X_j|_{X_i \cap X_j} \to X_i|_{X_i \cap X_j},$$

which were associated to the line bundles on $X_i \cap X_j$

$$\mathscr{L}_{ij} = \mathscr{O}_X(s_j - s_i).$$

In order to construct X^k we replace these translations by

$$\tau_{ij} : X_j|_{X_i \cap X_j} \to X_i|_{X_i \cap X_j},$$

associated to the line bundles

$$\mathscr{L}_{ij} = \mathscr{O}_X((k-1)(s_i - s_j)).$$

It is not hard to see that the τ_{ij}'s again satisfy the compatibility relations, so we can use them to reglue the slices X_i to a global space Y which has a natural map $Y \to S$, and consists of local slices Y_i over U_i. These slices are isomorphic to the initial slices X_i via isomorphisms (of spaces over S)

$$\rho_i : Y_i \to X_i,$$

which satisfy

$$\rho_i \circ \rho_j^{-1} = \tau_{ij}.$$

We claim that Y is globally isomorphic to X^k. The universal sheaves that we constructed before give isomorphisms

$$\varphi_i : Y_i \to X^k_i$$

(where $X^k_i = X^k \times_S U_i$) by the universal property of X^k, and all we need to prove is that the φ_i's glue together to a global isomorphism $\varphi : Y \to X^k$.

Let $t_i = \rho_i^{-1} \circ s_i : U_i \to Y_i$, the section of Y_i that corresponds to the original section s_i on X_i. Since the gluings are only translations, in order to show that φ_i and φ_j agree on Y_{ij} it suffices to show that $\varphi_i(t_i) = \varphi_j(t_i)$ on Y_{ij}.

It is easy to see that $\varphi_i(t_i) = [\mathscr{O}_{X_i}(ks_i)]$. (We have abused the notation seriously here: $\mathscr{O}_{X_i}(ks_i)$ is a line bundle of degree k on the fibers of $X_i \to U_i$, and as such it gives a map $U_i \to X_i^k$. The image of this map is $[\mathscr{O}_{X_i}(ks_i)]$.) Also, we have

$$\varphi_j(t_i) = [\mathscr{O}_{X_i \cap X_j}(\rho_j(t_i) + (k-1)s_j)].$$

Therefore we need to prove that

$$\mathscr{O}_{X_i \cap X_j}(\rho_j(t_i) + (k-1)s_j) \cong \mathscr{O}_{X_i \cap X_j}(ks_i).$$

Note that we have

$$\tau_{ij}\rho_j(t_i) = \rho_i(t_i) = s_i$$

and thus

$$\rho_j(t_i) = \tau_{ji}(s_i).$$

Hence we have

$$\begin{aligned} \mathscr{O}(\rho_j(t_i) + (k-1)s_j) &= \mathscr{O}(\tau_{ji}(s_i) + (k-1)s_j) \\ &= \tau_{ij}^* \mathscr{O}(s_i) \otimes \mathscr{O}((k-1)s_j) \\ &= \mathscr{O}(s_i + (k-1)(s_i - s_j) + (k-1)s_j) \\ &= \mathscr{O}(ks_i), \end{aligned}$$

which is what we needed.

Now we can prove the rest of the claims: since $X_s^k \cong X_s$ for all closed points $s \in S$, in order to prove that X^k and X have isomorphic Jacobians we only need to show that $p_{X^k,*}\omega_{X^k/S} \cong p_{X,*}\omega_{X/S}$ (use Proposition 4.2.2). But this follows at once from the fact that along any elliptic curve C, the action of a translation τ is trivial on a global section of ω_C. Since X^k is obtained by regluing slices of X along translations, this shows that $p_{X^k,*}\omega_{X^k/S}$ and $p_{X,*}\omega_{X/S}$ are glued together in the same way along the intersections U_{ij}.

Now let $\mu_i : J_{X,i} \to X_i$ be the isomorphisms that take the canonical section of the Jacobian of X, J_X to the local sections s_i in X_i. (These isomorphisms were called ρ_i in Section 4.2.) Also, let $\mu_i^k : J_{X^k,i} \to X_i^k$ be the corresponding morphisms that take the section of J_{X^k} to t_i. Composing, we get an isomorphism $\varphi : J_X \to J_{X^k}$ given by

$$\varphi = (\mu_i^k)^{-1} \circ \rho_i^{-1} \circ \mu_i$$

(it is not hard to see that these isomorphisms glue together.) We'll compare $\mathrm{Br}(J_{X^k})$ and $\mathrm{Br}(J_X)$ using this isomorphism.

The twisting in J associated to X^k is given by

$$\{(\mu_i^k)^* \mathscr{O}_{X_{ij}^k}(t_i - t_j), \}$$

Applying $(\rho_i^{-1})^*$ we get

$$\begin{aligned}(\rho_i^{-1})^*\mathscr{O}_{X_{ij}^k}(t_i - t_j) &= \mathscr{O}_{X_{ij}}(\rho_i(t_i) - \rho_i(t_j))\\ &= \mathscr{O}_{X_{ij}}(s_i - \tau_{ij}(s_j))\\ &= \mathscr{O}_{X_{ij}}(s_i) \otimes \mathscr{O}_{X_{ij}}(\tau_{ij}(s_j))^{-1}\\ &= \mathscr{O}_{X_{ij}}(s_i) \otimes \tau_{ji}^*\mathscr{O}_{X_{ij}}(-s_j)\\ &= \mathscr{O}_{X_{ij}}(s_i) \otimes \mathscr{O}_{X_{ij}}(-s_j) \otimes \mathscr{L}_{ji}^{-1}\\ &= \mathscr{O}_{X_{ij}}(s_i) \otimes \mathscr{O}_{X_{ij}}(-s_j) \otimes \mathscr{O}_{X_{ij}}((k-1)(s_i - s_j))\\ &= \mathscr{O}_{X_{ij}}(k(s_i - s_j)),\end{aligned}$$

and thus

$$\begin{aligned}\varphi^*(\mu_i^k)^*\mathscr{O}_{X_{ij}^k}(t_i - t_j) &= \mu_i^*(\rho_i^{-1})^*\mathscr{O}_{X_{ij}^k}(t_i - t_j)\\ &= \mu_i^*\mathscr{O}_{X_{ij}}(k(s_i - s_j)),\end{aligned}$$

which shows that if X is represented by $\alpha \in Ш_S(J_X)$, then X^k is represented by $(\varphi^{-1})^*\alpha^k$ in $Ш_S(J_{X^k})$.

The result about the existence of a universal sheaf is known (see, for example, [6]). But here is a quick proof of the existence of a universal sheaf when $(k, n) = 1$: we only need to show that we can twist the universal sheaves described in the beginning of the proof in a way that would make them glue. A computation similar to the one done in Section 4.3 shows that

$$\mathscr{M}_{ij} = \mathscr{U}_j|_{P_{ij}} \otimes \mathscr{U}_i^{-1}|_{P_{ij}}$$

is equal to the pull-back of a line bundle $\mathscr{F}_{ij}$ from X_{ij}^k, and $\mathscr{F}_{ij}$ can be taken to be of the form

$$\rho_i^*\mathscr{L}_{ij}^{-1} = \rho_i^*\mathscr{O}_{X_{ij}}((k-1)(s_j - s_i)).$$

Now let $\mathscr{S}$ be a line bundle on X, of fiber degree n (such a line bundle exists globally on X, by the comments at the end of the previous section), and consider the line bundles $\mathscr{S}_i' = \rho_i^*\mathscr{S}_i$ (where $\mathscr{S}_i = \mathscr{S}|_{X_i}$). We have

$$\begin{aligned}(\rho_j^{-1})^*(\mathscr{S}_j' \otimes (\mathscr{S}_i')^{-1}) &= \mathscr{S}_j \otimes \tau_{ij}^*\mathscr{S}_i^{-1}\\ &= \mathscr{S}_j \otimes \mathscr{S}_i^{-1} \otimes \mathscr{L}_{ij}^{\otimes -n}\\ &= \mathscr{O}_{X_{ij}}(-n(k-1)(s_i - s_j)).\end{aligned}$$

Therefore, considering the collections of line bundles $\{\mathscr{O}_{X_i^k}(t_i)\}$ and $\{\mathscr{S}_i'\}$, we see that their coboundaries are $\{\mathscr{O}(k(s_i - s_j))\}$ and $\{\mathscr{O}(-n(k-1)(s_i - s_j))\}$ (where we have omitted the subscript X_{ij} and the pull-backs by isomorphisms to X^k because these do not matter, since we are dealing with line bundles of degree 0). Since k and $-n(k-1)$ are coprime, we can obtain any multiple of $\{\mathscr{O}(s_i - s_j)\}$ out of them, in particular we can construct $\{\mathscr{O}((k-1)(s_j - s_i))\}$, which we have seen is the obstruction to gluing the $\mathscr{U}_i$'s together. Therefore this obstruction is trivial, and hence the $\mathscr{U}_i$'s can be glued to form a global universal sheaf, thus finishing the proof. □

Remark 4.5.3. Note that we have fixed a special isomorphism of J_X and J_{X^k} in order to obtain this result. This is what allows us to distinguish between X and X^{-1}, which are isomorphic, *even as fibrations over S!* Indeed, one can take $\mathscr{I}_\Delta$ as a universal sheaf on $X \times_S X$, where Δ is the diagonal in $X \times_S X$. But in this case, the isomorphism between J_X and $J_{X^{-1}}$ is doing a negation along the fibers, which also acts by -1 on the Brauer group. Note that this answers the question posed at the end of [13, Section 1].

Another consequence of this remark is that if the j invariant is not constant (so that the only automorphisms of J/S are given by translations and negation), and k is not equal to 1 or -1 in $\mathbf{Z}/n\mathbf{Z}$, then X^k is not isomorphic to X, even at the generic point of S. Thus if we can prove that there is only one elliptic fibration structure whose Jacobian is the same as that of X among all birational models of X, then X is not birational to X^k. (Because, if they were birational to each other, they would have to be isomorphic to one another because they have the same Jacobian, and this is impossible.) See also Section 6.7.

Chapter 5

K3 surfaces

In this chapter we apply the general results deduced in Part I to different situations related to K3 surfaces. The reason K3 surfaces provide an interesting set of examples is because, on one hand, we have a good explicit description of their moduli space (the Torelli theorem, 5.1.1) so we can handle them easily in terms of lattices and Hodge structures, and, on the other hand, moduli spaces of sheaves on K3's are well understood, primarily through the work of Mukai ([32]) (see the introductory section 5.1). Furthermore, it is also known when two K3 surfaces have equivalent derived categories (of untwisted sheaves), by the work of Orlov ([36]).

The first section presents the known results about K3 surfaces, moduli spaces of sheaves on them, and derived categories of untwisted sheaves. Everything else is built upon these facts. We have seen in Chapter 4 that one can recover the classical Ogg-Shafarevich theory of elliptic fibrations without a section by studying the obstruction to the existence of universal sheaves for certain moduli problems. For K3 surfaces, we can do better: in most cases we don't need an elliptic fibration on the K3 to play the same game. In Section 2 we study deformations of vector bundles as twisted sheaves, obtaining results that will be used in identifying the obstruction. These results are also interesting on their own, in the context of deformation theory of vector bundles.

Section 3 contains the main technical result of this chapter, an explicit description of the obstruction α (whose existence is asserted in Theorem 3.3.2) to the existence of a universal sheaf for an arbitrary moduli problem on a K3 X, when the moduli space is again a K3. This description is given in purely cohomological terms, via Fourier-Mukai transforms.

By analogy with Ogg-Shafarevich theory, one can also attempt to reverse the process of going from the pair (X, v) (where X is a K3, and v is the data for a moduli problem on X) to the pair (M, α) (where $M = M(v)$ is the moduli space, and α is the obstruction to the existence of a universal sheaf on $X \times M$). This is what would be needed in order to obtain a complete generalization of Ogg-Shafarevich theory in terms only of moduli problems and obstructions. Unfortunately, we cannot push this too far. The best we can do is to find the transcendental lattice of the candidate X, together with its Hodge structure. In most cases, this should allow us to get X; however, finding v seems a more difficult problem.

The last section deals with the relationship of the results of the previous sections with the topic of equivalences of derived categories. This provides an insight into why we can only get a hold of the transcendental lattice of X in generalizing Ogg-Shafarevich theory. As a consequence of this analysis we also observe the following curious phenomenon: when we can find X and v, X only depends on the cyclic subgroup of $\mathrm{Br}(M)$ generated by α. This allows us to infer that in some cases one has

$$\mathbf{D}^b_{\mathrm{coh}}(M, \alpha) = \mathbf{D}^b_{\mathrm{coh}}(M, \alpha^k)$$

for $(k, \mathrm{ord}(\alpha)) = 1$, a rather striking application of the theoretical results of the previous chapters. In the next chapter we'll see a similar result for elliptic Calabi-Yau threefolds.

5.1 General Facts

The main results about K3 surfaces: the Torelli theorem, Mukai's calculation of moduli spaces of sheaves on K3's, Orlov's characterization of derived equivalences for K3's – are all presented in this section, and put in the context of the results in the first part.

The Torelli Theorem

Let's fix notation: let X be a complex K3 surface, i.e., a complex manifold of complex dimension 2, having trivial canonical class ($K_X = 0$) and being simply connected (equivalent, using Hodge theory, to having $H^1(X, \mathscr{O}_X) = 0$).

We have $H^2(X, \mathbf{Z}) = \mathbf{Z}^{22}$, and considering this group with the intersection pairing we obtain a lattice which is isomorphic to

$$\mathcal{L}_{\mathrm{K3}} = (E_8)^{\oplus 2} \oplus (U(1))^{\oplus 3}.$$

Inside the $H^2(X, \mathbf{Z})$ lattice there are two natural sublattices, the Néron-Severi sublattice of X, $\mathrm{NS}(X)$, (consisting of first Chern classes of holomorphic vector bundles), and its orthogonal complement, the transcendental lattice $T_X = \mathrm{NS}(X)^{\perp}$. Both these lattices are primitive sublattices of $H^2(X, \mathbf{Z})$, but may be non-unimodular.

The complex structure of X is reflected in its Hodge decomposition

$$H^2(X, \mathbf{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X).$$

We have $\dim H^{2,0}(X) = 1$, so in order to specify this decomposition, it is enough to specify a one-dimensional complex subspace $\pi \subseteq H^2(X, \mathbf{C})$ (called the period of X): indeed, one can take $H^{2,0}(X)$ to be π, $H^{0,2}$ to be the complex conjugate $\overline{\pi}$ of π, and $H^{1,1}$ to be the orthogonal (with respect to the intersection pairing) to the span of π and $\overline{\pi}$. The complex subspace π satisfies the following extra property: if $v \in \pi$ is a non-zero vector, we have $v.v = 0$ and $v.\overline{v} > 0$.

Two lattices $\mathcal{L}, \mathcal{L}'$, endowed with Hodge structures, are said to be *Hodge isometric* if and only if there is an isometry between the two preserving the Hodge structures on them.

Theorem 5.1.1 (Torelli for K3's). *Two K3 surfaces, X and Y, are isomorphic if and only if the lattices $H^2(X, \mathbf{Z})$ and $H^2(Y, \mathbf{Z})$ are Hodge isometric. For all periods π in $\mathcal{L}_{\mathrm{K3}} \otimes \mathbf{C}$ satisfying $\pi.\pi = 0$ and $\pi.\overline{\pi} > 0$, there exists a K3 surface X whose period is π, under some choice of isomorphism between $H^2(X, \mathbf{Z})$ and $\mathcal{L}_{\mathrm{K3}}$.*

There is yet another part to this theorem, which explains the relationship between the group of Hodge isometries $H^2(X, \mathbf{Z}) \to H^2(X, \mathbf{Z})$ on a K3 X, and the group $\mathrm{Aut}(X)$ of automorphisms of X: the former group is generated by $\mathrm{Aut}(X)$, and by a certain class of automorphisms of $H^2(X, \mathbf{Z})$, called (-2)-reflections. For more details about the Torelli theorem for K3 surfaces, the reader should consult [3].

This shows how we can construct a moduli space for K3 surfaces: we take the subspace of vectors v in $\mathcal{L}_{\mathrm{K3}} \otimes \mathbf{C}$ satisfying $v.v = 0, v.\overline{v} > 0$, and then we projectivize it; it is a quasi-projective variety. Call it P. The automorphism group of the lattice $\mathcal{L}_{\mathrm{K3}}$ acts on P; let $M = P/\mathrm{Aut}(\mathcal{L}_{\mathrm{K3}})$. Then M is a coarse moduli space for K3 surfaces.

Note that many (most) complex K3 surfaces are not algebraic: in order to be algebraic they would have to be projective, i.e. have at least an ample divisor, and therefore at least a non-zero divisor. But most K3's don't have any non-zero divisors at all: these would correspond to classes in $H^{1,1}(X) \cap H^2(X, \mathbf{Z})$, and the set of periods for which this group is non-trivial is a countable union of codimension one subspaces of M.

Since we are mainly interested in algebraic K3's, we only make precise these statements in the context of primitively polarized K3 surfaces, described below. These results are taken from [25] and [32].

Definition 5.1.2. Let d be a positive integer. Let $\underline{\mathcal{K}}_d$ be the functor $(\mathfrak{Sch}/\mathbf{C})^\circ \to \mathfrak{Ens}$ that maps a scheme Y to the set of equivalence classes of pairs $(f : X \to Y, \mathscr{L})$ such that $f : X \to Y$ is a smooth family of K3 surfaces, $\mathscr{L}$ is a line bundle on X, and for any $t \in Y$, the restriction $\mathscr{L}_t$ of $\mathscr{L}_t$ to the fiber $X_t = f^{-1}(t)$ is an ample primitive line bundle with $c_1^2(\mathscr{L}_t) = 2d$. Two pairs $(f : X \to Y, \mathscr{L})$ and $(f' : X' \to Y, \mathscr{L}')$ are equivalent if and only if there exists a Y-isomorphism $g : X \to X'$ and a line bundle $\mathscr{N}$ on Y such that $g^*\mathscr{L}' \cong \mathscr{L} \otimes f^*\mathscr{N}$.

Theorem 5.1.3. *The functor $\underline{\mathcal{K}}_d$ is corepresented by a coarse moduli space $\mathcal{K}_d$ which is a quasi-projective, irreducible scheme.*

Moduli Problems and Mukai's Results

Definition 5.1.4. Let X be a K3 surface. The *Mukai lattice* of X is defined to be

$$\tilde{H}(X, \mathbf{Z}) = H^0(X, \mathbf{Z}) \oplus H^2(X, \mathbf{Z}) \oplus H^4(X, \mathbf{Z}),$$

endowed with the product

$$((r,l,s),(r',l',s')) = \int_X r.s' + r'.s - l.l',$$

where the dot product on the right hand side is the cup product in $H^*(X,\mathbf{Z})$.

The *Hodge decomposition* on $\tilde{H}(X,\mathbf{Z})$ is given by

$$\begin{aligned}
\tilde{H}^{2,0}(X) &= H^{2,0}(X) \\
\tilde{H}^{1,1}(X) &= H^0(X,\mathbf{C}) \oplus H^{1,1}(X) \oplus H^4(X,\mathbf{C}) \\
\tilde{H}^{0,2}(X) &= H^{0,2}(X)
\end{aligned}$$

Elements in $\tilde{H}^{1,1}(X)$ will be called *algebraic*.

We'll sometimes also consider $\tilde{H}(X,\mathbf{Q}) = \tilde{H}(X,\mathbf{Z}) \otimes \mathbf{Q}$, with intersection product and Hodge decomposition defined in a similar fashion.

For a coherent sheaf $\mathscr{E}$ (or, more generally, an element of $\mathbf{D}^b_{\mathrm{coh}}(X)$), define $v(\mathscr{E}) \in \tilde{H}(X,\mathbf{Z})$ by

$$v(\mathscr{E}) = \mathrm{ch}(\mathscr{E}).\sqrt{\mathrm{td}(X)} = (\mathrm{rk}(\mathscr{E}), c_1(\mathscr{E}), \mathrm{rk}(\mathscr{E})\omega + \frac{1}{2}c_1(\mathscr{E})^2 - c_2(\mathscr{E})),$$

(where $\omega \in H^4(X,\mathbf{Z})$ is the fundamental class of X). This element is called the *Mukai vector* of $\mathscr{E}$.

Remark 5.1.5. Note that the intersection product and the Mukai vector are the same as the more general ones introduced in Section 3.1, restricted to the lattice $\tilde{H}(X,\mathbf{Z})$. Therefore all the results obtained there apply.

The reason one studies these objects is because one is interested in studying moduli spaces of semistable sheaves on X. The following results are mainly due to Mukai ([32]). (Some have been rephrased in the more convenient language of sheaves of modules over an Azumaya algebra.)

Theorem 5.1.6. *Let X be a K3 surface, with a fixed polarization $\mathscr{P}$. Let $v \in \tilde{H}(X,\mathbf{Z})$ be a primitive (indivisible) vector that lies in the algebraic part of $\tilde{H}(X)$. Assume that v is isotropic (i.e. $(v,v)=0$), and that the moduli space of semistable sheaves of Mukai vector v, $M(v)$, is non-empty and does not contain any properly semistable points. Then $M(v)$ is a K3 surface.*

Example 5.1.7. Just for the purpose of having an example in mind, we include here the following construction due to Mukai ([34, 2.2]). Consider a net of quadrics in $\mathbf{P}^5$, generated by three non-singular quadrics Q_1, Q_2, Q_3, and assume that these intersect transversely. Taking the net general enough the intersection of the quadrics in this net is a smooth K3 surface.

Each smooth quadric Q in the net is isomorphic to $G(2,4)$, the Grassmanian of planes in $\mathbf{C}^4$ and as such it has two vector bundles of rank 2 on it, $\mathscr{T}_Q$ and $\mathscr{Q}_Q$ (the tautological and the quotient bundles). Consider their restrictions $\mathscr{T}_Q|_X$ and

$\mathscr{Q}_Q|_X$. After a fixed twist of one of them by a line bundle, it can be arranged that they have the same Chern classes, call them c_0, c_1, c_2.

Mukai proves that these vector bundles are stable, and that they are all different for different quadrics (and among themselves). However, when the quadric becomes singular, the two bundles become isomorphic with each other. These are all the semistable sheaves with Chern classes c_0, c_1, c_2.

The geometric picture one gets from this is that the moduli space of semistable sheaves with these Chern classes consists of a double cover of the space of quadrics, branched over the discriminant locus. The space of quadrics in question is $\mathbf{P}^2$ (we are talking about a net!), and the discriminant locus is a sextic curve in this $\mathbf{P}^2$. Therefore the moduli space is isomorphic to a double cover of $\mathbf{P}^2$ branched over a sextic, which is again a K3 surface.

Definition 5.1.8. Under the assumptions of the previous theorem, let $M = M(v)$, and let $\alpha \in \mathrm{Br}(M)$ be the obstruction of Definition 3.3.3 for this moduli problem. We'll denote α by $\mathrm{Obs}(X, v)$. Theorem 3.3.2 shows the existence of a $\pi_M^*\alpha$-twisted universal sheaf $\mathscr{U}$ on $X \times M$ (where $\pi_M : X \times M \to M$ is the projection). Let $\mathscr{A}$ be an Azumaya algebra on M representing α, and using the equivalence between twisted sheaves and modules over an Azumaya algebra, represent $\mathscr{U}$ as a sheaf $\tilde{\mathscr{U}}$ of modules over $\pi_M^*\mathscr{A}$.

Then $\tilde{\mathscr{U}}$ is called a *quasi-universal sheaf.* It has the property that

$$\tilde{\mathscr{U}}|_{X\times[\mathscr{F}]} \cong \mathscr{F}^{\oplus n}$$

for some n that only depends on $\tilde{\mathscr{U}}$ (the isomorphism is as sheaves of $\mathscr{O}_X$-modules). This n is called the *similitude* of $\tilde{\mathscr{U}}$, denoted by $s(\tilde{\mathscr{U}})$. (Here, $\mathscr{F}$ is a semistable sheaf on X with Mukai vector v, and $[\mathscr{F}]$ is the point of M that corresponds to it.)

In fact, $\tilde{\mathscr{U}}$ satisfies a certain universal property which is very similar to that enjoyed by a universal sheaf; see [32, Appendix 2].

Definition 5.1.9. The cohomological Fourier-Mukai transform associated to $\tilde{\mathscr{U}}$ is the homomorphism

$$\varphi = \varphi_{X\to M}^{\tilde{\mathscr{U}}^\vee} : \tilde{H}(X, \mathbf{Q}) \to \tilde{H}(M, \mathbf{Q})$$

given by

$$\varphi(\,\cdot\,) = \frac{1}{s(\tilde{\mathscr{U}})}\pi_{M,*}(\pi_X^*(\,\cdot\,).v(\tilde{\mathscr{U}}^\vee)),$$

where

$$\tilde{\mathscr{U}}^\vee = \mathbf{R}\underline{\mathrm{Hom}}(\tilde{\mathscr{U}}, \mathscr{O}_{X\times M}),$$

and

$$v(\tilde{\mathscr{U}}^\vee) = \mathrm{ch}(\tilde{\mathscr{U}}^\vee).\sqrt{\mathrm{td}(X \times M)}.$$

Theorem 5.1.10. *Under the hypotheses of Theorem 5.1.6, the map φ is a Hodge isometry between $\tilde{H}(X, \mathbf{Q})$ and $\tilde{H}(M, \mathbf{Q})$. It maps $v \in \tilde{H}(X, \mathbf{Q})$ to the vector $(0, 0, \omega) \in \tilde{H}(M, \mathbf{Q})$, and it therefore induces a Hodge isometry*

$$v^{\perp}/v \cong H^2(M, \mathbf{Q}),$$

the latter being computed inside $\tilde{H}(X, \mathbf{Q})$. Restricted to $v^{\perp}/v$, this isometry is independent of the choice of quasi-universal bundle and is integral, i.e. it takes integral vectors to integral vectors. It therefore induces a Hodge isometry

$$H^2(M, \mathbf{Z}) \cong v^{\perp}/v,$$

the latter now computed in $\tilde{H}(X, \mathbf{Z})$.

Remark 5.1.11. Using the Torelli theorem, this gives a complete description of the moduli space M.

Proof. When the moduli problem is fine, one first proves that $\mathscr{U}$ (the universal sheaf) induces an equivalence of derived categories $\mathbf{D}^b_{\mathrm{coh}}(X) \cong \mathbf{D}^b_{\mathrm{coh}}(M)$. We'll do this in a more general context, in Theorem 5.5.1. Then the result follows immediately from the results in Section 3.1. The general case (when the moduli problem is not fine) is done by a deformation argument in [25, Chapter 6]. $\square$

Theorem 5.1.12. *Let n be the greatest common divisor of the numbers (u, v), where u runs over all $\tilde{H}^{1,1}(X) \cap \tilde{H}(X, \mathbf{Z})$. Then the following statements hold:*

1. *There exist α-twisted locally free sheaves on M of ranks $r_1, r_2, \ldots, r_k$, with $\gcd(r_1, r_2, \ldots, r_k) = n$, and therefore α is n-torsion. (α is the obstruction described in Definition 5.1.8.)*

2. *Denote by φ any cohomological Fourier-Mukai transform defined above (for some choice of quasi-universal sheaf). Then φ maps T_X into T_M, (viewing T_X as a sublattice of $\tilde{H}(X, \mathbf{Z})$ via the inclusion $\lambda \mapsto (0, \lambda, 0)$), and $\varphi|_{T_X}$ is independent of the choice of quasi-universal bundle used to define it.*

3. *There exists $\lambda \in T_X$ such that $v + \lambda$ is divisible by n (in $\tilde{H}(X, \mathbf{Z})$); for such a λ we have $\varphi(\lambda)$ divisible by n (in T_M).*

4. *φ is injective, and its cokernel is a finite, cyclic group of order n, generated by $\varphi(\lambda)/n$ for any λ satisfying the condition in (3).*

Example 5.1.13. In Example 5.1.7 the number $n = 2$. Therefore the obstruction α (which lives in the Brauer group of the moduli space, which is the double cover of $\mathbf{P}^2$ branched over a sextic) is of order 2.

Proof. For the first statement, see [32, Remark A.7] and the description of twisted sheaves in the first chapter of this work. The other statements are [32, 6.4]. Note that n is defined in a slightly different fashion than in [32, 6.4], but this is the correct version of it: otherwise one could have, for example, a Mukai vector $v = (r, l, s)$ for which $\gcd(r, s) = 1$, but $\gcd(l.d) > 1$ when d runs over all $d \in \mathrm{NS}(X)$. In this case, the proof of [loc.cit.] fails unless one takes the correct value of n, which is 1. □

Theorem 5.1.14 (Orlov). *Two K3 surfaces, X and M, have equivalent derived categories, $\mathbf{D}^b_{\mathrm{coh}}(X) \cong \mathbf{D}^b_{\mathrm{coh}}(M)$, if and only if the lattices T_X and T_M are Hodge isometric.*

Proof. See [36]. Here is a sketch of the proof: when $\mathbf{D}^b_{\mathrm{coh}}(X) \cong \mathbf{D}^b_{\mathrm{coh}}(M)$, the equivalence must be given by a Fourier-Mukai transform by Theorem 3.1.16 (this is the core of Orlov's paper). Then, the results in Section 3.1 give Hodge isometries between T_X and T_M.

In the other direction, one proves that if T_X and T_M are Hodge isometric, one can extend this isometry to an isometry $i : \tilde{H}(X, \mathbf{Z}) \cong \tilde{H}(M, \mathbf{Z})$, and this can be done in such a way that if $v = i^{-1}(0, 0, \omega_M)$, one has on X a polarization such that $M = M(v)$. Since $n = 1$ in this case (by Theorem 5.1.12), there exists a universal sheaf on $X \times M$, which induces a Fourier-Mukai transform $\mathbf{D}^b_{\mathrm{coh}}(X) \cong \mathbf{D}^b_{\mathrm{coh}}(M)$ (by Theorem 5.5.1). □

5.2 Deformations of Twisted Sheaves

In this section we consider what happens when one tries to deform a rank n vector bundle $\mathscr{E}_0$ on the central fiber X_0 of a deformation, under the hypothesis that $c_1(\mathscr{E}_0)$ does *not* extend to an algebraic class on neighboring fibers. This analysis will be used for identifying the obstruction α from Definition 5.1.8. We are working over $\mathbf{C}$.

Let's first set up the context. We start with $f : X \to S$, a proper, smooth morphism of schemes or analytic spaces, and with 0 a closed point of S. The Brauer group we consider is $\mathrm{Br}'_{\mathrm{an}}(X)_{\mathrm{tors}}$, which is the natural generalization to the analytic setting of the étale Brauer group used in the algebraic case. Throughout this section we'll be loose in our notation and refer to $\mathrm{Br}'_{\mathrm{an}}(X)_{\mathrm{tors}}$ as the Brauer group of X, or $\mathrm{Br}'(X)$.

Let X_0 be the fiber of f over 0. We consider an element $\alpha \in \mathrm{Br}(X)$, such that $\alpha|_{X_0}$ is trivial as an element of $\mathrm{Br}(X_0)$, and $\mathscr{E}$ a locally free α-twisted sheaf on X. Since $\alpha|_{X_0} = 0$, we can find a modification of the transition functions of $\mathscr{E}|_{X_0}$ by a coboundary in $H^2(X_0, \mathscr{O}^*_{X_0})$ such that the transition functions glue, to get an untwisted locally free sheaf $\mathscr{E}_0$ on X_0. We want to understand what happens to $c_1(\mathscr{E}_0)$ in the neighboring fibers.

Note that since the morphism f is smooth, by possibly restricting first the base S to a smaller open set (analytic or étale), we can identify the cohomology groups of different fibers. Let's explain this better (in the analytic cohomology, where things are simpler): since the morphism is smooth, the topology of X is necessarily that of $X_0 \times S$, after restricting the base enough to make it simply connected. Then the maps $i_s^* : H^*(X, \mathbf{Z}) \to H^*(X_s, \mathbf{Z})$ are isomorphisms, where $i_s : X_s \to X$ is the natural embedding of the fiber X_s of f over a point $s \in S$. The composition of one of these morphisms with the inverse of another gives a canonical way to identify the integral cohomology groups of the fibers of X.

It is worthwhile observing that for any space X we have

$$H^2(X, \mathbf{Q}/\mathbf{Z}) = H^2(X, \mathbf{Z}) \otimes \mathbf{Q}/\mathbf{Z} \oplus H^3(X, \mathbf{Z})_{\mathrm{tors}},$$

from the universal coefficient theorem. We saw (Theorem 1.1.3) that $\mathrm{Br}'(X)$ is the quotient of $H^2(X, \mathbf{Q}/\mathbf{Z})$ by the image of $\mathrm{Pic}(X) \otimes \mathbf{Q}/\mathbf{Z}$. But classes in $H^3(X, \mathbf{Z})_{\mathrm{tors}}$ cannot become zero in this quotient, so (in view of the fact that cohomology groups of the fibers are locally constant over the base) the only way an element of $\mathrm{Br}'(X)$ can become zero in a central fiber X_0 without being zero in the neighboring fibers is if it belongs to $H^2(X, \mathbf{Z}) \otimes \mathbf{Q}/\mathbf{Z}$, and in the central fiber it is also in the image of $\mathrm{Pic}(X_0) \otimes \mathbf{Q}/\mathbf{Z}$.

This situation (which is typical for, say, K3 surfaces, for which $H^3(X, \mathbf{Z})_{\mathrm{tors}} = 0$) should be contrasted with the situation for Calabi-Yau threefolds. The two cases are quite opposite, when it comes to deformations: in the case of K3 surfaces, we can have a class in $H^2(X, \mathbf{Q}/\mathbf{Z})$ which becomes zero in the Brauer group of a central fiber, without being zero in the neighboring fibers, by "suddenly becoming algebraic". In the case of Calabi-Yau threefolds, this cannot happen, since we are only dealing with classes in $H^3(X, \mathbf{Z})_{\mathrm{tors}}$. However, it can happen that all neighboring fibers X_s, $s \neq 0$, have trivial Brauer group, while the central fiber X_0 has a non-trivial Brauer group, by having a non-torsion class in $H^3(X_s, \mathbf{Z})$ become torsion in $H^3(X_0, \mathbf{Z})$. Obviously, in this case the morphism $X \to S$ can not be smooth. The reader should consult Chapter 6 for more details, and for an example when this phenomenon occurs.

We now return to the situation we started studying in the beginning of this section, that of a smooth morphism $f : X \to S$ and an element $\alpha \in \mathrm{Br}(X)$. Because of the previous considerations, assume that α belongs to $H^2(X, \mathbf{Z}) \otimes \mathbf{Q}/\mathbf{Z}$, and therefore it can be written as $\frac{1}{n}c$, for some class $c \in H^2(X, \mathbf{Z})$. The fact that $\alpha|_{X_0} = 0$ means that $c|_{X_0}$ belongs to $\mathrm{Pic}(X_0)$. We'd like to identify $c|_{X_0}$ (which is the same as finding c, because of the existence of the isomorphism $H^*(X, \mathbf{Z}) \cong H^*(X_0, \mathbf{Z})$). This is the content of the following theorem:

Theorem 5.2.1. *Let $\mathscr{E}$ be a locally free α-twisted sheaf on X, and let $\mathscr{E}_0 = \mathscr{E}|_{X_0}$. Assume that S is small enough (say, contractible), so that we have an identification $H^2(X, \mathbf{Z}) \cong H^2(X_0, \mathbf{Z})$. Since $\alpha|_{X_0} = 0$, we can view $\mathscr{E}_0$ as an usual sheaf on X_0, by Lemma 1.2.8. Then*

$$\alpha = -\frac{1}{\mathrm{rk}(\mathscr{E}_0)} c_1(\mathscr{E}_0),$$

under the identification $H^2(X, \mathbf{Z}) \cong H^2(X_0, \mathbf{Z})$ *and the inclusion*

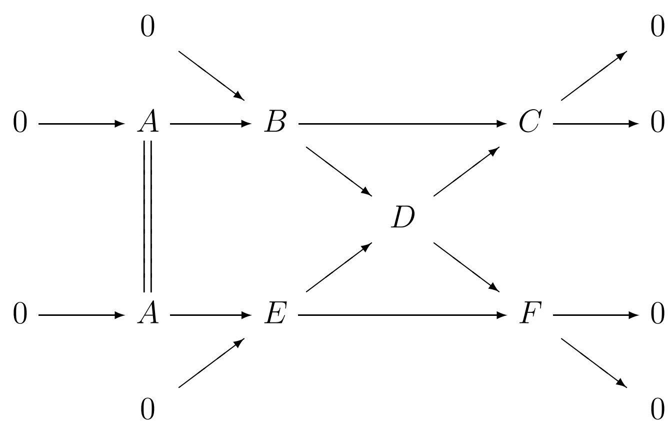
$$(H^2(X, \mathbf{Z})/NS(X)) \otimes \mathbf{Q}/\mathbf{Z} \hookrightarrow \mathrm{Br}'(X).$$

The interpretation of this theorem is the following: if we try to deform a vector bundle $\mathscr{E}_0$, given on the central fiber X_0, in a family in which the class $c_1(\mathscr{E}_0)$ is not algebraic in neighboring fibers X_t, the only hope to be able to do this is to deform $\mathscr{E}_0$ as a twisted sheaf, and then the twisting should be precisely

$$-\frac{1}{\mathrm{rk}(\mathscr{E}_0)} c_1(\mathscr{E}_0).$$

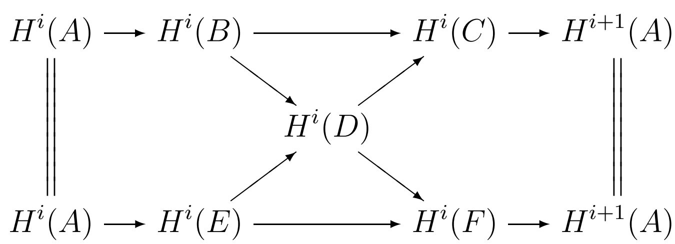
We first need a couple of propositions:

Lemma 5.2.2 (The Roman 9 Lemma). *Let* $\mathcal{A}$ *be an abelian category with e-nough injectives,* F *a left-exact functor, and let* H^i *be the right derived functors of* F*. Consider a commutative diagram*



with exact rows and diagonals $(A, B, C, D, E, F \in \mathrm{Ob}(\mathscr{A}))$*.*

Then the induced diagram in cohomology



commutes as well, except for the right pentagon which anti-commutes.

Proof. The only non-trivial issue is the anti-commutativity of the right pentagon; we'll prove it only for the case when $\mathcal{A}$ is the category of quasi-coherent sheaves on a scheme X, and one uses Čech cohomology. In the general case of an abelian category, this is an easy exercise in homological algebra (done for example in [44]).

Let d be an element of $H^i(D)$, and represent it as a Čech cocycle $\{d_{j_0 \cdots j_i}\}$ over some fine enough cover of X. From here on we'll omit the indices, and just refer to this collection as d.

Let c be the image of d under the natural map $D \to C$ (as Čech cocycles). By possibly refining the cover, one can lift c to a cochain $b \in \check{C}^i(B)$. It is the failure of b to be a cocycle that gives the coboundary map $H^i(C) \to H^{i+1}(A)$.

Let d' be the cochain obtained by applying to b the map $B \to D$. By the commutativity of the triangle that contains B, C and D, d' maps to c as well, so that $d - d'$ maps to 0 in c, so $d - d'$ lifts to a cochain in E. Let this cochain be e.

Now e is a lift of f (where f is the image of d under the map $D \to F$): d' comes from b, so it maps to 0 in F.

Computing, we have $\partial e = \partial(d - d') = -\partial d' = -\partial b$, which is what we needed. □

Definition 5.2.3. Consider the two exact sequences on a space X

$$0 \to \mathscr{O}_X^* \to \mathrm{GL}(n) \to \mathrm{PGL}(n) \to 0$$

and

$$0 \to \mathbf{Z}/n\mathbf{Z} \to \mathrm{SL}(n) \to \mathrm{PGL}(n) \to 0,$$

which we've already seen in Section 1.1 (as usual, we are using the étale or analytic topologies). For an element p of $H^1(X, \mathrm{PGL}(n))$, let $a(p)$ and $t(p)$ be the images of p under the two coboundary maps

$$H^1(X, \mathrm{PGL}(n)) \to H^2(X, \mathscr{O}_X^*)$$

and

$$H^1(X, \mathrm{PGL}(n)) \to H^2(X, \mathbf{Z}/n\mathbf{Z}),$$

respectively. We'll call $a(p)$ the *analytic twisting class* of p and $t(p)$ the *topological twisting class* of p. The first one belongs to $\mathrm{Br}'(X)$, and as such depends on the complex structure of X, while the second one is in $H^2(X, \mathbf{Z}/n\mathbf{Z})$ and depends only on the topology of X.

If $Y \to X$ is a $\mathbf{P}^{n-1}$-bundle over X, then the analytic and topological twisting classes of Y/X, $t(Y/X)$ and $a(Y/X)$, are the classes of the element p of $H^1(X, \mathrm{PGL}(n))$ associated to the bundle $Y \to X$.

Proposition 5.2.4. *Let X be a scheme, $\mathscr{E}$ a rank n locally free sheaf on X and let $Y = \mathbf{Proj}(\mathscr{E}) \to X$ be the associated projective bundle. Then*

$$t_1(Y/X) = -c_1(\mathscr{E}) \bmod n.$$

(Here, and in the sequel, by reducing mod n we mean applying the natural map $H^2(X, \mathbf{Z}) \to H^2(X, \mathbf{Z}/n\mathbf{Z})$).

Proof. Consider the commutative diagram with exact rows and diagonals

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & 0 \\
 & & & \searrow & & & & \nearrow & \\
0 & \longrightarrow & \mathbf{Z}/n\mathbf{Z} & \longrightarrow & \mathrm{SL}(n) & \longrightarrow & \mathrm{PGL}(n) & \longrightarrow & 0 \\
 & & \| & & \searrow & & \nearrow & & \\
 & & \| & & & \mathrm{GL}(n) & & & \\
 & & \| & & \nearrow & & \searrow^{\det} & & \\
0 & \longrightarrow & \mathbf{Z}/n\mathbf{Z} & \longrightarrow & \mathscr{O}_X^* & \xrightarrow{\cdot n} & \mathscr{O}_X^* & \longrightarrow & 0 \\
 & & & \nearrow & & & & \searrow & \\
 & & 0 & & & & & & 0
\end{array}$$

and apply the Roman 9 lemma for $i = 1$. We get the anti-commutative diagram

$$\begin{array}{ccccc}
 & & H^1(X, \mathrm{PGL}(n)) & \xrightarrow{t_1} & H^2(X, \mathbf{Z}/n\mathbf{Z}) \\
 & \nearrow & & & \| \\
H^1(X, \mathrm{GL}(n)) & & & & \| \\
 & \searrow^{\det} & & & \| \\
 & & H^1(X, \mathscr{O}_X^*) & \xrightarrow{c_1 \bmod n} & H^2(X, \mathbf{Z}/n\mathbf{Z})
\end{array}$$

which is exactly what we need. □

Proposition 5.2.5. *For any integer n consider the following two natural maps: one, the map $\varphi : H^2(X, \mathbf{Z}/n\mathbf{Z}) \to \mathrm{Br}'(X)$ obtained from the short exact sequence*

$$0 \to \mathbf{Z}/n\mathbf{Z} \to \mathscr{O}_X^* \xrightarrow{\cdot n} \mathscr{O}_X^* \to 0.$$

The other one is mapping $H^2(X, \mathbf{Z})$ to $\mathrm{Br}'(X)$ by taking $x \in H^2(X, \mathbf{Z})$ to $\frac{1}{n}x \in H^2(X, \mathbf{Z}) \otimes \mathbf{Q}/\mathbf{Z}$, which naturally maps to $\mathrm{Br}'(X)$.

Then, the following diagram commutes:

$$\begin{array}{ccc}
H^2(X, \mathbf{Z}) & \xrightarrow{\bmod n} & H^2(X, \mathbf{Z}/n\mathbf{Z}) \\
 & {\scriptstyle x \mapsto \frac{1}{n}x} \searrow \quad \swarrow {\scriptstyle \varphi} & \\
 & \mathrm{Br}'(X). &
\end{array}$$

Furthermore, if $Y \to X$ is a $\mathbf{P}^{n-1}$-bundle over X, and $t(Y/X)$ and $a(Y/X)$ are the topological and analytic twisting classes associated to it, then we have

$$\varphi(t(Y/X)) = a(Y/X).$$

Proof. Trivial chase through the definitions. The last statement follows from the commutativity of the diagram

$$\begin{array}{ccc} H^1(X, PGL(n)) & \xrightarrow{t} & H^2(X, \mathbf{Z}/n\mathbf{Z}) \\ \| & & \downarrow \\ H^1(X, PGL(n)) & \xrightarrow{a} & \mathrm{Br}'(X), \end{array}$$

which is deduced from the map of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{Z}/n\mathbf{Z} & \longrightarrow & \mathrm{SL}(n) & \longrightarrow & \mathrm{PGL}(n) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathscr{O}_X^* & \longrightarrow & \mathrm{GL}(n) & \longrightarrow & \mathrm{PGL}(n) & \longrightarrow & 0. \end{array}$$

□

Proof. (of Theorem 5.2.1) Let $n = \mathrm{rk}(\mathscr{E})$, and consider the projective bundle associated to $\mathscr{E}$, $Y = \mathrm{Proj}(\mathscr{E}) \to X$. As such, it has an associated topological twisting class, $t = t(Y/X) \in H^2(X, \mathbf{Z}/n\mathbf{Z})$, for which we have $\varphi(t) = \alpha$, where φ is the map from the previous proposition. Since the topological twisting class is natural and is a topological invariant, we must have $t|_{X_s} = t(Y_s/X_s)$ for every $s \in S$. But on X_0, we must have $t|_{X_0} = -c_1(\mathscr{E}_0) \bmod n$ by Proposition 5.2.4. Since the map φ is compatible with restriction, we have

$$\alpha = \varphi(t) = \varphi(t|_{X_0}) = \varphi(-c_1(\mathscr{E}_0) \bmod n) = -\frac{1}{n}c_1(\mathscr{E}_0).$$

□

Remark 5.2.6. In particular, this shows that on X_s we must have

$$\alpha|_{X_s} = -\frac{1}{n}c_1(\mathscr{E}_0)$$

as well.

5.3 Identifying the Obstruction

In this section we compute, for a moduli problem on a K3 surface, the obstruction α to the existence of a universal sheaf (Definition 5.1.8), in terms of the cohomological Fourier-Mukai transform described in Definition 5.1.9.

The setup is, as before, an algebraic K3 surface X, with a fixed polarization $\mathscr{P}$, and v a primitive, isotropic vector in $\tilde{H}^{1,1}(X) \cap \tilde{H}(X, \mathbf{Z})$, such that the moduli space of semistable sheaves of Mukai vector v, $M(v)$, is non-empty and does not contain any properly semistable points. Then, according to Theorem 5.1.6, $M = M(v)$ is a K3 surface, and choosing a quasi-universal sheaf $\tilde{\mathscr{U}}$ we obtain a Fourier-Mukai transform

$$\varphi^{\tilde{\mathscr{U}}^\vee}_{X\to M} : \tilde{H}(X,\mathbf{Q}) \to \tilde{H}(M,\mathbf{Q})$$

(which depends on $\tilde{\mathscr{U}}$).

There is a natural element $\alpha \in \mathrm{Br}(M)$ which expresses the obstruction to the existence of a universal sheaf on $X \times M$. The following theorem gives a precise identification of α in terms of the initial data X and v.

Theorem 5.3.1. *Let $u \in \tilde{H}(X,\mathbf{Z})$ be any vector such that $(u,v) = 1 \bmod n$, (where n is the one defined in Theorem 5.1.12) and let $w = \varphi^{\mathscr{U}^\vee}_{X\to M}(u)$. Denote by w_2 the $H^2(M,\mathbf{Q})$ component of w, and let $\bar{w}$ be the element of $\mathrm{Br}(M)$ that is the image of w_2 under the composite homomorphism*

$$H^2(M,\mathbf{Q}) \to H^2(M,\mathbf{Q}/\mathbf{Z}) \to H^2(M,\mathbf{Q}/\mathbf{Z})/(NS(X)\otimes\mathbf{Q}/\mathbf{Z}) = \mathrm{Br}(M).$$

Then $\bar{w}$ is the obstruction to the existence of a universal sheaf on $X \times M$, as described in Definition 5.1.8.

Remark 5.3.2. Since $\tilde{H}(X,\mathbf{Z})$ is a unimodular lattice, one can always find a vector u satisfying $(u,v) = 1 \bmod n$, so that one can always compute the obstruction $\alpha = \bar{w}$ using this theorem.

Proof. Let's start off by analyzing the Fourier-Mukai transform when the moduli problem is in fact fine, but we are using a quasi-universal sheaf instead of the universal sheaf.

So assume that we are in the above setup, with $n = 1$, and therefore there exists a universal sheaf $\mathscr{E}$ on $X \times M$. If $\mathscr{O}_X(1)$ is an ample line bundle on X, and k is large enough, then

$$\mathscr{V} = \Phi^{\mathscr{E}^\vee}_{X\to M}(\mathscr{O}_X(-k))[2]$$

is a locally free sheaf on M. Indeed, this is a relative version of the fact that on a smooth, projective scheme of dimension r, we have

$$\mathrm{Ext}^{r-i}(\mathscr{F}(k),\omega) = H^i(\mathscr{F}(k)) = 0$$

for any coherent sheaf $\mathscr{F}$, $i > 0$, and large enough k. So let k be large enough to have $\mathscr{V}$ locally free. We are interested in the quasi-universal sheaf $\mathscr{U} = \mathscr{E}\otimes\pi_M^*\mathscr{V}$, of similitude $\mathrm{rk}(\mathscr{V})$.

It is important to note that we have

$$\begin{aligned} v(\mathscr{V}) &= v(\Phi^{\mathscr{E}^\vee}_{X\to M}(\mathscr{O}_X(-k))[2]) \\ &= v(\Phi^{\mathscr{E}^\vee}_{X\to M}(\mathscr{O}_X(-k))) \\ &= \varphi^{\mathscr{E}^\vee}_{X\to M}(v(\mathscr{O}_X(-k))), \end{aligned}$$

and therefore

$$\begin{aligned}\operatorname{rk}(\mathscr{V}) &= (v(\mathscr{V}), (0,0,\omega_M))\\ &= (\varphi^{\mathscr{E}^\vee}_{X\to M}(v(\mathscr{O}_X(-k))), \varphi^{\mathscr{E}^\vee}_{X\to M}(v))\\ &= (v(\mathscr{O}_X(-k)), v),\end{aligned}$$

because the Fourier-Mukai transform is an isometry.

Let $\mathscr{F}$ be a line bundle on X, anti-ample enough so that

$$\mathscr{G} = \Phi^{\mathscr{U}^\vee}_{X\to M}(\mathscr{F})[2]$$

is a locally free sheaf on M. We have

$$\mathscr{G} = \Phi^{\mathscr{E}^\vee}_{X\to M}(\mathscr{F})[2] \otimes \mathscr{V}$$

by the projection formula, so that

$$c_1(\mathscr{G}) = \operatorname{rk}(\mathscr{V}) \cdot c_1(\Phi^{\mathscr{E}^\vee}_{X\to M}(\mathscr{F})[2]) + \operatorname{rk}(\Phi^{\mathscr{E}^\vee}_{X\to M}(\mathscr{F})[2]) \cdot c_1(\mathscr{V}).$$

But

$$\operatorname{rk}(\Phi^{\mathscr{E}^\vee}_{X\to M}(\mathscr{F})[2]) = (v(\mathscr{F}), v)$$

by a computation identical to the one above, and therefore

$$\begin{aligned}\varphi^{\mathscr{U}^\vee}_{X\to M}(v(\mathscr{F}))_2 &= \frac{1}{\operatorname{rk}(\mathscr{V})} c_1(\mathscr{G})\\ &= \frac{(v(\mathscr{F}), v)}{\operatorname{rk}(\mathscr{V})} c_1(\mathscr{V}) \ + \ \text{integral part},\end{aligned}$$

where the subscript 2 denotes the $H^2(M, \mathbf{Q})$-part.

We now return to the case when $n \neq 1$, and we start by proving that $\bar{w}$ is independent of the choice of $\mathscr{U}$ and u. If $\mathscr{U}$ and $\mathscr{U}'$ are two quasi-universal sheaves, then there exists another quasi-universal sheaf $\mathscr{U}''$ and vector bundles $\mathscr{F}$ and $\mathscr{F}'$ on M such that

$$\mathscr{U}'' \cong \mathscr{U} \otimes \pi_M^* \mathscr{F} \cong \mathscr{U}' \otimes \pi_M^* \mathscr{F}',$$

(remark before Theorem A.5 in [33]), so it is enough to prove that $\bar{w}$ is the same when using $\mathscr{U}$ and $\mathscr{U}' = \mathscr{U} \otimes \pi_M^* \mathscr{F}$ (for some vector bundle $\mathscr{F}$ on M) in computing the Fourier-Mukai transform.

A standard calculation shows that

$$\varphi^{\mathscr{U}'^\vee}_{X\to M}(u)_2 = \varphi^{\mathscr{U}^\vee}_{X\to M}(u)_2 + \frac{(u,v)}{\operatorname{rk}\mathscr{F}} c_1(\mathscr{F}),$$

(see [25, Definition 6.1.12], or the previous computations), and therefore

$$\varphi^{\mathscr{U}'^\vee}_{X\to M}(u)_2 = \varphi^{\mathscr{U}^\vee}_{X\to M}(u)_2$$

in $H^2(M,\mathbf{Q})/(NS(M)\otimes\mathbf{Q})$. Thus $\bar{w}$ is independent of the choice of $\mathscr{U}$, and hence from now on we'll omit the superscript $\mathscr{U}^\vee$ in $\varphi^{\mathscr{U}^\vee}_{X\to M}$.

Let $u,u'\in\tilde{H}(X,\mathbf{Z})$ be such that

$$(u,v)=(u',v)=1 \bmod n.$$

Then $(u-u',v)$ is divisible by n, and since n is the greatest common divisor of (t,v) where t runs over all algebraic vectors in $\tilde{H}(M,\mathbf{Z})$ we conclude that there exists a $t\in\tilde{H}(M,\mathbf{Z})$, algebraic, and such that $(u-u'-t,v)=0$. Using [33, Theorem 1.5], we conclude that

$$\varphi_2(u)-\varphi_2(u')-\varphi_2(t)$$

is integral. Also, since t was algebraic, $\varphi(t)$ is also algebraic; therefore, $\varphi_2(u)$ and $\varphi_2(u')$ must be equal in

$$\tilde{H}(M,\mathbf{Q})/((\tilde{H}^{1,1}(M)\cap\tilde{H}(M,\mathbf{Q}))+\tilde{H}(M,\mathbf{Z})),$$

which is $\mathrm{Br}(M)$. Thus $\bar{w}$ is also independent of the choice of u.

From here on proceed as in the proof of (2) and (3) of Theorem 1.5 in [33] (pp. 385-386). Fix an ample line bundle $\mathscr{O}_X(1)$ on X, let l be its class in $H^2(X,\mathbf{Z})$, and let v_2 be the projection of v onto $H^2(X,\mathbf{Z})$. Let T_0 be the moduli space of K3 surfaces X' with an isometric marking $i: H^2(X',\mathbf{Z})\to H^2(X,\mathbf{Z})$ such that i^*v_2 and i^*l are algebraic on X', and $i^*\mathscr{O}_X(1)$ is ample. T_0 contains a point 0 corresponding to (X,id) and is of dimension 18 or 19 according to whether l and v_2 are linearly independent in $H^2(X,\mathbf{Z})$ or not. Over an analytic neighborhood of the origin in T_0 we have a smooth family $\mathscr{X}\to T_0$. On each fiber X_t ($t\in T_0$), fix the polarization given by $i^*\mathscr{O}_X(1)$, and consider the moduli space M_t of stable sheaves on X_t, having Mukai vector i^*v, with respect to this polarization.

The spaces M_t form a flat family $\mathscr{M}\to T_0$, which is smooth over some analytic neighborhood of the origin in T_0, and by restricting T_0 further we can also assume that M_t is compact for all $t\in T_0$. On the other hand, since the set of points $t\in T_0$ for which M_t is a fine moduli space is dense in T_0, we can assume that there exists $1\in T_0$ for which M_1 is the fine moduli space of semistable sheaves on X_1. (M_t was taken to be the moduli space of *stable* sheaves on X_t; however, the compactness assumption on M_t implies that there are no properly semistable points in M_t, so that the notions of semistable and stable coincide.)

Using Theorems 3.3.2 and 3.3.4, we get an $\alpha\in\mathrm{Br}(\mathscr{M})$ and an α-twisted universal sheaf $\mathscr{E}$ on $\mathscr{X}\times_{T_0}\mathscr{M}$. By restricting T_0 even further (to an open set of a component of T_0), we can assume T_0 is smooth, and $\mathrm{Br}(T_0)=0$. Using this last property, we can find a relatively ample sheaf $\mathscr{O}_{\mathscr{X}}(1)$ on $\mathscr{X}$, which restricts to the fixed polarization $i^*\mathscr{O}_X(1)$ on each fiber of $\mathscr{X}$.

As before, consider

$$\mathscr{V}=\mathbf{R}\pi_{\mathscr{M},*}(\mathbf{L}\pi^*_{\mathscr{X}}\mathscr{O}(-k)\overset{\mathbf{L}}{\otimes}\mathscr{E}^\vee)[2]$$

for some $k \gg 0$. By the same argument as before, $\mathscr{V}$ consists of a single α^{-1}-twisted locally free sheaf. (The twisting is by α^{-1} because we have dualized $\mathscr{E}$.) We are thus in the situation studied in the previous section: we have a smooth family $\mathscr{M} \to T_0$ over a smooth base, an element $\alpha \in \mathrm{Br}(\mathscr{M})$ such that over M_1 we have $\alpha^{-1}|_{M_1} = 0$, and an α^{-1}-lffr $\mathscr{V}$ over $\mathscr{M}$. Under these circumstances, we can apply Theorem 5.2.1 and conclude that

$$\alpha^{-1} = -\frac{1}{\mathrm{rk}(\mathscr{V}_1)} c_1(\mathscr{V}_1),$$

and hence

$$\alpha = \frac{1}{\mathrm{rk}(\mathscr{V}_1)} c_1(\mathscr{V}_1),$$

where $\mathscr{V}_1$ is some gluing of $\mathscr{V}|_{\mathscr{M}_1}$ to a global locally free sheaf (under the identifications discussed in Section 5.2).

As a quasi-universal sheaf on $\mathscr{X} \times_{T_0} \mathscr{M}$ we can take

$$\mathscr{U} = \mathscr{E} \otimes \pi_{\mathscr{M}}^* \mathscr{V},$$

which naturally glues to a sheaf on $\mathscr{X} \times_{T_0} \mathscr{M}$. Form here on this will be the universal sheaf we'll be using for defining the Fourier-Mukai transforms.

Of course, as an element of $\mathrm{Br}(M_1)$, α is zero (being algebraic), as expected from the fact that the moduli problem is fine at $t = 1$. However, the right hand side of the equality

$$\alpha = \frac{1}{\mathrm{rk}(\mathscr{V}_1)} c_1(\mathscr{V}_1),$$

is defined solely in topological terms, and is therefore constant in the family $\mathscr{M} \to T_0$. Also, since $\mathscr{U}$ is defined in the whole family, $\varphi^{\mathscr{U}^\vee}_{X_t \to M_t}(u)$ is constant as a function of $t \in T_0$, for any $u \in \tilde{H}(X, \mathbf{Z})$. Therefore, if we prove that

$$\varphi^{\mathscr{U}^\vee}_{X_1 \to M_1}(u)_2 = \frac{(u, v)}{\mathrm{rk}(\mathscr{V}_1)} c_1(\mathscr{V}_1) \ + \ \text{integral part},$$

for some $u \in \tilde{H}(X_1, \mathbf{Z})$ with $(u, v) = 1 \bmod n$, then we can conclude that the same equality holds at $t = 0$, after identifying $\tilde{H}(M_0, \mathbf{Z}) \cong \tilde{H}(M_1, \mathbf{Z})$. The right hand side of the above equality reduces to $(u, v)\alpha$ in $\mathrm{Br}(M_0)$, therefore using the fact that α is n-torsion (Theorem 5.1.12) we get

$$\bar{w} = \varphi^{\mathscr{U}^\vee}_{X_1 \to M_1}(u)_2 = (u, v)\alpha = \alpha.$$

But let's now look at what is happening over $t = 1 \in T_0$; we are precisely in the situation discussed in the beginning of the proof: the moduli problem is fine, and we have a polarization $\mathscr{O}_{X_1}(1)$ of which we take a sufficiently negative multiple and apply the Fourier-Mukai transform to get $\mathscr{V}_1$ on M_1. Therefore, if we consider a line bundle $\mathscr{F}$ on X_1, which is sufficiently anti-ample, we have

$$\varphi^{\mathscr{U}^\vee}_{X_1 \to M_1}(v(\mathscr{F}))_2 = \frac{(v(\mathscr{F}), v)}{\mathrm{rk}(\mathscr{V}_1)} c_1(\mathscr{V}_1) \ + \ \text{integral part}.$$

All we need to do now is show that we can find an $\mathscr{F}$ as above for which $(v(\mathscr{F}), v) = 1 \bmod n$. Indeed, we can then take $u = v(\mathscr{F})$ (using the fact the we have the freedom of choosing u as long as $(u, v) = 1 \bmod n$) and get

$$\begin{aligned}\varphi^{\mathscr{U}^\vee}_{X_1\to M_1}(u)_2 &= \varphi^{\mathscr{U}^\vee}_{X_1\to M_1}(v(\mathscr{F}))_2 \\ &= \frac{(v(\mathscr{F}), v)}{\mathrm{rk}(\mathscr{V}_1)} c_1(\mathscr{V}_1) \;+\; \text{integral part} \\ &= \frac{(u, v)}{\mathrm{rk}(\mathscr{V}_1)} c_1(\mathscr{V}_1) \;+\; \text{integral part},\end{aligned}$$

which is what we want.

Since we assume that $n = 1$ at $t = 1$ we can find a line bundle $\mathscr{F}$ on X_1 with $c_1(\mathscr{F}) = d$ such that $\gcd(v_0, v_2.d, v_4) = 1$ (where $v = (v_0, v_2, v_4)$). Since n divides v_0 and v_4, we must have $\gcd(v_2.d, n) = 1$. By possibly replacing $\mathscr{F}$ by some tensor power of itself, we can assume $(v_2.d) = 1 \bmod n$. Then we have

$$v(\mathscr{F}).v = v_2.d - v_4 - v_0 - v_0(d.d) = 1 \bmod n,$$

as desired. Twisting $\mathscr{F}$ by $\mathscr{O}_{X_1}(-k)$ for k large enough, we get the $\mathscr{F}$ we were looking for, thus finishing the proof. (Note that twisting by $\mathscr{O}(-k)$ does not change $(v(\mathscr{F}), v) \bmod n$, because $(c_1(\mathscr{O}_{X_1}(1)).v_2)$ must be divisible by n, since it equals $(c_1(\mathscr{O}_{X_0}(1)).v_2)$.)

□

5.4 The Map on Brauer Groups

We have seen in the previous section that we can identify the obstruction α solely in terms of the Fourier-Mukai transform. In this section we prove that there is in fact a map between the Brauer groups of X and of M, whose kernel is generated by α.

Lemma 5.4.1. *Let M be a K3 surface, and let T_M be the transcendental lattice of M, i.e. the orthogonal lattice to $\mathrm{NS}(M)$ inside $H^2(M, \mathbf{Z})$ (endowed with the intersection pairing). Then there is a natural isomorphism*

$$\mathrm{Br}(M) = T_M^\vee \otimes_{\mathbf{Z}} \mathbf{Q}/\mathbf{Z},$$

where $T_M^\vee$ is the dual lattice to T_M.

Proof. Consider the exact sequence of Proposition 1.1.3,

$$0 \to \mathrm{NS}(M) \otimes \mathbf{Q}/\mathbf{Z} \to H^2(M, \mathbf{Q}/\mathbf{Z}) \to \mathrm{Br}(M) \to 0,$$

(note that for a K3 M we have $\mathrm{NS}(M) = \mathrm{Pic}(M)$); using the universal coefficient theorem, we also know that

$$H^2(M, \mathbf{Q}/\mathbf{Z}) = H^2(M, \mathbf{Z}) \otimes \mathbf{Q}/\mathbf{Z},$$

and therefore

$$\operatorname{Br}(M) = (H^2(M, \mathbf{Z}) / \operatorname{NS}(M)) \otimes \mathbf{Q}/\mathbf{Z}.$$

Now consider the map

$$p : H^2(M, \mathbf{Z}) \to T_M^\vee$$

given by

$$p(v)(w) = (v, w)$$

for $w \in T_M$. It is obviously a linear map, it is surjective (because $H^2(M, \mathbf{Z})$ is unimodular) and its kernel consists of all $v \in H^2(M, \mathbf{Z})$ such that $(v, w) = 0$ for all $w \in T_M$, which is $T_M^\perp$. Since $\operatorname{NS}(M)$ is a primitive sublattice in $H^2(M, \mathbf{Z})$, we have

$$T_M^\perp = (\operatorname{NS}(M)^\perp)^\perp = \operatorname{NS}(M),$$

and therefore we conclude that

$$T_M^\vee = H^2(M, \mathbf{Z}) / \operatorname{NS}(M),$$

thus obtaining the result of the lemma. □

Lemma 5.4.2. *Let H be a unimodular lattice, let N be a primitive sublattice in H, and let n be a fixed integer. Define N_n to be the sublattice of N consisting of all $x \in N$ such that (x, y) is divisible by n for all $y \in N$. Also, let $T = N^\perp$, and let T_n be defined just as N_n. Then there is a natural group isomorphism $N_n/nN \cong T_n/nT$ defined by sending an element $v \in N_n/nN$ to the unique element λ of T_n/nT such that $v - \lambda$ is divisible by n inside H.*

Proof. Begin by proving that for every $v \in N_n$ there exists a $\lambda \in T_n$ such that $v - \lambda$ is divisible by n. The idea is from [32]: consider the functional

$$\frac{1}{n} v^\vee = \frac{1}{n}(v, \cdot) \in N^\vee$$

(it takes integer values because $v \in N_n$). Since N is a primitive sublattice in the unimodular lattice H, there exists $w \in H$ such that $w^\vee|_N = \frac{1}{n} v^\vee$. Thus $(v - nw, x) = 0$ for all $x \in N$, and thus $\lambda = v - nw \in T$. Obviously, we have $v - \lambda$ divisible by n inside H (equaling nw). The fact that λ is in fact in T_n is immediate:

$$(\lambda, y) = (\lambda - v, y) = -n(w, y)$$

for any $y \in T$.

It is obvious that the map is well-defined, bijective, and a group homomorphism, so we're set. □

Theorem 5.4.3. *Let X be a K3 surface, $v \in \tilde{H}^{1,1}(X) \cap \tilde{H}(X, \mathbf{Z})$ a primitive isotropic Mukai vector, and assume that the moduli space $M = M(v)$ of semistable sheaves on X of Mukai vector v is non-empty and does not contain any properly semistable points, so that M is again a K3 surface by Theorem 5.1.6. Then there is a natural exact sequence*

$$0 \to \mathbf{Z}/n\mathbf{Z} \to \mathrm{Br}(M) \to \mathrm{Br}(X) \to 0,$$

(where n is the one defined in Theorem 5.1.12), induced by the map $\varphi : T_X \to T_M$ of the same theorem. The cyclic kernel of the map $\mathrm{Br}(M) \to \mathrm{Br}(X)$ is generated by the obstruction $\alpha \in \mathrm{Br}(M)$ to the existence of a universal sheaf on $X \times M$, and this particular generator of the kernel is in fact singled out by the construction of the exact sequence.

Remark 5.4.4. The last statement of the proposition should be interpreted as follows: the exact sequence between the Brauer groups is fully determined by just knowing the cyclic subgroup $\langle v \rangle$ of $\tilde{H}^{1,1}(X, \mathbf{Z}/n\mathbf{Z})$ generated by v. However, having singled out a particular generator v of this subgroup, this singles out a particular generator α of the kernel of the map $\mathrm{Br}(M) \to \mathrm{Br}(X)$, and it is this particular generator that is the obstruction to the existence of a universal sheaf on $X \times M$ that is asserted in Definition 5.1.8. See the actual proof for more details.

Proof. Using Lemma 5.4.2, we obtain a natural correspondence between elements of $H^{1,1}(X, \mathbf{Z})_n / nH^{1,1}(X, \mathbf{Z})$ and elements of $T_{X,n}/nT_X$. Since $v \in H^{1,1}(X, \mathbf{Z})_n$, we get an element $\lambda \in T_{X,n}$ such that $v - \lambda$ is divisible by n in $\tilde{H}(X, \mathbf{Z})$. (Here, λ is uniquely defined by v up to an n-multiple of T_X.)

Consider the exact sequence

$$0 \to T_X \xrightarrow{\varphi} T_M \to \mathbf{Z}/n\mathbf{Z} \to 0$$

from Theorem 5.1.12, where φ is a Fourier-Mukai transform defined by means of quasi-universal sheaf (it does not depend on this choice). The cokernel of the first map is generated by the element $\mu = \frac{1}{n}\varphi(\lambda)$. Dualizing, we get

$$0 \to T_M^\vee \to T_X^\vee \to \mathbf{Z}/n\mathbf{Z} = \mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Z}/n\mathbf{Z}, \mathbf{Z}) \to 0,$$

where the last map is

$$\psi \in T_X^\vee \mapsto \psi(\lambda) \bmod n.$$

Tensoring this exact sequence with $\mathbf{Q}/\mathbf{Z}$ we get

$$0 \to \mathbf{Z}/n\mathbf{Z} = \mathrm{Tor}_1^{\mathbf{Z}}(\mathbf{Z}/n\mathbf{Z}, \mathbf{Z}) \to T_M^\vee \otimes \mathbf{Q}/\mathbf{Z} \to T_X^\vee \otimes \mathbf{Q}/\mathbf{Z} \to 0,$$

or, in view of Lemma 5.4.1,

$$0 \to \mathbf{Z}/n\mathbf{Z} \to \mathrm{Br}(M) \to \mathrm{Br}(X) \to 0.$$

In order to identify the kernel, note that the map φ becomes invertible when tensored with $\mathbf{Q}$, and the same holds also for $\varphi^\vee$. If $\psi \in T_X^\vee$ is such that $\psi(\lambda) = 1 \bmod n$, then $\psi' = \psi \circ (\varphi \otimes \mathbf{Q})^{-1}$ is an element of $T_M^\vee \otimes \mathbf{Q}$ whose image in $T_M^\vee \otimes \mathbf{Q}/\mathbf{Z}$ generates the kernel. Indeed, its image in $T_X^\vee \otimes \mathbf{Q}$ is ψ, which is integral (and hence becomes zero in $T_X^\vee \otimes \mathbf{Q}/\mathbf{Z}$), so it belongs to the kernel. Since ψ generates the cokernel of the map $T_M^\vee \to T_X^\vee$, ψ' generates the kernel of $T_M^\vee \otimes \mathbf{Q}/\mathbf{Z} \to T_X^\vee \otimes \mathbf{Q}/\mathbf{Z}$.

It is easy to see that all these constructions are natural, and that having chosen a particular generator μ for the cokernel of $T_X \to T_M$, we have singled out a generator of the kernel of $\mathrm{Br}(M) \to \mathrm{Br}(X)$.

Now consider the result of Theorem 5.3.1: it states that in order to obtain α, we have to find $u \in \tilde{H}(X, \mathbf{Z})$ such that $(u, v) = 1 \bmod n$, and consider the image

$$\alpha = \varphi(u) \in \mathrm{Br}(M).$$

Tracing through the identifications of $\mathrm{Br}(M)$ with $T_M^\vee \otimes \mathbf{Q}/\mathbf{Z}$, we find that $\varphi(u) \in \mathrm{Br}(M)$ corresponds to the functional $\psi' \in T_M^\vee \otimes \mathbf{Q}/\mathbf{Z}$ given by

$$\psi'(\,\cdot\,) = (\varphi(u), \,\cdot\,),$$

where $\varphi(u)$ is now viewed as an element of $\tilde{H}(M, \mathbf{Q})$. But it is easy to see that

$$\psi'(\varphi(\lambda)) = (\varphi(u), \varphi(\lambda)) = (u, \lambda) = 1 \bmod n,$$

the last equality following from the fact that $v - \lambda$ is divisible by n by the construction of λ, and $(u, v) = 1 \bmod n$.

Therefore

$$\psi(\,\cdot\,) = \psi' \circ \varphi(\,\cdot\,) = (\varphi(u), \varphi(\,\cdot\,)) = (u, \,\cdot\,)$$

is an integral functional on T_X, satisfying $\psi(\lambda) = 1 \bmod n$, and thus ψ' is the distinguished generator of the kernel of the map

$$T_M^\vee \otimes \mathbf{Q}/\mathbf{Z} \to T_X^\vee \otimes \mathbf{Q}/\mathbf{Z}.$$

We conclude that α, the obstruction to the existence of a universal sheaf on $X \times M$, is precisely the distinguished generator of the kernel of the map $\mathrm{Br}(M) \to \mathrm{Br}(X)$. $\square$

Remark 5.4.5. This result should be contrasted with [11, 5.3.5], where the authors observe that in the case of an elliptically fibered K3 X, $\mathrm{Br}(X) = \mathrm{Br}(J)$, where J is the relative Jacobian. Of course, J can be viewed as a moduli space of semistable sheaves on X (Chapter 4), so that one should in fact get

$$0 \to \mathbf{Z}/n\mathbf{Z} \to \mathrm{Br}(J) \to \mathrm{Br}(X) \to 0,$$

with the kernel generated by $\alpha \in \mathrm{Br}(J)$ which is the obstruction to the existence of a universal sheaf on $X \times J$, which coincides with the element $\alpha \in Ш_S(J)$ that represents X. The explanation for this apparent mismatch is the following: $\mathrm{Br}(X)$

and Br(J) are isomorphic, as abstract groups. Indeed, they are isomorphic to $T_X^\vee \otimes \mathbf{Q}/\mathbf{Z}$ and $T_J^\vee \otimes \mathbf{Q}/\mathbf{Z}$, respectively, and T_X is isomorphic to T_J as abstract groups. However, if one takes into account the lattice structure on T_X and T_J, one obtains the more precise formulation of Theorem 5.4.3.

Remark 5.4.6. Identifying $\operatorname{Br}(M)$ with $T_M^\vee \otimes \mathbf{Q}/\mathbf{Z}$, we can view any $\alpha \in \operatorname{Br}(M)$ as a functional on T_M, with values in $\mathbf{Q}/\mathbf{Z}$. Theorem 5.4.3 states that T_X is Hodge isometric (via the Fourier-Mukai transform) to $\operatorname{Ker}(\alpha)$, viewed as a Hodge sublattice of T_M. The striking thing about this is the following: if one takes α^k (we keep the multiplicative notation for the operation in $\operatorname{Br}(M)$) for $\gcd(k, \operatorname{ord}(\alpha)) = 1$, then obviously $\operatorname{Ker}(\alpha^k) = \operatorname{Ker}(\alpha)$. Thus, if we have X and X' K3 surfaces, and $v \in \tilde{H}^{1,1}(X, \mathbf{Z})$, $v' \in \tilde{H}^{1,1}(X', \mathbf{Z})$ such that $M_X(v) = M = M_{X'}(v')$, and $\operatorname{Obs}(M, v) = \alpha$, $\operatorname{Obs}(M', v') = \alpha^k$, then T_X is Hodge isometric to $T_{X'}$. This will be used in the following section.

5.5 Relationship to Derived Categories

Whenever one deals with a moduli problem of semistable sheaves on a K3 X, and the moduli space M is 2-dimensional and has no properly semistable points, then one gets an equivalence of derived categories between some derived category of twisted sheaves on M, and the derived category of (usual) coherent sheaves on X. This is the content of the following theorem.

Theorem 5.5.1. *Let X be a K3 surface, let $v \in \tilde{H}^{1,1}(X, \mathbf{Z})$ be a primitive, isotropic Mukai vector, and assume that there is a polarization on X such that the moduli space $M = M_X(v)$ of semistable sheaves with Mukai vector v is non-empty and does not contain any properly semistable points. Let $\alpha = \operatorname{Obs}(X, v)$. Then we have*

$$\mathbf{D}^b_{\mathrm{coh}}(X) \cong \mathbf{D}^b_{\mathrm{coh}}(M, \alpha^{-1}).$$

Proof. Using Theorem 3.3.2 we conclude that there exists a $\pi_M^*\alpha$-twisted universal sheaf $\mathscr{U}$ on $X \times M$. Therefore we can consider the integral functor

$$F = \Phi^{\mathscr{U}}_{M \to X} : \mathbf{D}^b_{\mathrm{coh}}(M, \alpha^{-1}) \to \mathbf{D}^b_{\mathrm{coh}}(X)$$

defined by $\mathscr{U}$, and in order to check that F is an equivalence, all we have to do is check that the conditions of Theorem 3.2.1 are satisfied.

For a point $[\mathscr{F}] \in M$ that corresponds to a stable sheaf $\mathscr{F}$ on X with Mukai vector v, we have $F(\mathscr{O}_{[\mathscr{F}]}) = \mathscr{F}$, from the very definition of a universal sheaf. Thus the conditions of Theorem 3.2.1 can be restated as:

1. if $\mathscr{F} \neq \mathscr{G}$, $\operatorname{Ext}^i(\mathscr{F}, \mathscr{G}) = 0$ for all i;

2. $\operatorname{Hom}(\mathscr{F}, \mathscr{F}) = \mathbf{C}$, and $\operatorname{Ext}^i(\mathscr{F}, \mathscr{F}) = 0$ for all $i < 0$ or $i > 2$;

3. $\mathscr{F} \otimes \omega_X \cong \mathscr{F}$;

for all $\mathscr{F}, \mathscr{G}$ stable sheaves on X with Mukai vector v.

Since X is smooth of dimension 2, $\mathrm{Ext}^i(\mathscr{F},\mathscr{G}) = 0$ for $i < 0$ or $i > 2$ anyway. Since $\mathscr{F}$ and $\mathscr{G}$ are stable, $\mathrm{Hom}(\mathscr{F},\mathscr{G}) = 0$ if $\mathscr{F} \neq \mathscr{G}$. Therefore $\mathrm{Ext}^2(\mathscr{F},\mathscr{G}) = 0$ by Serre duality (use the fact that ω_X is trivial). Since

$$\chi(\mathscr{F},\mathscr{G}) = (v(\mathscr{F}), v(\mathscr{G})) = (v, v) = 0$$

by Lemma 3.1.7, we conclude that $\mathrm{Ext}^1(\mathscr{F},\mathscr{G}) = 0$ as well. $\mathrm{Hom}(\mathscr{F},\mathscr{F}) = \mathbf{C}$ because a stable sheaf is simple. Finally, the last condition holds automatically because ω_X is trivial. □

As an application of this we prove the following:

Theorem 5.5.2. *Let M be a K3 surface, let $\alpha \in \mathrm{Br}(M)$, and let $k \in \mathbf{Z}$ be such that $\gcd(k, \mathrm{ord}(\alpha)) = 1$. Assume that there exist K3 surfaces X and X', and Mukai vectors $v \in \tilde{H}^{1,1}(X, \mathbf{Z})$, $v' \in \tilde{H}^{1,1}(X', \mathbf{Z})$ such that we have isomorphisms*

$$M \cong M_X(v) \cong M_{X'}(v'),$$

and such that under these isomorphisms, $\mathrm{Obs}(X, v)^{-1} = \alpha$ and $\mathrm{Obs}(X', v')^{-1} = \alpha^k$. Then we have

$$\mathbf{D}^b_{\mathrm{coh}}(M, \alpha) \cong \mathbf{D}^b_{\mathrm{coh}}(M, \alpha^k).$$

Proof. We have seen (Remark 5.4.6) that under these assumptions T_X is Hodge isometric to $T_{X'}$. Orlov's theorem (Theorem 5.1.14) then tells us that

$$\mathbf{D}^b_{\mathrm{coh}}(X) \cong \mathbf{D}^b_{\mathrm{coh}}(X').$$

But Theorem 5.5.1 shows that $\mathbf{D}^b_{\mathrm{coh}}(M, \alpha) \cong \mathbf{D}^b_{\mathrm{coh}}(X)$ and $\mathbf{D}^b_{\mathrm{coh}}(M, \alpha^k) \cong \mathbf{D}^b_{\mathrm{coh}}(X')$, and thus

$$\mathbf{D}^b_{\mathrm{coh}}(M, \alpha) \cong \mathbf{D}^b_{\mathrm{coh}}(M, \alpha^k).$$

□

It is tempting (especially in view of the results of Chapter 6) to conjecture that the conclusion of the previous theorem holds more generally:

Conjecture 5.5.3. *For any K3 surface M, and any twisting $\alpha \in \mathrm{Br}(M)$, we have*

$$\mathbf{D}^b_{\mathrm{coh}}(M, \alpha) \cong \mathbf{D}^b_{\mathrm{coh}}(M, \alpha^k)$$

for any k with $\gcd(k, \mathrm{ord}(\alpha)) = 1$.

The difficulty in proving such a theorem would lie on one hand in the fact that if one takes $T_X = \mathrm{Ker}(\alpha)$, then T_X may not be embeddable in the K3 lattice $\mathscr{L}_{\mathrm{K3}}$. Even if one assumes this embeddability condition, there are problems with finding the Mukai vector v.

Remark 5.5.4. There is one more thing to note here: having started with M and α, an easy way to find X would be to consider $\mathbf{D}^b_{\mathrm{coh}}(M, \alpha)$, and take X to be the "spectrum" of this derived category (see [39] for details on the spectrum of a triangulated category and related topics). This would work, of course, only if X existed, and by Theorem 5.4.3 we know that in this case $T_X = \mathrm{Ker}(\alpha)$. This would mean in particular that $\mathrm{Ker}(\alpha)$ can be primitively embedded in $\mathscr{L}_{\mathrm{K3}}$. This does not in general hold, but Theorem 5.4.3 gives us *anyway* a candidate for T_X! The conclusion of this is that $\mathbf{D}^b_{\mathrm{coh}}(M, \alpha)$ should be viewed as a derived category of a "virtual" K3 X, where X itself might not necessarily exist, but it makes sense to speak of its transcendental lattice.

Another interesting conjecture which relates to these topics is the following:

Conjecture 5.5.5. *If X and X' are K3 surfaces with $\mathbf{D}^b_{\mathrm{coh}}(X) \cong \mathbf{D}^b_{\mathrm{coh}}(X')$, then Theorem 5.4.3 gives us a natural isomorphism $\varphi : \mathrm{Br}(X) \cong \mathrm{Br}(X')$. Is it then the case that $\mathbf{D}^b_{\mathrm{coh}}(X, \alpha) \cong \mathbf{D}^b_{\mathrm{coh}}(X', \varphi(\alpha))$?*

Chapter 6

Elliptic Calabi-Yau Threefolds

In this chapter we pursue further the results of Chapter 4, to the case of some elliptic fibrations with singular fibers. We are mainly interested in elliptic Calabi-Yau threefolds, and we only consider a number of generic examples here. By "generic" we mean that the singularities of the fibers are the simplest possible to still get a Calabi-Yau: the discriminant locus Δ is a curve with only nodes and cusps as singularities, and the curve over the generic point of Δ is a rational curve with one node (I_1). (See Section 6.1 for details on this topic.)

We provide three examples of such fibrations in Section 6.2. The first one is folklore, but the second and third ones are, to the best of my knowledge, new.

Given a fibration with the above properties, our first concern is understanding the relative Jacobian, which is defined as a relative moduli space (Section 6.4). In order to do this, we need to understand moduli spaces of semistable sheaves on the fibers of X, and we do this in Section 6.3. Having done that, we go on to study the relative Jacobian in Section 6.4. We note here an interesting phenomenon: although the space X we start with is smooth, the relative Jacobian has singularities. These singularities can not be removed without losing the Calabi-Yau property (J has trivial canonical bundle, but a small – or crepant – resolution does not exist in general).

The existence of these singularities would seem to prevent us from finding an equivalence of derived categories (since derived categories "detect smoothness", as well as triviality of canonical bundle), but by working in the analytic category we are able to bypass this problem. Indeed, in this category we find a small resolution $\bar{J}$ of the singularities of J, and we are able to show that there exists a twisted "pseudo-universal" sheaf on $X \times_S \bar{J}$ (where S is the base of the fibrations). This twisted pseudo-universal sheaf is equal to the usual twisted universal sheaf over the stable part of $\bar{J}$ (which equals the stable part of J), but also parametrizes *some* of the semistable sheaves on X, over the properly semistable part of $\bar{J}$ (which is the resolution of the singular, semistable points of J). Using this pseudo-universal sheaf, we can define an integral transform, which turns out to be an equivalence. This is done in Section 6.5.

We then note, in Section 6.6 another occurrence of the same surprising phenomenon observed in Section 5.5: on $\bar{J}$, we get an equivalence of derived categories

$$\mathbf{D}^b_{\mathrm{coh}}(\bar{J}, \alpha) \cong \mathbf{D}^b_{\mathrm{coh}}(\bar{J}, \alpha^k),$$

where $\alpha \in \mathrm{Br}(\bar{J})$ is the twisting associated to X via Ogg-Shafarevich theory, and $\gcd(k, \mathrm{ord}(\alpha)) = 1$.

In the next section we discuss the relationship of derived categories and the Torelli problem for Calabi-Yau threefolds. We show that Calabi-Yau threefolds with equivalent derived categories have the same Hodge data, and hence they provide a counterexample to the generalization of Torelli from K3 surfaces. In particular, we provide such an example based on the elliptic Calabi-Yau's studied in Section 6.2.

We finish this chapter with a section on the relationship of these examples with the example of Vafa-Witten ([43]) and Aspinwall-Morrison-Gross ([1]). We give a quick overview of their example, and suggest a possible explanation for the occurrence of stable singularities in string theory.

6.1 Generic Elliptic Calabi-Yau Threefolds

In this section we define the objects of study in this chapter: a certain class of elliptic Calabi-Yau three-folds, in which we have very good control over the degenerations of the elliptic fibers. The results in this section could probably be easily generalized to arbitrary elliptic fibrations without multiple fibers, but since the results here are intended only to give life to some large class of examples, we'll limit ourselves to this sufficiently general situation.

Recall the definition of a Calabi-Yau threefold:

Definition 6.1.1. A smooth, projective algebraic variety or analytic space X is called a Calabi-Yau threefold if it has (complex) dimension 3, and if it satisfies $K_X = 0$ and $H^1(X, \mathscr{O}_X) = 0$. (Here, and in the future, K_X denotes the canonical divisor of X.)

We will be focused on studying the case of complex threefolds, in which case the Hodge structure of X will come into play. It is described by the Hodge diamond

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{2,2} & & 0 & \\ 1 & & h^{2,1} & & h^{1,2} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1. & & & \end{array}$$

Of course, $h^{1,1} = h^{2,2}$, and $h^{2,1} = h^{1,2}$.

The following definitions are from [13]:

Definition 6.1.2. A projective morphism $f : X \to S$ of algebraic varieties or analytic spaces is called an elliptic fibration if its generic fiber E is a regular curve of genus one and all fibers are geometrically connected. If X is a Calabi-Yau threefold, then f is called an elliptic Calabi-Yau threefold.

Remark 6.1.3. Most of the time we'll abuse the notation, by saying "let X be an elliptic Calabi-Yau threefold." By this we mean "let X be a Calabi-Yau threefold, and let $f : X \to S$ be an elliptic fibration on X."

Definition 6.1.4. Let $f : X \to S$ be an elliptic fibration. The locus $\Delta \subseteq S$ of points $s \in S$ such that f is not smooth at some point $x \in X$ with $f(x) = s$, is called the *discriminant locus* of f. It is a Zariski closed subset of S. The closed subset of Δ

$$\Delta^m = \{s \in S \mid f \text{ is not smooth at any } x \in f^{-1}(s)\}$$

is called the multiple locus of f. A fiber over a point $s \in \Delta^m$ is called a *multiple fiber*.

We will only be interested in studying elliptic fibrations without multiple fibers, so from now on we make the assumption that $\Delta^m = \emptyset$.

Definition 6.1.5. A *section* (resp. a *rational section*) of an elliptic fibration $f : X \to S$ is a closed subscheme Y of X for which the restriction of f to Y is an isomorphism (resp. a birational morphism). A degree n multisection (or an n-section) is a closed subscheme Y of X such that the restriction of f to Y is a finite morphism of degree n. A *rational degree n multisection* is a closed subscheme Y of X such that the restriction of f to Y is generically finite of degree n.

When working in the algebraic category, it is well-known that any elliptic fibration has a rational multisection. In the analytic category this does not necessarily hold true any more, but we will always assume not only the existence of a rational multisection, but even more: that the elliptic fibrations under consideration have multisections (not only rational).

Definition 6.1.6. For the rest of this chapter, define a *generic elliptic Calabi-Yau threefold* to be an elliptic threefold $f : X \to S$, with X and S smooth algebraic varieties over $\mathbf{C}$ or complex manifolds, X Calabi-Yau, and satisfying the following extra properties:

1. f is flat (i.e. all fibers are 1-dimensional);
2. f does not have any multiple fibers;
3. f admits a multisection;
4. the discriminant locus Δ is a reduced, irreducible curve in S, having only nodes and cusps as singularities;

5. the fiber over a general point of Δ is a rational curve with one node.

Remark 6.1.7. The reason for calling such an elliptic fibration "generic" is the fact that, in many families of elliptic Calabi-Yau threefolds, these properties are shared by the general members of the family. See the examples in Section 6.2 for more details.

From here on we fix a generic elliptic Calabi-Yau threefold $f : X \to S$, whose structure we want to investigate.

Theorem 6.1.8. *Locally, in the étale or analytic topology, f has a section.*

Proof. This is an immediate consequence of the fact that f has no multiple fibers. □

Theorem 6.1.9. *The fibers of f are as follows:*

(0) *over $s \in S \setminus \Delta$, X_s is a smooth elliptic curve;*

(I_1) *over a smooth point s of Δ, X_s is a rational curve with one node;*

(I_2) *over a node s of Δ, X_s is a reducible curve of type I_2, i.e. two smooth $\mathbf{P}^1$'s meeting transversely at two points;*

(II) *over a cusp s of Δ, X_s is a rational curve with one cusp.*

Furthermore, in the case of the I_2 fiber, each component C of the fiber has normal bundle

$$\mathscr{N}_{C/X} \cong \mathscr{O}_C(-1) \oplus \mathscr{O}_C(-1).$$

Proof. See [31]. □

Remark 6.1.10. Let $s \in \Delta$ be a node of the discriminant locus, and let U be a neighborhood of s such that $X_U \to U$ has a section. Fix such a section. Then, according to [13, 2.3] and [35, 2.1], there exists a birational morphism $\pi : X_U \to X'$ of schemes over U, which contracts the component of the I_2 fiber which is not met by the chosen section of $X_U \to U$, and such that $X' \to U$ is a Weierstrass model (see, for example, [13] for a review of the theory of Weierstrass models). Most of the work in the next few sections will consist of understanding this map better, from the point of view of moduli spaces of semistable sheaves.

6.2 Examples

In this section we give three examples of generic elliptic Calabi-Yau threefolds. The first example is mainly folklore, while the last two examples are original, and they are used in Section 6.7 to construct possible counterexamples to the Torelli problem.

Example 6.2.1. We've already seen a first example of a generic elliptic Calabi-Yau threefold in Example 4.1.2. Indeed, the only thing we need to check is that the map $f : X \to \mathbf{P}^2$ satisfies the conditions for being a generic elliptic threefold. The facts that the fibers are 1-dimensional (so that f is flat), there are no multiple fibers, and f admits a multisection are immediate. The discriminant locus is a reduced curve of degree 36, with 216 cusps and 189 nodes. (This can be checked directly using the software package Macaulay [28], or by using Euler characteristic computations.) As we have seen before, this elliptic fibration has $n = 3$ (smallest degree of a multi-section) and the only twisted powers of it (as in Section 4.5) are its relative Jacobian and itself.

Example 6.2.2. In this example we will construct a generic elliptic Calabi-Yau threefold $X \to \mathbf{P}^2$. The ambient space under consideration is $\mathbf{P}^2 \times \mathbf{P}^4$, with coordinates $x_0, \ldots, x_2, y_0, \ldots, y_4$. Let M be a generic 5×5 skew-symmetric matrix whose (i, j)-th entry is a polynomial of bi-degree $(1 - \delta_{j5}, 1)$ (in other words, the bi-degree is $(1, 1)$ everywhere except the last row and column, where it is $(0, 1)$). According to [15, 0.1], the 4×4 Pfaffians of this matrix define a degeneracy locus X, which has a symmetric locally free resolution

$$0 \to \mathscr{L} \to \mathscr{E} \xrightarrow{\varphi} \mathscr{E}^\vee(\mathscr{L}) \to \mathscr{O}_{\mathbf{P}^2 \times \mathbf{P}^4} \to \mathscr{O}_X \to 0,$$

where

$$\begin{aligned} \mathscr{L} &= \omega_{\mathbf{P}^2 \times \mathbf{P}^4} = \mathscr{O}(-3, -5), \\ \mathscr{E} &= \bigoplus_{i=1}^{5} \mathscr{O}(a_i, -3), \\ (a_i) &= (-2, -2, -2, -2, -1), \end{aligned}$$

and the map φ is given by the matrix M. Then it can be easily checked using the results in [15] that X is a smooth Calabi-Yau three-fold.

The projection of X to $\mathbf{P}^2$ is surjective and flat, and the fibers are degree 5 curves in $\mathbf{P}^4$ given by Pfaffians of a skew-symmetric 5×5 matrix. Therefore the fibers are (generically) elliptic curves, and it can be checked by computer that this exhibits $X \to \mathbf{P}^2$ as a generic elliptic fibration.

The projection of X to $\mathbf{P}^4$ maps to a quintic threefold Q in $\mathbf{P}^4$, contracting 52 lines and a conic, to 53 ordinary double points in Q. It can now be checked using standard techniques that the Picard number of Q (and therefore that of X) is 2. Let D and H be pull-backs of hyperplane sections from $\mathbf{P}^2$ and $\mathbf{P}^4$, respectively. It is easy to compute intersection numbers. They are:

$$D^3 = 0, \quad D^2H = 5, \quad DH^2 = 9, \quad H^3 = 5.$$

Since D^2H and DH^2 are coprime, D and H must be primitive in $\mathrm{NS}(X)$, so $\mathscr{O}_X(D)$ and $\mathscr{O}_X(H)$ generate $\mathrm{Pic}(X)$. If $F = D^2$ is a fiber of $X \to \mathbf{P}^2$, then we have $DF = 0$ and $HF = 5$, so we conclude that $n = 5$ (smallest degree of a

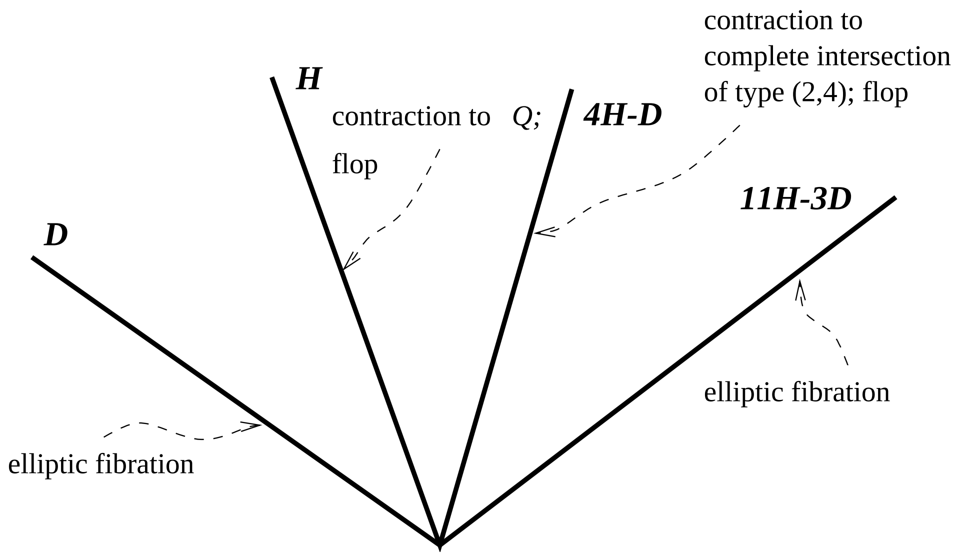


Figure 6.1: The Kähler cones in Example 6.2.2

multi-section). One can take H for a multisection. There is one more interesting Calabi-Yau threefold that can be constructed from X (see Sections 4.5 and 6.6), namely X^2. We'd like to be able to claim that X^2 is not birational to X, and for this we analyze the Kähler cones of X.

One can draw the Kähler cones for X (Figure 6.2.2). The walls are given by

1. D – map to $\mathbf{P}^2$; elliptic fibration;

2. H – map to $\mathbf{P}^4$, image is a quintic with 53 ordinary double points; passing through this wall is a flop;

3. $4H-D$ – map to $\mathbf{P}^5$, image is a complete intersection of type $(2,4)$ with 41 ordinary double points; passing through this wall is a flop;

4. $11H-3D$ – map to $\mathbf{P}^2$; elliptic fibration.

Note that the first elliptic fibration is embedded in $\mathbf{P}^2 \times \mathbf{P}^4$, while the second one in $\mathbf{P}^2 \times \mathbf{P}^5$ (although the fibers are degree 5 each time). If we could prove that X^2 can also be embedded in $\mathbf{P}^2 \times \mathbf{P}^4$, this would prove that X^2 is not birational to X. We are unable to do this in the course of this work due to lack of time.

Example 6.2.3. The construction is almost identical to that in the previous example, except that we take the base of the fibration to be $S = \mathbf{P}^1 \times \mathbf{P}^1$. (We want to do this in order to have K_S divisible by 2; see Section 6.7.) The ambient space is now $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^4$, and M is a 5×5 skew-symmetric matrix of polynomials of tri-degree $(a_{ij}, b_{ij}, 1)$, where $a_{ij} = 0$ except when $i = 4$ or $j = 4$, in which case $a_{ij} = 1$, and $b_{ij} = 0$ except when $i = 5$ or $j = 5$, in which case $b_{ij} = 1$.

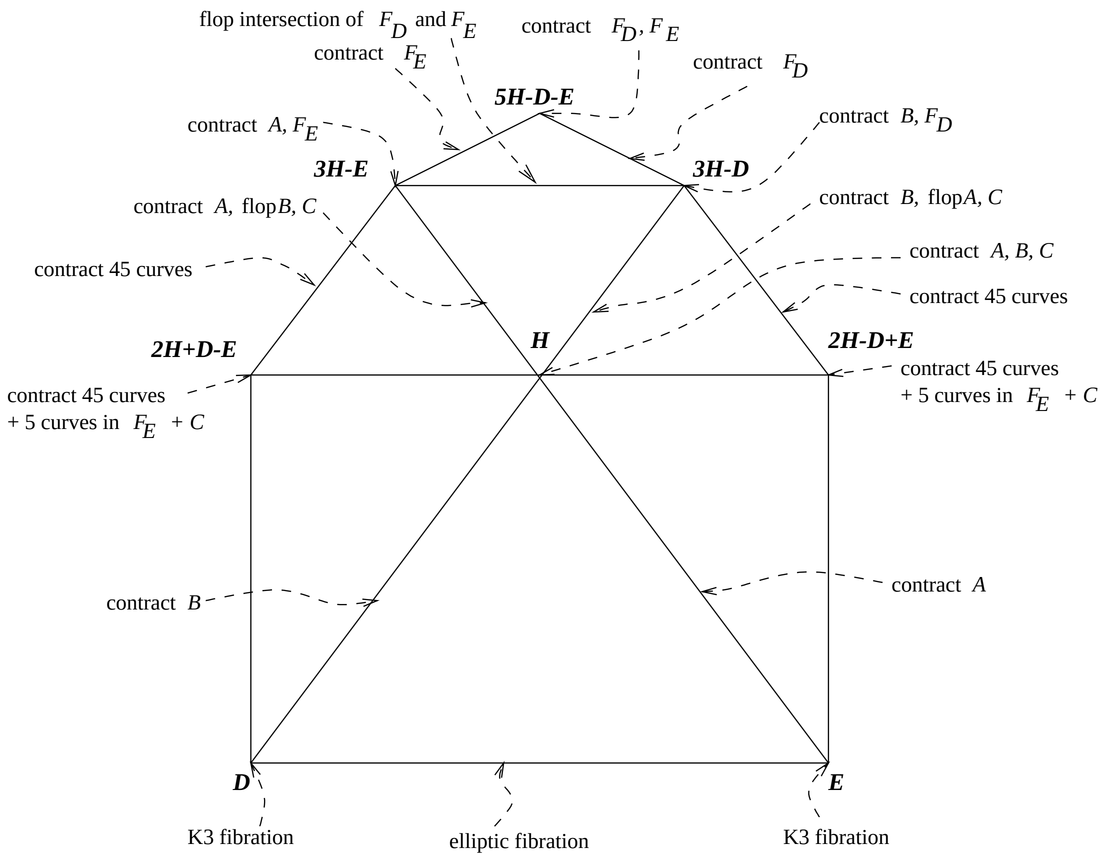


Figure 6.2: The Kähler cones in Example 6.2.3

As before, the 4×4 Pfaffians of M define a subvariety X of $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^4$ which is a smooth Calabi-Yau manifold. The projection of X to $\mathbf{P}^1 \times \mathbf{P}^1$ makes X into a generic elliptic Calabi-Yau manifold, with fibers of degree 5. The Picard number is 3, and the smallest degree of a multisection is again 5.

We can only do a partial analysis of the Kähler cones in this case, because of lack of time. A cross-section through them looks like Figure 6.2.3, where we have included the divisors corresponding to the walls and the behavior obtained by crossing these walls. Here, D, E and H are the pull-backs of the hyperplane sections from the two $\mathbf{P}^1$'s and the $\mathbf{P}^4$, respectively. The important curves and divisors that get contracted by various maps are as follows: the map given by $|H|$ contracts 61 curves, of which 25 (A_i) of type $(1,0)$, 25 (B_i) of type $(0,1)$, and 11 (C_i) of type $(1,1)$. The image of X under $|H|$ is a quintic Q in $\mathbf{P}^4$. The image of D in Q is a complete intersection of type $(2,3)$ in $\mathbf{P}^4$, which is linked in X to a Del Pezzo surface F_D (complete intersection of type $(2,2)$ in $\mathbf{P}^4$). A second Del Pezzo, F_E, is associated to E.

The geometry of this situation seems to suggest again that even if there is another elliptic fibration among the birational models of X (which we actually believe is true), it should not be isomorphic to X^2, but at the moment we are unable to prove this.

6.3 Jacobians of Curves of Genus 1

Given a generic elliptic Calabi-Yau threefold $X \to S$, we will be primarily interested in understanding its relative Jacobian, which is defined to be the relative moduli space of semistable sheaves of rank 1, degree 0 on the fibers of $X \to S$. Therefore, it is fundamentally important to study the corresponding moduli problems on the curves that can occur as fibers of $X \to S$. We do this in this section.

One note about the definition we use for stability of sheaves. We use Gieseker stability, in its generalized form for pure sheaves used by Simpson in [41]. For a quick account of these notions, see [25, Section 1.2].

Theorem 6.3.1. *Let C be an irreducible curve of arithmetic genus 1 (i.e. either of the following: a smooth elliptic curve, or a rational curve with one singular point which is either a node or a cusp), and let J be the moduli space of semistable sheaves on C of rank 1, degree 0. (Since C is irreducible, the rank does not depend on the choice of polarization.) Then we have:*

1. *J is isomorphic to C;*
2. *there are no properly semistable points in J;*
3. *$\mathscr{O}_C$ is a stable sheaf of rank 1, degree 0, and is therefore represented by a point $[\mathscr{O}_C]$;*
4. *choose any isomorphism $\varphi : C \to J$, let $\Gamma \subseteq C \times J$ be the graph of φ, and let $P = \varphi^{-1}([\mathscr{O}_C])$, which is a smooth point of C. Since P is a Cartier divisor on C, it makes sense to speak of the line bundle $\mathscr{O}_C(P)$ on C. Then*
$$\mathscr{U} = \mathscr{I}_\Gamma \otimes \pi_C^*(\mathscr{O}_C(P))$$
is a universal sheaf on $C \times J$, where $\mathscr{I}_\Gamma$ denotes the ideal sheaf of Γ. Any smooth point of C can occur as P.

Proof. The result is well-known (see [22, II, 6.10.1, 6.11.4 and Ex. 6.7]). The only fine point is, in the case of one of the singular curves, the fact that the compactification J of $\mathrm{CaCl}^\circ(C)$ is given by the addition of one extra point, which corresponds to the stable sheaf

$$\mathscr{O}_C(-P) \otimes \mathscr{O}(Q),$$

where Q is the singular point and $\mathscr{O}(Q)$ is defined to be the dual of $\mathscr{I}_Q$. But this is also standard (the proof would be entirely similar to the one given in the analysis of the moduli problem on an I_2 curve, given in the sequel). $\square$

Remark 6.3.2. This theorem basically tells us that, as long as we do not have reducible fibers, nothing very interesting happens: the relative moduli space will only consist of some rearrangement of the fibers of the original space.

Note that the usual way to parametrize line bundles of degree zero on an elliptic curve (or the other singular curves that the theorem deals with) is to associate to a point $Q \in C$ the line bundle $\mathscr{O}(Q-P)$, where P is the origin. Our construction associates $\mathscr{O}(P-Q)$ to $Q \in C$, an entirely isomorphic construction. We chose this particular one for ease of notation later on. (When Q is singular, it is easy to refer to $\mathscr{O}(-Q)$ as $\mathscr{I}_Q$, but there is no easy way to refer to its dual.) Note that this convention is different from the one used in Chapter 4.

From here on we concentrate on understanding what is a corresponding result to Theorem 6.3.1, for a curve of type I_2, i.e. a curve which consists of two components, l_1 and l_2, each isomorphic to $\mathbf{P}^1$, and meeting transversely at two points. Fix such a curve C for the rest of the section.

On C fix the polarization whose restriction to each component $\mathbf{P}^1$ is $\mathscr{O}_{\mathbf{P}^1}(1)$. We'll study sheaves $\mathscr{F}$ on C which are semistable (with respect to this polarization) and have Hilbert polynomial $P(\mathscr{F};t) = 2t$ (note that this is the Hilbert polynomial of $\mathscr{O}_C$ by a simple calculation). This corresponds to studying semistable sheaves of rank 1 and degree 0, under the correct definition of rank (see, for example, [25, 1.2.2]).

This may seem like an arbitrary choice of polarization, but in fact the next lemma shows that the notion of (semi)stability does not depend on the choice of polarization, and therefore this particular choice is just a convenience.

Recall the definition of the reduced Hilbert polynomial: the (usual) Hilbert polynomial of a sheaf $\mathscr{F}$ of dimension n (the dimension of a sheaf is the dimension of its support) is always of the form

$$P(\mathscr{F};t) = \sum_{i=0}^{n} a_i t^i$$

with $a_n > 0$, and the reduced Hilbert polynomial of $\mathscr{F}$ is defined to be

$$p(\mathscr{F};t) = \frac{1}{a_n} P(\mathscr{F};t).$$

Reduced Hilbert polynomials are compared lexicographically: $p(\mathscr{F};t) \le p(\mathscr{G};t)$ if and only if the leading coefficient of $p(\mathscr{G};t) - p(\mathscr{F};t)$ is non-negative.

Lemma 6.3.3. *Consider C polarized by two polarizations; for a coherent sheaf $\mathscr{F}$ on C, write $P(\mathscr{F};t)$ and $p(\mathscr{F};t)$ for the usual and the reduced Hilbert polynomials, respectively, with respect to the first polarization, and $P'(\mathscr{F};t)$ and $p'(\mathscr{F};t)$ with respect to the second one. Then we have, for any coherent sheaf $\mathscr{F}$ on C,*

$$p(\mathscr{F};t) \le t \quad \Leftrightarrow \quad p'(\mathscr{F};t) \le t,$$

and

$$p(\mathscr{F};t) < t \qquad \Leftrightarrow \qquad p'(\mathscr{F};t) < t,$$

Therefore, if $\mathscr{F}$ is a semistable sheaf on C with $p(\mathscr{F};t) = t$, then $\mathscr{F}$ is also semistable with respect to the second polarization, and the same statement holds with "semistable" replaced by "stable" instead.

Proof. There are two possibilities: $\dim \mathscr{F}$ is 0 or 1. The first case is trivial: $p(\mathscr{F};t) = p'(\mathscr{F};t) = 1$. Assume $\dim \mathscr{F} = 1$, so that $P(\mathscr{F};t) = a_1 t + \chi(\mathscr{F})$, $P'(\mathscr{F};t) = a_1' t + \chi(\mathscr{F})$, with $a_1, a_1' > 0$. Since $p(\mathscr{F};t) < t$, $\chi(\mathscr{F}) < 0$, so that $p'(\mathscr{F};t) < t$ as well.

If $\mathscr{F}$ is semistable with $p(\mathscr{F};t) = t$, then clearly $p'(\mathscr{F};t) = t$, and any destabilizing sheaf with respect to the second polarization would be destabilizing with respect to the first one as well. □

On C there is a special class of sheaves, which is described below.

Lemma 6.3.4. *For any two distinct points $P, Q \in C$, or for $P = Q$ smooth points of C, there exists a unique non-trivial extension*

$$0 \to \mathscr{I}_Q \to \mathscr{F}(P,Q) \to \mathscr{O}_P \to 0,$$

which is precisely $\mathscr{O}_C(P) \otimes \mathscr{I}_Q$ if P is smooth. (Note that P is a Cartier divisor on C in this case, so it makes sense to talk about $\mathscr{O}_C(P)$, which is a line bundle on C.)

Proof. All we need to do is compute $\mathrm{Ext}^1(\mathscr{O}_P, \mathscr{I}_Q)$ and show it is equal to $\mathbf{C}$. A trivial local computation shows that

$$\underline{\mathrm{Ext}}^i(\mathscr{O}_P, \mathscr{I}_Q) = \begin{cases} \mathscr{O}_P & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Since the cohomology $H^i(C, \mathscr{O}_P)$ vanishes except when $i = 0$, the local-to-global spectral sequence for Ext's gives the claim. □

Proposition 6.3.5. *Any extension $\mathscr{F} = \mathscr{F}(P,Q)$ given by the previous lemma is semistable with Hilbert polynomial $P(\mathscr{F};t) = 2t$. Such a sheaf is stable if and only if P and Q are smooth points of C belonging to the same component of C. If it is semistable but not stable, its Jordan-Hölder filtration has factors $\mathscr{O}_{l_1}(-1)$ and $\mathscr{O}_{l_2}(-1)$ (and hence all semistable sheaves among the $\mathscr{F}(P,Q)$'s are in the same S-equivalence class).*

Proof. The exact sequence $0 \to \mathscr{I}_Q \to \mathscr{O}_C \to \mathscr{O}_Q \to 0$ together with $P(\mathscr{O}_C;t) = 2t$ shows that $P(\mathscr{I}_Q;t) = 2t - 1$, and therefore the Hilbert polynomial $P(\mathscr{F};t) = 2t$.

Let $\mathscr{G}$ be a dimension zero subsheaf of $\mathscr{F}$, and let $\mathscr{H}$ be the kernel of the map $\mathscr{G} \to \mathscr{O}_P$. It is a subsheaf of $\mathscr{I}_Q$, and has dimension zero, so it must be zero because $\mathscr{I}_Q$ is pure. Therefore $\mathscr{G}$ is a subsheaf of $\mathscr{O}_P$, so it could only be $\mathscr{O}_P$.

But this is impossible, because then $\mathscr{F}$ would be the trivial extension $\mathscr{O}_P \oplus \mathscr{I}_Q$, contradiction. We conclude that $\mathscr{F}$ has no zero-dimensional subsheaves, hence it is pure.

Let $\mathscr{H}$ be a subsheaf of $\mathscr{I}_Q$. Viewing $\mathscr{H}$ as a subsheaf of $\mathscr{O}_C$ via the natural inclusion $\mathscr{I}_Q \to \mathscr{O}_C$, we can consider the closed subscheme H that is determined by $\mathscr{H}$. Note that H is a subscheme of C and hence it is reduced at the generic point of each component of C.

H could be one of the following:

1. $\operatorname{Supp} H = C$; in this case $\mathscr{H}$ is a nilpotent ideal sheaf, hence 0 because C is reduced. We have $P(\mathscr{H}; t) = 0$.

2. H is supported on one component, say l_1, and possibly at some other isolated points on l_2; then $P(\mathscr{O}_H; t) = t + 1 + k$ where $k \geq 0$ takes into account the extra isolated points on l_2, as well as the possible nonreducedness of H at the singular points of C. (Recall that we have $P(\mathscr{O}_{\mathbf{P}^1}; t) = t+1$ by Riemann-Roch.) Therefore $P(\mathscr{H}; t) = t - k - 1$. Since $Q \in H$, if Q is on $l_2 - l_1$ then necessarily $k \geq 1$.

3. H is supported at a number of points; then $P(\mathscr{H}, t) = 2t - k$ for some $k \geq 1$, and if $\mathscr{H} \neq \mathscr{I}_Q$ then $k \geq 2$.

Let $\mathscr{G}$ be a nonzero proper subsheaf of $\mathscr{F}$, and consider the composite map $\mathscr{G} \to \mathscr{F} \to \mathscr{O}_P$. There are two cases to consider, when this map is zero or when it is surjective. In the first case, $\mathscr{G}$ is a subsheaf of $\mathscr{I}_Q$ so by the previous analysis $p(\mathscr{G}; t) = t - k/n$ where n is 1 or 2 and $k \geq 1$ and hence $p(\mathscr{G}; t) < t = p(\mathscr{F}; t)$, so that $\mathscr{G}$ cannot be a destabilizing sheaf for $\mathscr{F}$.

Assume we're in the second case, and consider $\mathscr{H}$ to be the kernel of the composite map $\mathscr{G} \to \mathscr{F} \to \mathscr{O}_P$, which is a subsheaf of $\mathscr{I}_Q$. Since we have the exact sequence

$$0 \to \mathscr{H} \to \mathscr{G} \to \mathscr{O}_P \to 0$$

we see that we have $P(\mathscr{G}; t) = P(\mathscr{H}; t) + 1$. Checking each possibility for $\mathscr{H}$ in turn, we see that indeed $p(\mathscr{G}) \leq t = p(\mathscr{F})$.

Since we proved that $p(\mathscr{G}; t) \leq p(\mathscr{F}; t)$ for all proper subsheaves $\mathscr{G}$ of $\mathscr{F}$, we conclude that $\mathscr{F}$ is semistable.

Now assume P and Q are lying in the same component of C, and none is a singular point of C. Without loss of generality, assume $P, Q \in l_1$. In order to have $p(\mathscr{G}; t) = t$ for some proper subsheaf $\mathscr{G}$ of $\mathscr{F}$, the map $\mathscr{G} \to \mathscr{O}_P$ must be surjective, and $P(\mathscr{H}; t) = t - 1$ ($\mathscr{H}$ is, as before, the kernel of $\mathscr{G} \to \mathscr{O}_P$). (If $P(\mathscr{H}; t) = 2t - 1$, then $\mathscr{H} = \mathscr{I}_Q$, and hence $\mathscr{G} = \mathscr{F}$, contradicting our assumption that $\mathscr{G}$ is a *proper* subsheaf of $\mathscr{F}$.) Therefore $\mathscr{H} = \mathscr{I}_{l_1}$ or $\mathscr{H} = \mathscr{I}_{l_2}$. Since $\mathscr{H} \subseteq \mathscr{I}_Q$, and $Q \notin l_2$, $\mathscr{H}$ must be $\mathscr{I}_{l_1}$. But then $\mathscr{H}$ is zero around P, hence $\mathscr{G}$ is the trivial extension between $\mathscr{H}$ and $\mathscr{O}_P$. This is a contradiction, because if this were the case $\mathscr{O}_P$ would be a subsheaf of $\mathscr{F}$, contradicting the fact that $\mathscr{F}$ is pure. We conclude that $p(\mathscr{G}; t) < t$ for all proper subsheaves $\mathscr{G}$ of $\mathscr{F}$.

This shows that $\mathscr{F}$ is stable when P, Q are smooth points of C belonging to the same component of C. To finish, we need to prove that $\mathscr{F}$ is not stable when this condition is not true. First, assume P and Q are not the two singular points of C, and P is smooth. Then we can choose a component of C that contains Q but not P, say l_2. Consider $\mathscr{F}|_{l_2}$, which is the same as $\mathscr{I}_Q|_{l_2}$: it is either $\mathscr{O}_{l_2}(-1)$ or $\mathscr{O}_{l_2}(-1) \oplus \mathscr{O}_Q$, according to whether Q is singular or not. In any case, after possibly projecting on $\mathscr{O}_{l_2}(-1)$, we get a surjective map $\mathscr{F} \to \mathscr{F}|_{l_2} \to \mathscr{O}_{l_2}(-1)$. Its kernel is a subsheaf of $\mathscr{F}$ of Hilbert polynomial t, hence destabilizing.

If P is one of the singular points of C and Q is a smooth point, consider the component of C that contains Q. Without loss of generality assume it is l_2. The restriction of the exact sequence

$$0 \to \mathscr{I}_Q \to \mathscr{F} \to \mathscr{O}_P \to 0$$

to l_2 gives

$$0 \to \mathscr{O}_{l_2}(-1) \to \mathscr{F}|_{l_2} \to \mathscr{O}_P \to 0,$$

(note that $\mathrm{Tor}_1(\mathscr{O}_P, \mathscr{O}_{l_2})$ is torsion, hence the first map in the above sequence is indeed injective) so that $\mathscr{F}|_{l_2}$ is either $\mathscr{O}_{l_2}(-1) \oplus \mathscr{O}_P$ or $\mathscr{O}_{l_2}$. But a local computation shows that $\mathscr{F}|_{l_2}$ must have torsion at P, so $\mathscr{F}|_{l_2} = \mathscr{O}_{l_2}(-1) \oplus \mathscr{O}_P$, and the kernel of the projection $\mathscr{F} \to \mathscr{F}|_{l_2} \to \mathscr{O}_{l_2}(-1)$ is a destabilizing sheaf for $\mathscr{F}$.

Now assume P and Q are the two singular points of C. Tensor the exact sequence

$$0 \to \mathscr{I}_Q \to \mathscr{F} \to \mathscr{O}_P \to 0$$

with $\mathscr{O}_{l_2}$ to get

$$\mathscr{O}_P = \mathrm{Tor}_1(\mathscr{O}_P, \mathscr{O}_{l_2}) \to \mathscr{I}_Q|_{l_2} = \mathscr{O}_{l_2}(-1) \oplus \mathscr{O}_Q \to \mathscr{F}|_{l_2} \to \mathscr{O}_P \to 0.$$

The first map must be zero, since $\mathscr{O}_{l_2}(-1)$ is torsion-free. But a local computation shows that $\mathscr{F}|_{l_2}$ must be a direct sum between an invertible sheaf on l_2 and $\mathscr{O}_P$ and $\mathscr{O}_Q$. We conclude that this invertible sheaf must be $\mathscr{O}_{l_2}(-1)$, so we have the surjective composite map $\mathscr{F} \to \mathscr{F}|_{l_2} = \mathscr{O}_{l_2}(-1) \oplus \mathscr{O}_P \oplus \mathscr{O}_Q \to \mathscr{O}_{l_2}(-1)$, whose kernel is a destabilizing sheaf for $\mathscr{F}$.

We note that in all these three cases there exists a map $\mathscr{F} \to \mathscr{O}_{l_2}(-1) \to 0$. Computing its kernel, we see that it is $\mathscr{O}_{l_1}(-1)$. Since both $\mathscr{O}_{l_1}(-1)$ and $\mathscr{O}_{l_2}(-1)$ are stable, we conclude that the Jordan-Hölder filtration of $\mathscr{F}$ has these two sheaves as factors. □

Lemma 6.3.6. *Let P be a smooth point of C, and let $Q \in C$ be an arbitrary point. Then we have*

$$\mathrm{Hom}_C(\mathscr{F}(P,Q), \mathscr{F}(P,Q')) = \begin{cases} 0 & \text{if } Q \neq Q' \\ \mathbf{C} & \text{if } Q = Q' \end{cases}$$

Proof. We have $\mathscr{F}(P,Q) = \mathscr{I}_Q \otimes \mathscr{O}(P)$ and $\mathscr{F}(P,Q') = \mathscr{I}_{Q'} \otimes \mathscr{O}(P)$, and $\mathscr{O}(P)$ is a line bundle on C, therefore $\mathrm{Hom}_C(\mathscr{F}(P,Q), \mathscr{F}(P,Q')) = \mathrm{Hom}_C(\mathscr{I}_Q, \mathscr{I}_{Q'})$.

We have $H^0(C, \mathscr{I}_Q) = 0$ and $\chi(\mathscr{I}_Q) = -1$, so we must have $H^1(C, \mathscr{I}_Q) = \mathbf{C}$. By Serre duality, since $\omega_C = \mathscr{O}_C$, it follows that $\operatorname{Hom}_C(\mathscr{I}_Q, \mathscr{O}_C) = \mathbf{C}$, and thus any non-zero homomorphism $\mathscr{I}_Q \to \mathscr{O}_C$ is, up to a scalar multiple, the inclusion of $\mathscr{I}_Q$ into $\mathscr{O}_C$.

Assume $\mathscr{I}_Q \to \mathscr{I}_{Q'}$ is a non-zero homomorphism. Composing with the inclusion $\mathscr{I}_{Q'} \to \mathscr{O}_C$ we get a non-zero homomorphism $\mathscr{I}_Q \to \mathscr{O}_C$, thus by the previous reasoning it must have $\mathscr{O}_Q$ as the cokernel. But this shows that the cokernel of $\mathscr{I}_{Q'} \to \mathscr{O}_C$ has Q in its support, and therefore $Q = Q'$. This shows that $\operatorname{Hom}_C(\mathscr{I}_Q, \mathscr{I}_{Q'}) = 0$ if $Q \neq Q'$. To finish, note that any non-zero homomorphism $\mathscr{I}_Q \to \mathscr{I}_Q$ becomes, after composing with $\mathscr{I}_Q \to \mathscr{O}_C$, the inclusion of $\mathscr{I}_Q$ into $\mathscr{O}_C$, up to a scalar multiple. This shows that any homomorphism $\mathscr{I}_Q \to \mathscr{I}_Q$ is multiplication by a scalar multiple, finishing the proof. □

Lemma 6.3.7. *Let $f : X \to S$ be a flat morphism, let $\Delta \subseteq X \times_S X$ be the diagonal and let $\mathscr{I}_\Delta$ be the ideal sheaf of Δ in $X \times_S X$. Then $\mathscr{I}_\Delta$ is flat over X (via projection on the second component), and for any point $x \in X$, if $t = f(x) \in S$ then the fiber $\mathscr{I}_\Delta|_{X \times_S \{x\}}$ of $\mathscr{I}_\Delta$ over $X \times_S \{x\} \cong X_t \times_S \{x\} \cong X_t$ is isomorphic to the ideal sheaf of x in X_t.*

Proof. Consider the exact sequence

$$0 \to \mathscr{I}_\Delta \to \mathscr{O}_{X \times_S X} \to \mathscr{O}_\Delta \to 0.$$

Obviously, $\mathscr{O}_\Delta$ and $\mathscr{O}_{X \times_S X}$ are flat over X, so we conclude that $\mathscr{I}_\Delta$ is also flat over X. Restricting this exact sequence to $X \times_S \{x\}$ is the same as doing a base change $\{x\} \to X$ under the map $X \times_S X \to X$ which is projection on the second component. Since $\mathscr{O}_\Delta$ is flat over X, we get the exact sequence

$$0 \to \mathscr{I}_\Delta|_{X \times_S \{x\}} \to \mathscr{O}_{X \times_S \{x\}} \to \mathscr{O}_{\{x\} \times_S \{x\}} \to 0$$

(where the zero on the left is a consequence of this flatness), which shows that

$$\mathscr{I}_\Delta|_{X \times_S \{x\}} \cong \mathscr{I}_{\{x\} \times_S \{x\}},$$

which is just $\mathscr{I}_x$ in X_t. □

Proposition 6.3.8. *Let P be a smooth point of C. Then on $C \times C$ there exists a flat family $\mathscr{F}(P)$ such that the fiber of $\mathscr{F}$ over a point $Q \in C$ is isomorphic to the sheaf $\mathscr{F}(P, Q)$ described in the Proposition 6.3.5.*

Proof. Let $\Delta \subseteq C \times C$ be the diagonal, and let $\mathscr{I}_\Delta$ be the ideal sheaf of Δ. The fiber of $\mathscr{I}_\Delta$ over a point $Q \in C$ is isomorphic to $\mathscr{I}_Q$ (by the previous lemma). It is easy to see that $p_2^* \mathscr{O}_C(P) \otimes \mathscr{I}_\Delta$ is the sheaf we're looking for (where $p_2 : C \times C \to C$ is the projection on the second component). □

Theorem 6.3.9. *Let M be the moduli space of semistable sheaves on C with Hilbert polynomial $2t$. Then the point $[\mathscr{O}_C]$ that corresponds to $\mathscr{O}_C$ lies in a unique irreducible component M' of M, which is a rational curve with one node.*

Proof. Consider the restriction of the family described in the previous proposition to the component on which P lies (say, l_1); since all the sheaves in the family are semistable of Hilbert polynomial $2t$ (Proposition 6.3.5), we get by the universal property of M a map $l_1 \to M$, and $[\mathscr{O}_C]$ is in its image (because if $Q = P$, we have $\mathscr{F}(P,Q) \cong \mathscr{O}_C$). Since l_1 is irreducible, this map factors through a map $l_1 \to M'$ for some irreducible component M' of M.

Note that if $Q \neq Q'$, then $\mathscr{F}(P,Q)$ is not isomorphic to $\mathscr{F}(P,Q')$ by Lemma 6.3.6. Since the sheaves $\mathscr{F}(P,Q)$ are stable for $Q \in l_1 - l_2$, we see that the map $l_1 \to M'$ is injective on $l_1 - l_2$. But

$$\dim T_{[\mathscr{F}(P,Q)]}M = \dim \operatorname{Ext}^1_C(\mathscr{F}(P,Q), \mathscr{F}(P,Q)) = 1$$

for $Q \in l_1 - l_2$, so that M' must be of dimension 1 and smooth along the image of $l_1 - l_2$. In particular it is smooth at $[\mathscr{O}_C]$, and therefore it is the unique component of M containing $[\mathscr{O}_C]$. We conclude that the map $l_1 \to M'$ is surjective, and the only issue left is whether the two points of $l_1 \cap l_2$ map to different points of M' (in which case $M' \cong \mathbf{P}^1$) or to the same point of M' (in which case M' is a rational curve with one node). But since $\mathscr{F}(P,Q)$ and $\mathscr{F}(P,Q')$ are properly semistable when Q and Q' are the two singular points of C, they must be in the same S-equivalence class by Proposition 6.3.5, so the map $l_1 \to M'$ maps Q and Q' to the same point. We conclude that M' is a singular nodal curve. ☐

Remark 6.3.10. We made a choice here: we specifically chose the point $P \in l_1$, and considered sheaves of the form $\mathscr{F}(P,Q)$, for varying Q. Some of these are semistable (in fact all those for which $Q \in l_2$ would be semistable) and have the same Jordan-Hölder filtration, so they are all mapped to one point. The picture we see is that, having chosen $P \in l_1$, we contract l_2 to get the moduli space. However, had we chosen $P \in l_2$, we would have contracted l_1 to get the moduli space. Looking at this picture upside down, we see that there are two ways of going from the moduli space to C, and these two ways will be the two sides of a flop in a small resolution of the relative moduli space.

We summarize these results in the following theorem:

Theorem 6.3.11. *Let C be one of the following:*

- (0) *a smooth elliptic curve;*
- (I_1) *a rational curve with one node;*
- (I_2) *an I_2 curve (two smooth $\mathbf{P}^1$'s meeting transversely at two points);*
- (II) *a rational curve with one cusp.*

Then there is a unique component of the moduli space of semistable sheaves on C that contains $[\mathscr{O}_C]$, and that component is isomorphic either to the original curve in cases (0), (I_1) and (II), or to a rational curve with one node in case (I_2).

Having fixed a smooth point $P \in C$ there exists a flat family $\mathscr{U}$ on $C \times C$ such that the fiber of $\mathscr{U}$ over a point $Q \in C$ is $\mathscr{O}(P) \otimes \mathscr{I}_Q$. Therefore the fibers of $\mathscr{U}$ are semistable for all Q, and are stable for all Q in cases (0), (I_1) and (II), while in case (I_2) they are stable only when Q is a smooth point in the same component of C as P.

6.4 The Relative Jacobian

In this section we consider the problem of constructing the *relative Jacobian* of an elliptic fibration $X \to S$: we want it to be a projective, flat morphism $J \to S$, such that at least over the smooth part $S \setminus \Delta$, it agrees with the relative Jacobian constructed in Chapter 4. Furthermore, we'd like the construction to be in a certain sense natural.

A good framework for defining the relative Jacobian is that of relative moduli spaces. Indeed, given a projective, flat morphism $f : X \to S$, a relatively ample line bundle $\mathscr{O}(1)$ on X, and a Hilbert polynomial P, one can consider $M = M_{X/S}(P)$. It comes with a natural flat, projective map $M \to S$, and the fiber over each point $s \in S$, M_s, is naturally isomorphic to the (absolute) moduli space of semistable sheaves of Hilbert polynomial P on the fiber X_s, with the polarization given by $\mathscr{O}(1)$. (In fact, $M_{X/S}(P)$ satisfies a certain stronger corepresentability property. See Theorem 3.3.1.)

Therefore consider the following definition:

Definition 6.4.1. Let $X \to S$ be a generic elliptic fibration, and let $\mathscr{O}_X(1)$ be a line bundle on X ample relative to S. Fix a point $s \in S$, and let P be the Hilbert polynomial of $\mathscr{O}_{X_s}$ on X_s, with respect to the polarization given by $\mathscr{O}_X(1)|_{X_s}$. (P is independent of the choice of $s \in S$, by [22, III, 9.9].) Consider the relative moduli space $M = M_{X/S}(P) \to S$ of semistable sheaves of Hilbert polynomial P on the fibers of X/S. By the universal property of M, there exists a natural section $S \to M$ which takes $s \in S$ to the point $[\mathscr{O}_{X_s}]$, representing the sheaf $\mathscr{O}_{X_s}$ on X_s. Then there exists a unique component J of M that contains the image of this section. J/S is defined to be the *relative Jacobian* of X/S.

There is one thing that requires proof here: the fact that the section lies in a unique component. Let $t : S \to M$ denote the section. Since S is connected, the only thing that needs to be proved is that for some $s \in S$, $t(s)$ is a smooth point of M (in this case, $t(s)$ lies in a unique component of M, and we're done). But it is easy to see that if $s \in S \setminus \Delta$ (which is an open set in S), M is smooth over a neighborhood of s, so we're done.

Next, we study the properties of the relative Jacobian: the map $J \to S$ is obviously a flat, projective morphism. Since the fiber J_s over a point $s \in S$ is isomorphic to the absolute moduli space $M_{X_s}(P)$, we see that J is an elliptic fibration. (By the results in Section 6.3, J_s is an elliptic curve for $s \in S \setminus \Delta$.)

We start our study of the Jacobian by examining what happens locally. We are working with a generic elliptic fibration $f : X \to S$, with a small open set $U \subseteq S$, and with the restriction $f : X_U \to U$ of X to U. We assume U to be small enough so that f has a section $s : U \to X_U$. (See Theorem 6.1.8.)

Proposition 6.4.2. *For every point $Q \in X_U$, let X_Q be the fiber of f containing Q ($X_{f(Q)}$) and let $P = s(f(Q))$ be the corresponding point of the section in X_Q.*

Then there exists a sheaf $\mathscr{U}$ on $X_U \times_U X_U$, flat over the second factor, such that the fiber of $\mathscr{U}$ on X_Q is isomorphic to $\mathscr{O}(P) \otimes \mathscr{I}_Q$.

Proof. Note that since s is a section, P is always a smooth point of the fiber X_Q. The image of the section s is a Cartier divisor on X_U and as such defines a line bundle $\mathscr{O}(s)$ on X_U. Take $\mathscr{U} = \mathscr{I}_\Delta \otimes \mathscr{O}(s)$, where $\mathscr{I}_\Delta$ is the ideal sheaf of the diagonal Δ in $X_U \times_U X_U$. Using Lemma 6.3.7, it is easy to see that this sheaf satisfies the property we're looking for. □

It is this sheaf $\mathscr{U}$ that will provide the local model for the α-twisted pseudo-universal sheaf whose existence is asserted in the introduction to this chapter. Note that if C is a reducible fiber of f, the section s picks up a distinguished component of C; changing the section will correspond to moving from l_1 to l_2 in Remark 6.3.10, and thus will correspond globally to performing a flop on the small resolution of the moduli space.

Theorem 6.4.3. *The family $\mathscr{U}$ of Proposition 6.4.2 (which depends on the choice of the section s) induces by the universal property of J_U/U a surjective map $X_U \to J_U$ of schemes over U which is the contraction of those components of the fibers that do not meet the section s.*

Proof. The sheaf $\mathscr{U}$ constructed before is semistable in each fiber, hence gives by the universal property of M_U/U a map $X_U \to M_U$ of schemes over U. The image of this map is irreducible, and contains each $[\mathscr{O}_{X_p}]$ for all $p \in S$, so the image is in fact contained inside J_U. Also, the image is closed (because X/S is proper) and three-dimensional (the map is locally injective around a point corresponding to a stable line bundle on a fiber of X_U/U), which equals $\dim J$. We conclude that the map $X_U \to J_U$ is surjective, and it is locally an isomorphism around the stable points of J_U.

Using the information from Theorem 6.3.11, and looking fiberwise at this map, we see that it contracts precisely the components of the fibers that are not hit by the section s (Remark 6.3.10). □

Theorem 6.4.4. *Consider the full generic elliptic Calabi-Yau threefold $X \to S$, and let J/S be its relative Jacobian. Then J/S is an elliptic fibration, and the only singularities of J are isolated ordinary double points (ODP's), one over each I_2 degeneration.*

Proof. Since X/S admits a local section (in the analytic topology), everything follows from Theorem 6.4.3 except the fact that the singularities of J are ODP's. But this becomes obvious once we remark that each component of a reducible fiber has normal bundle $\mathscr{O}(-1) \oplus \mathscr{O}(-1)$ (Theorem 6.1.9), and it is well known that the contraction of such a curve in a threefold gives an ODP. □

Remark 6.4.5. There are a number of things to note here, and they are all related to the fact that J has singularities. One, is to note that the reason the singularities occur is the fact that the moduli space under consideration has properly semistable points, and these are precisely the singular points of J. Obviously, there is no hope in finding a universal sheaf on all of $X \times_S J$ (even allowing twistings), because there is no "good" candidate to put over the singular points of J (since they represent a whole S-equivalence class of sheaves). Also, it is worthwhile noting that it is not possible to have an equivalence $\mathbf{D}^b_{\mathrm{coh}}(X) \cong \mathbf{D}^b_{\mathrm{coh}}(J, \alpha)$ for some twisting α; indeed, this is because "derived categories detect smoothness" (this should be considered more as a slogan than a theorem), and $\mathbf{D}^b_{\mathrm{coh}}(X)$ is "smooth" while $\mathbf{D}^b_{\mathrm{coh}}(J, \alpha)$ is not. We'll see in the next section how to remedy this situation. The solution is to replace J by a small resolution of it $\bar{J}$, by working in the analytic category.

Finally, it is worthwhile noting that we cannot hope to find a small resolution of the singularities of J in the algebraic category. In order for a small resolution of J to exist, one would have to have $\operatorname{rk}\operatorname{Cl}(J) > \operatorname{rk}\operatorname{Pic}(J)$ (we need to blow-up a Weil divisor which is not even $\mathbf{Q}$-Cartier). One can now prove that $\operatorname{rk}\operatorname{Cl}(J) = \operatorname{rk}\operatorname{Cl}(X) = \operatorname{rk}\operatorname{Pic}(X)$. Thus if one takes an X with $\operatorname{rk}\operatorname{Pic}(X) = 2$ (for example any of the examples in Section 6.2), then J cannot have a small resolution because we clearly have $\operatorname{rk}\operatorname{Pic}(J) \geq 2$ (there is the section, which is a Cartier divisor, as well as divisors coming from $\operatorname{Pic}(S)$).

Theorem 6.4.6. *For any analytic small resolution $\rho : \bar{J} \to J$ of the singularities of J, there exists a covering $\{U_i\}$ of S by analytic open sets such that for each i, X_{U_i} and $\bar{J}_{U_i}$ are isomorphic, as analytic spaces over U_i.*

Proof. Let T be the collection of points in S over which in X there are degenerations of type I_2. Cover S with analytic open sets $\{U_i\}$ small enough to have the following properties:

- for each i the map $X_{U_i} \to U_i$ has a section s_i;
- each U_i contains at most one point of T.

Let $X_i = X_{U_i}$, $J_i = J_{U_i}$ and $\bar{J}_i = \bar{J}_{U_i}$.

If U_i does not contain any points in T, the statement is clear: $X_i \cong J_i \cong \bar{J}_i$ since in this case J_i is smooth. Now assume U_i contains $t \in T$. The family $\mathscr{U}$ on $X_i \times_{U_i} X_i$ that we constructed in Proposition 6.4.2 gives, by the universal property of J_i/U_i, a map $\varphi_i : X_i \to J_i$ that is a small resolution of J_i. Therefore X_i is isomorphic, over U_i, to one of the two small resolutions of J at x. If $\bar{J}_i$ happens to be that particular resolution, we're done. Otherwise, replace the section s_i by another section s'_i that intersects the component of X_t that is *not* hit by s_i. (Each section intersects precisely one component of X_t, and we're just switching components.) One can do this, by possibly restricting to a smaller open set, because X_t is reduced. Moving to a different component X_t corresponds to performing a flop to the map $X_i \to J_i$, and therefore we get an isomorphism (over U_i) $X_i \cong \bar{J}_i$. □

6.5 The Twisted Pseudo-Universal Sheaf and Derived Equivalences

Let $f : X \to S$ be a generic elliptic Calabi-Yau threefold, and let $J \to S$ be its relative Jacobian. Let Δ be the discriminant locus of $X \to S$, let $U = S \setminus \Delta$, and consider the smooth elliptic fibration $X_U \to U$. Of course, $J_U \to U$ is the relative Jacobian of $X_U \to U$, and we obtain by the results in Chapter 4 a unique $\alpha \in \mathrm{Br}(J_U)$ that corresponds to $X_U \to U$ (as an obstruction to the existence of a universal sheaf on $X_U \times_U J_U$). Let $p_2 : X \times_S J \to J$ and $\bar{p}_2 : X \times_S \bar{J} \to \bar{J}$ be the projections.

Theorem 6.5.1. *There exists a unique $\alpha' \in H^2_{\mathrm{an}}(J, \mathscr{O}_J^*)$, whose restriction to J_U equals α. For any analytic small resolution $\rho : \bar{J} \to J$ of the singularities of J, let $\bar{\alpha} = \rho^*\alpha$. Then*

1. $\bar{\alpha} \in \mathrm{Br}(\bar{J})$;
2. *there exists a $\bar{p}_2^*\bar{\alpha}$-sheaf $\mathscr{U}$ on $X \times_S \bar{J}$, flat over $\bar{J}$, whose restriction to*

$$X_U \times_U \bar{J}_U = X_U \times_U J_U$$

is isomorphic to the $p_2^*\alpha$-twisted universal sheaf of Proposition 3.3.2.

Proof. Throughout this proof, let J^s denote the stable part of J (which is just the smooth part of J), and $\bar{J}^s = \rho^{-1}(J^s)$. Obviously, $\bar{J}^s \cong J^s$. Sometimes we'll identify $\bar{J}^s$ with J^s, when there is no danger of confusion.

Using Theorem 6.4.6, find a covering $\{U_i\}$ of S by analytic open sets, that satisfies the conditions in the proof of that theorem, and isomorphisms

$$\varphi_i : X_i \to \bar{J}_i$$

of analytic spaces over U_i, where $X_i = X_{U_i}$, and similarly for J_i, $\bar{J}_i$. Let $\mathscr{U}_i$ be the pull-back by $\mathrm{id} \times_{U_i} \varphi_i^{-1}$ to $X_i \times_S \bar{J}_i$ of the family constructed in Proposition 6.4.2.

Over $\bar{J}^s$, the sheaves $\mathscr{U}_i$ are local universal sheaves: indeed, this follows from the very way the maps φ_i is constructed, and the fact that φ_i is an isomorphism between $X_i - C_i$ and $J_i - x_i$ when there is a curve C_i to be contracted to an ODP x_i, and an isomorphism between X_i and J_i otherwise. Therefore, restricting to $\bar{J}^s$, the collection $\{\mathscr{U}_i|_{\bar{J}^s}\}$ forms an α-twisted universal sheaf.

Therefore, α can be represented on the cover $\{J^s \cap J_i\}$. Indeed, over each open set in this cover we have a universal sheaf $(\mathscr{U}_i|_{J_s})$, and therefore we can construct α as in Proposition 3.3.2. Note that if $i \neq j$, $J_i \cap J_j \subseteq J^s$. Thus, if we set $\alpha'_{iii} = 1$, and $\alpha'_{ijk} = \alpha_{ijk}$ when not all three of i, j, k are equal, we get a well defined element α' of $\check{H}^2(J, \mathscr{O}_J^*)$, whose restriction to J^s is α. (The conditions for being a cocycle have to be verified only along fourfold intersections, and there they are already satisfied by α.) Note that this α' is obviously unique.

The same situation holds for $\bar{J}$: if $i \neq j$, $\bar{J}_i \cap \bar{J}_j \subseteq \bar{J}^s$ by the very construction of $\bar{J}_i$ and of the cover $\{U_i\}$. Since the only conditions for a collection of sheaves and isomorphisms to form a twisted sheaf happen on the intersections of the open sets in the cover, and in our case all these intersections lie inside $\bar{J}^s$, we conclude that $\{\mathscr{U}_i\}$ form a $\bar{p}_2^*\bar{\alpha}$-twisted sheaf on $\bar{J}$, where $\bar{\alpha} = \rho^*\alpha$. (Here we could have used the standard purity theorem for cohomological Brauer groups, [18, III, 6.2].) The fact that $\bar{\alpha}$ is in fact in $\mathrm{Br}(\bar{J})$ follows immediately from Proposition 3.3.4. $\square$

Remark 6.5.2. This result should be compared with the following analysis: by looking at the smooth part, the initial fibration corresponds to an element $\alpha \in Ш_U(J_U)$. Since there are no multiple fibers involved, α is actually in $Ш_S(J)$. One can blow up S to a new base S_1, and pull-back J over S_1 to get a resolution J_1 of J. By [13, 2.18], we have an inclusion $Ш_S(J) \subseteq Ш_{S_1}(J_1)$, and using [13, 1.17], we have

$$Ш_{S_1}(J_1) = \mathrm{Br}'(J_1)/\,\mathrm{Br}'(S_1).$$

Using an analytic form of [18, III, 7.3], we have

$$\mathrm{Br}'(J_1) = \mathrm{Br}'_{\mathrm{an}}(\bar{J})_{\mathrm{tors}}$$
$$\mathrm{Br}'(S_1) = \mathrm{Br}'_{\mathrm{an}}(S)_{\mathrm{tors}}.$$

Since X is an elliptic Calabi-Yau threefold, S must be a rational or Enriques surface ([17, 2.3]), and thus we have $\mathrm{Br}'(S) = \mathrm{Br}'_{\mathrm{an}}(S)_{\mathrm{tors}} = 0$. We conclude that the element $\alpha \in Ш_S(J)$ can be viewed as an element in

$$Ш_{S_1}(J_1) = \mathrm{Br}'_{\mathrm{an}}(\bar{J})_{\mathrm{tors}}$$

via Ogg-Shafarevich theory. This element coincides with α' by the analysis in Chapter 4, and the above theorem claims that it is actually an element of the Brauer group (i.e. it is represented by an Azumaya algebra).

Theorem 6.5.3. *Let $\mathscr{U}^0$ be the extension by zero (push-forward by the inclusion map) of the $\bar{p}_2^*\bar{\alpha}$-sheaf $\mathscr{U}$ defined in the previous theorem, from $X \times_S \bar{J}$ to $X \times \bar{J}$. ($\mathscr{U}^0$ is naturally a $(p_2')^*\bar{\alpha}$-twisted sheaf on $X \times \bar{J}$, where $p_2' : X \times \bar{J} \to \bar{J}$ is the projection.) Then the Fourier-Mukai transform $D(X) \to D(\bar{J}, \bar{\alpha})$ determined by $\mathscr{U}^0$ is an equivalence of categories.*

Proof. For $P \in \bar{J}$, let $\mathscr{F}(P)$ be the fiber of $\mathscr{U}$ over $X \times_S \{P\}$, and let $\mathscr{F}^0(P)$ be the extension by zero of $\mathscr{F}(P)$ to X (which is also the fiber of $\mathscr{U}^0$ over $X \times \{P\}$). Using the criterion of Theorem 3.2.1, and remarking that $\mathscr{U}^0$ is flat over $\bar{J}$, all we need to check is that

- if $P \neq Q$, $\mathrm{Ext}^i_X(\mathscr{F}^0(P), \mathscr{F}^0(Q)) = 0$, for all i;
- $\mathrm{Ext}^i_X(\mathscr{F}^0(P), \mathscr{F}^0(P)) = 0$, for all P, and for $i > 3$;
- $\mathrm{Hom}_X(\mathscr{F}^0(P), \mathscr{F}^0(P)) = \mathbf{C}$, for all P;
- $\mathscr{F}^0(P) \otimes \omega_X \cong \mathscr{F}^0(P)$, for all P.

The last property follows easily from the adjunction formula and the fact that the fibers of X have trivial canonical bundles.

If P and Q lie in different fibers of $\bar{J} \to S$, there is nothing to prove as $\mathrm{Supp}\,\mathscr{F}^0(P)$ and $\mathrm{Supp}\,\mathscr{F}^0(Q)$ are disjoint. So assume that P and Q lie in the same fiber $\bar{J}_s$ (over some $s \in S$). Let $i : X_s \to X$ be the inclusion, so that $\mathscr{F}^0(P) = i_*\mathscr{F}(P)$ and $\mathscr{F}^0(Q) = i_*\mathscr{F}(Q)$.

Using Proposition 3.3.5 we reduce to checking that

- if $P \neq Q$, $\mathrm{Ext}^i_{X_s}(\mathscr{F}(P), \mathscr{F}(Q)) = 0$ for all i;
- $\mathrm{Ext}^i_{X_s}(\mathscr{F}(P), \mathscr{F}(P)) = 0$ for all P, and for $i > 1$;
- $\mathrm{Hom}_{X_s}(\mathscr{F}(P), \mathscr{F}(P)) = \mathbf{C}$ for all P.

For irreducible fibers (smooth, nodal or cuspidal), the sheaves $\mathscr{F}(P)$ are stable, so the first and third statements follow from [25, 1.2.7 and 1.2.8], while for I_2 fibers they follow from Lemma 6.3.6. Since X_s is Cohen-Macaulay and has trivial dualizing sheaf, we get by Serre duality

$$\mathrm{Ext}^i_{X_s}(\mathscr{F}(P), \mathscr{F}(Q)) = \mathrm{Ext}^{1-i}_{X_s}(\mathscr{F}(Q), \mathscr{F}(P)).$$

which proves the second statement, thus finishing the proof. □

Since the actual result is quite central to the theory, we restate it in a slightly different form:

Theorem 6.5.4. *Let $X \to S$ be a generic elliptic Calabi-Yau threefold, let $J \to S$ be its relative Jacobian, and let $\bar{J}$ be an analytic small resolution of the singularities of J. Let $U = S \setminus \Delta$, where Δ is the discriminant locus of $X \to S$. If $\alpha \in \mathrm{Br}(J_U)$ is the obstruction to the existence of a universal sheaf on $X_U \times_U J_U$ (or, alternatively, the element of the Tate-Shafarevich group representing $X \to S$), then there is a unique extension of α to an element $\bar{\alpha}$ of $\mathrm{Br}(\bar{J})$. For this $\bar{\alpha}$, there is an equivalence of derived categories*

$$\mathbf{D}^b_{\mathrm{coh}}(X) \cong \mathbf{D}^b_{\mathrm{coh}}(\bar{J}, \bar{\alpha}^{-1}).$$

Remark 6.5.5. The fact that X is a Calabi-Yau threefold is irrelevant for the actual proof. We only needed that K_X is trivial along the fibers of $X \to S$, which is an immediate consequence of the adjunction formula, if $X \to S$ has fibers which are plane curves of arithmetic genus 1.

As an easy corollary of the above theorem, we obtain the following result:

Corollary 6.5.6. *Under the hypotheses of the previous theorem, $\bar{J}$ has trivial canonical class.*

Proof. The result follows at once when one notes that the Serre functors in $\mathbf{D}^b_{\mathrm{coh}}(X)$ and $\mathbf{D}^b_{\mathrm{coh}}(\bar{J}, \bar{\alpha}^{-1})$ must be the same (being defined categorically), and these determine the respective canonical classes. □

6.6 Equivalent Twistings

The result we want to focus on in this section is the following:

Theorem 6.6.1. *Let $X \to S$ be a generic elliptic Calabi-Yau threefold, and let $J \to S$ be its relative Jacobian. Let $\bar{J} \to J$ be any analytic small resolution of the singularities of J, let $\bar{\alpha} \in \mathrm{Br}(\bar{J})$ be the twisting defined in Theorem 6.5.4, and let n be the order of $\bar{\alpha}$ in $\mathrm{Br}(\bar{J})$. Then we have*

$$\mathbf{D}^b_{\mathrm{coh}}(\bar{J}, \bar{\alpha}) \cong \mathbf{D}^b_{\mathrm{coh}}(\bar{J}, \bar{\alpha}^k),$$

for any k coprime to n.

This result should be compared to the similar one for K3 surfaces (Theorem 5.5.2), and contrasted to the corresponding situation for spectra of local rings (Theorem 1.3.19).

Theorem 6.6.2. *Let $X \to S$ be a generic elliptic Calabi-Yau threefold, and let $M \to S$ be the relative moduli space of semistable sheaves of rank 1, degree k on the fibers of $X \to S$, for an appropriate choice of relatively ample sheaf on $X \to S$. Let Y be the union of the components of M that contain a point corresponding to a stable line bundle on a fiber of $X \to S$. Then Y is in fact smooth, irreducible, a universal sheaf exists on $X \times_S Y$, and*

$$\mathbf{D}^b_{\mathrm{coh}}(X) \cong \mathbf{D}^b_{\mathrm{coh}}(Y).$$

Notation 6.6.3. In keeping with the notation introduced in Section 4.5, we will denote Y by X^k.

Before we proceed to the proof of this theorem, let us remark that Theorem 6.6.1 is an immediate consequence of this result: indeed, it is easy to see that Y is again a generic elliptic Calabi-Yau threefold: the equivalence of categories $\mathbf{D}^b_{\mathrm{coh}}(X) \cong \mathbf{D}^b_{\mathrm{coh}}(Y)$ implies $K_Y = 0$ by the uniqueness of the Serre functor. Also, the isometry induced by the Fourier-Mukai transform (Corollary 3.1.13 and Proposition 3.1.14), induces an isomorphism

$$H^{2,0}(X) \oplus H^{4,0}(X) \cong H^{2,0}(Y) \oplus H^{4,0}(Y),$$

and therefore $H^{2,0}(Y) = 0$. By Serre duality, this implies $H^1(Y, \mathscr{O}_Y) = 0$, so that Y is Calabi-Yau. Since X and Y are locally isomorphic over S, they obviously have the same singular fibers, and their discriminant loci are the same. The Jacobian of Y can be identified with J (Proposition 4.5.2), and therefore we get an isomorphism

$$\mathbf{D}^b_{\mathrm{coh}}(Y) \cong \mathbf{D}^b_{\mathrm{coh}}(\bar{J}, \bar{\beta}),$$

where $\bar{\beta}$ is the unique element of $\mathrm{Br}(\bar{J})$ that extends β, the element of $\mathrm{Br}(J^s)$ that corresponds to $Y \to S$. But Theorem 4.5.2 and Remark 6.5.2 tell us that $\beta = \alpha^k$, and thus $\bar{\beta} = \bar{\alpha}^k$. Therefore, we have

$$\mathbf{D}^b_{\mathrm{coh}}(\bar{J}, \bar{\alpha}) \cong \mathbf{D}^b_{\mathrm{coh}}(X) \cong \mathbf{D}^b_{\mathrm{coh}}(Y) \cong \mathbf{D}^b_{\mathrm{coh}}(\bar{J}, \bar{\alpha}^k),$$

which is what we wanted.

Proof. Choose a relatively ample sheaf $\mathscr{O}_X(1)$ such that the degree of its restriction to any fiber of $X \to S$ is n. (The fact that this can be done is a well known fact in the theory of elliptic fibrations. See, for example, [13, Section 1], where n is denoted by δ_S.) We'll consider M under this particular polarization.

The important result (which we prove later) is that if C is an I_2 curve, polarized by a polarization of degree n, then there is a family on $C \times C$, flat over the second component, whose fibers are stable sheaves of rank 1, degree k on C, and thus C is naturally embedded in the moduli space $M_C(1, k)$ of semistable sheaves of rank 1, degree k on C. Therefore, there is no contraction of one component, as was the case when $k = 0$.

The whole analysis in Sections 6.3, 6.4, and 6.5 carries through without any significant modification, except for the extra simplification caused by the fact that there are no contractions on the I_2 fibers. Thus X and Y are locally isomorphic over S, and hence Y is smooth.

The standard technique of Mukai ([32, Appendix A]) can then be used to prove that the moduli problem is fine in this case. Another way to see this fact is the following: the obstruction to the existence of a universal sheaf is an element γ of $\mathrm{Br}(Y)$ (since there are no properly semistable sheaves). Let $U = S - \Delta$, where Δ is the discriminant locus of $X \to S$ (or $Y \to S$). There is a natural restriction map $\mathrm{Br}'(Y) \to \mathrm{Br}'(Y_U)$, which is injective by the results in [18]. (Using the standard purity statement [18, III, 6.2], we can remove the singular locus of Δ from the picture. Then we have the situation of removing a smooth divisor from a smooth scheme, for which we get injectivity by a corrected version of [18, III, 6.2].) But the results in Section 4.5 show that the restriction $\gamma|_{Y_U}$ is zero, and thus $\gamma = 0$. Hence the moduli problem is fine.

A universal sheaf then exists, and it induces an integral transform for which the usual criterion for being an equivalence applies, giving us

$$\mathbf{D}^b_{\mathrm{coh}}(X) \cong \mathbf{D}^b_{\mathrm{coh}}(Y).$$

□

Remark 6.6.4. We could also prove this theorem as follows: the moduli problem is fine for numerical reasons (see, for example, [6, 4.2]), and so we can apply the results in [7] to conclude that Y is smooth and there is an equivalence of derived categories $\mathbf{D}^b_{\text{coh}}(X) \cong \mathbf{D}^b_{\text{coh}}(Y)$. But our proof also shows explicitly why one needs to have the coprimality condition.

Proposition 6.6.5. *Let C be an I_2 curve, polarized by an ample line bundle of degree n. Let k be an integer coprime to n. Then there exists a family $\mathscr{U}$ on $C \times C$, whose fiber $\mathscr{U}_Q$ over $Q \in C$ (viewed as the second component of the product $C \times C$) is stable of rank 1, degree k on the first component. All the sheaves $\mathscr{U}_Q$ are distinct.*

Proof. We'll only sketch the proof, since it is mostly similar to the proofs of Propositions 6.3.5, 6.3.6, and 6.3.8.

As before, let l_1 and l_2 be the two components of C. Let $\mathscr{L}$ be the given polarization, and let p_i be the degrees of the restriction of $\mathscr{L}$ to l_i. Define

$$s_i = \left\lceil \frac{kp_i}{p_1 + p_2} \right\rceil$$

for $i = 1, 2$.

We claim that the expression whose integral part we take is never an integer: indeed, this is immediate once one notes that $p_1 + p_2 = n$ is coprime to k. Therefore $s_1 + s_2 = k + 1$.

Choose an effective divisor D on C, whose support is entirely contained in the smooth part of C, and such that it has degree s_i on l_i. The family $\mathscr{U}$ on $C \times C$ that we shall consider is

$$\mathscr{U} = \mathscr{I}_\Delta \otimes \pi_1^* \mathscr{O}_C(D).$$

Over a point $Q \in C$, the fiber $\mathscr{U}_Q$ of $\mathscr{U}$ is isomorphic to $\mathscr{O}_C(D) \otimes \mathscr{I}_Q$, and therefore has rank 1, degree k. For ease of notation, let $\mathscr{F} = \mathscr{U}_Q$. There is a natural surjection

$$\mathscr{F} \to (\mathscr{F}|_{l_i})^{\text{free}} \to 0,$$

where the superscript "free" means removing any zero dimensional subsheaves $\mathscr{F} \otimes \mathscr{O}_{l_i}$ might have. (This surjection comes from the exact sequence

$$0 \to \mathscr{O}_{l_1}(-2) \to \mathscr{O}_C \to \mathscr{O}_{l_2} \to 0,$$

or the similar one with l_1 and l_2 interchanged, by tensoring with $\mathscr{F}$. If $\mathscr{F}$ is locally free at the singular points of C (i.e. Q is a smooth point of C), we get

$$0 \to \mathscr{F}|_{l_1}(-2) \to \mathscr{F} \to \mathscr{F}|_{l_2} = (\mathscr{F}|_{l_2})^{\text{free}} \to 0.$$

If Q is a singular point of C, we get

$$0 \to \mathscr{O}_Q \to \mathscr{F}|_{l_1}(-2) \to \mathscr{F} \to \mathscr{F}|_{l_2} \to 0,$$

where now $\mathscr{F}|_{l_2} = (\mathscr{F}|_{l_2})^{\text{free}} \oplus \mathscr{O}_Q$. Composing the surjection $\mathscr{F} \to \mathscr{F}|_{l_2}$ with the surjection $\mathscr{F}|_{l_2} \to (\mathscr{F}|_{l_2})^{\text{free}}$, we get the desired surjection in this case.) Denote by $\mathscr{G}_i$ the kernel of this surjection.

Following through the analysis of Proposition 6.3.5, we find that the only possible subsheaves of $\mathscr{F}$ that could destabilize it are of the form $\mathscr{G}_i$. The degree of $\mathscr{G}_i$ on l_i is $s_i - 2 - \delta_{iq}$, where δ_{iq} is 1 if Q is in l_i and a smooth point of C, and 0 otherwise. Thus the Hilbert polynomial of $\mathscr{G}_i$ is

$$P(t; \mathscr{G}_i) = p_i t + s_i - 1 - \delta_{iq}.$$

The reduced Hilbert polynomial of $\mathscr{G}_i$ is then

$$p(t; \mathscr{G}_i) = t + \frac{s_i - 1 - \delta_{iq}}{p_i}.$$

On the other hand, the Hilbert polynomial of $\mathscr{F}$ is

$$P(t; \mathscr{F}) = (p_1 + p_2)t + s_1 + s_2 - 1$$

and its reduced Hilbert polynomial is

$$p(t; \mathscr{F}) = t + \frac{s_1 + s_2 - 1}{p_1 + p_2}.$$

In order for $\mathscr{G}_i$ to be destabilizing, we should have

$$\frac{s_i - 1 - \delta_{iq}}{p_i} \geq \frac{k}{p_1 + p_2}.$$

But this is equivalent to

$$s_i - 1 - \delta_{iq} \geq \frac{kp_i}{p_1 + p_2},$$

which is impossible since $\delta_{iq} \geq 0$, and

$$s_i = \left\lceil \frac{kp_i}{p_1 + p_2} \right\rceil.$$

(Note that if kp_i is divisible by $p_1 + p_2$, we can have properly semistable sheaves, as we have seen before.)

Therefore we conclude that all the sheaves $\mathscr{U}_Q$ for $Q \in C$ are stable, and a computation similar to Lemma 6.3.6 shows that they are all distinct. □

6.7 Applications to the Torelli Problem

Proposition 6.7.1. *Let X and Y be Calabi-Yau threefolds, (complex projective manifolds, simply connected and with trivial canonical class) such that $\mathbf{D}^b_{\text{coh}}(X) \cong \mathbf{D}^b_{\text{coh}}(Y)$. Then there exists a Hodge isometry*

$$H^3(X, \mathbf{Z}[1/2])_{\text{free}} \cong H^3(Y, \mathbf{Z}[1/2])_{\text{free}},$$

where the intersection form on these groups is given by the cup product, followed by evaluation against the fundamental class of the space, and the subscript "free" denotes the torsion-free part of the corresponding group.

If $U^\cdot$ is an element of $\mathbf{D}^b_{\mathrm{coh}}(X \times Y)$ that induces an isomorphism $\mathbf{D}^b_{\mathrm{coh}}(X) \cong \mathbf{D}^b_{\mathrm{coh}}(Y)$, denote by $c_3'(U^\cdot)$ the component of $c_3(U^\cdot)$ that lies in $H^3(X,\mathbf{Z}) \otimes H^3(Y,\mathbf{Z})$ under the Künneth decomposition. If $c_3'(U^\cdot)$ is divisible by 2, the isometry between the H^3 groups described above is integral, i.e. it restricts to a Hodge isometry

$$H^3(X,\mathbf{Z})_{\mathrm{free}} \cong H^3(Y,\mathbf{Z})_{\mathrm{free}}.$$

Warning: Throughout this section we will use a non-standard notation, by using $H^{p,q}$ for the subgroup $H^p(X,\mathbf{C}) \otimes H^q(Y,\mathbf{C})$ of $H^{p+q}(X \times Y,\mathbf{C})$ (from the Künneth decomposition), instead of the one that comes from the Hodge decomposition.

Proof. Using Theorem 3.1.16, we can assume that the isomorphism $\mathbf{D}^b_{\mathrm{coh}}(X) \cong \mathbf{D}^b_{\mathrm{coh}}(Y)$ is given by a Fourier-Mukai transform $\Phi^{U^\cdot}_{X\to Y}$ for some $U^\cdot \in \mathbf{D}^b_{\mathrm{coh}}(X \times Y)$. Corollary 3.1.13, combined with Proposition 3.1.14 then give us a Hodge isometry

$$H^*(X,\mathbf{C}) \cong H^*(Y,\mathbf{C}),$$

(the two groups endowed with the inner product described in 3.1.2), which must take $H^3(X,\mathbf{C})$ to $H^3(Y,\mathbf{C})$ and vice-versa, so it gives an isometry

$$H^3(X,\mathbf{C}) \cong H^3(Y,\mathbf{C}).$$

It is easy to see that the restriction of the product on $H^*(X,\mathbf{C})$ to $H^3(X,\mathbf{C})$ gives the usual inner product (up to a $-i$ sign).

However, recall that the correspondence is

$$\varphi : H^*(X,\mathbf{C}) \to H^*(Y,\mathbf{C})$$

given by

$$\varphi(\,\cdot\,) = \pi_{Y,*}(\pi_X^*(\,\cdot\,).\,\mathrm{ch}(U^\cdot).\sqrt{\mathrm{td}(X \times Y)}).$$

Write u_i for $c_i(U^\cdot)$, and r for $\mathrm{rk}(U^\cdot)$.

Observe that the only part that contributes to the $H^3(Y,\mathbf{C})$ component of $\varphi(x)$ (for some $x \in H^3(X,\mathbf{C})$) is the $H^{6,3}$ part of

$$\pi_X^*(x).\,\mathrm{ch}(U^\cdot).\sqrt{\mathrm{td}(X \times Y)},$$

and thus, since $\pi_X^*(x) \in H^{3,0}$, we are only interested in the $H^{3,3}$ component of

$$\mathrm{ch}(U^\cdot).\sqrt{\mathrm{td}(X \times Y)}.$$

Now

$$\sqrt{\mathrm{td}(X \times Y)} = 1 + \frac{1}{24}(\pi_X^* c_2(X) + \pi_Y^* c_2(Y)) + \text{ higher order terms},$$

so that the $H^{3,3}$ component of

$$\mathrm{ch}(U^{\cdot}).\sqrt{\mathrm{td}(X\times Y)}$$

is just $\mathrm{ch}_3(U^{\cdot})$. Indeed, $\pi_X^*c_2(X)\in H^{4,0}$, so it can not give an element of $H^{3,3}$ by multiplication with anything. We have

$$\mathrm{ch}_3(U^{\cdot})=\frac{1}{6}(c_1^3(U^{\cdot})-3c_1(U^{\cdot})c_2(U^{\cdot})+3c_3(U^{\cdot}));$$

$c_1(U^{\cdot})\in H^2(X\times Y,\mathbf{C})$, and since $H^1(X,\mathbf{C})=H^1(Y,\mathbf{C})=0$, it must belong to $H^{2,0}\oplus H^{0,2}$. Therefore $c_1^3(U^{\cdot})$ can not have any $H^{3,3}$ component. Similarly for $c_1(U^{\cdot})c_2(U^{\cdot})$. Hence we conclude that the only contribution to the map $H^3(X,\mathbf{C})\to H^3(Y,\mathbf{C})$ comes from $c_3(U^{\cdot})$, in the form

$$\varphi|_{H^3(X,\mathbf{C})}(\,\cdot\,)=\frac{1}{2}\pi_{Y,*}(\pi_X^*(\,\cdot\,).c_3'(U^{\cdot})),$$

and hence the result. □

Proposition 6.7.2. *Just as before, let X and Y be Calabi-Yau threefolds with equivalent derived categories. Assume that there exists a codimension 2 subvariety $i:Z\hookrightarrow X\times Y$ which is locally a complete intersection, and an element $U^{\cdot}\in\mathbf{D}^b_{\mathrm{coh}}(Z)$ such that if one takes $i_*U^{\cdot}\in\mathbf{D}^b_{\mathrm{coh}}(X\times Y)$, then $\Phi^{i_*U^{\cdot}}_{X\to Y}:\mathbf{D}^b_{\mathrm{coh}}(X)\to\mathbf{D}^b_{\mathrm{coh}}(Y)$ is an equivalence of categories. Let*

$$K_Z=c_1(\omega_Z)\in H^2(X\times Y,\mathbf{Z})$$

be the first Chern class of the dualizing sheaf of Z, (which exists since Z is locally a complete intersection). If K_Z is divisible by 2 in $H^2(X\times Y,\mathbf{Z})$, then the isometry of Proposition 6.7.1 is integral, i.e. it restricts to a Hodge isometry

$$H^3(X,\mathbf{Z})_{\mathrm{free}}\cong H^3(Y,\mathbf{Z})_{\mathrm{free}}.$$

Proof. As seen before, to prove the integrality of the isomorphism we would need to show that $c_3(i_*U^{\cdot})$ is divisible by 2. We use Grothendieck-Riemann-Roch to compute $c_3(i_*U^{\cdot})$ (we can apply it because i is a locally complete intersection projective morphism). We have

$$\mathrm{ch}(i_*U^{\cdot})=i_*(\mathrm{ch}(U^{\cdot}).\,\mathrm{td}(-\mathscr{N})),$$

where $\mathscr{N}$ is the normal sheaf of Z in $X\times Y$, and the negation is taken in the Grothendieck group (see [21, Exposé 0]).

The left hand side of the above equality is

$$r+c_1+\frac{1}{2}(c_1^2-c_2)+\frac{1}{6}(c_1^3-3c_1c_2+3c_3)+\dots,$$

where $c_i = c_i(i_* U^{\cdot})$. Obviously, $r = c_1 = 0$ because Z has complex codimension 2 (also from the right hand side of the equality), so the real dimension 6 part of the left hand side consists of just

$$\frac{1}{2} c_3(i_* U^{\cdot}).$$

On the other hand, the real dimension 6 part of the right hand side comes from the dimension 2 part of

$$\mathrm{ch}(U^{\cdot}).\,\mathrm{td}(-\mathscr{N}),$$

which consists of

$$c_1(U^{\cdot}) - \frac{1}{2}\,\mathrm{rk}(U^{\cdot}) c_1(\mathscr{N}).$$

We conclude that

$$c_3(i_* U^{\cdot}) = 2 i_* c_1(U^{\cdot}) - \mathrm{rk}(U^{\cdot}) i_* c_1(\mathscr{N}).$$

The first term is obviously divisible by 2. Using the adjunction formula and the fact that X and Y have trivial canonical bundles, we have

$$K_Z = c_1(\omega_Z) = c_1(\mathscr{N}),$$

so the result follows. □

Theorem 6.7.3. *Let X and Y be the elliptic Calabi-Yau threefolds considered in Theorem 6.6.2, assuming that the base S has canonical class divisible by 2 (for example, if $S = \mathbf{P}^1 \times \mathbf{P}^1$). Then the Fourier-Mukai transform given by the universal sheaf induces a Hodge isometry*

$$H^3(X, \mathbf{Z})_{\mathrm{free}} \cong H^3(Y, \mathbf{Z})_{\mathrm{free}}.$$

Proof. Obviously the support of the universal sheaf $\mathscr{U}$ is $X \times_S Y$, which is of codimension 2 in $X \times Y$. It is also a locally complete intersection, because it is the pull-back of the diagonal in $S \times S$ (which is a locally complete intersection) under the natural map $X \times Y \to S \times S$. Therefore the conditions of the previous theorem apply, and we only need to show that $K_{X \times_S Y}$ is divisible by 2.

We have the commutative square

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{\pi_Y} & Y \\ \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle p_Y} \\ X & \xrightarrow{p_X} & S \end{array}$$

Since all the spaces involved are Cohen-Macaulay, we have equalities (in the corresponding K-groups, or integral cohomology groups)

$$\begin{aligned} K_X &= p_X^* K_S + K_{X/S} \\ K_Y &= p_Y^* K_S + K_{Y/S} \\ K_{X \times_S Y} &= \pi_X^* K_X + K_{X \times_S Y/X}. \end{aligned}$$

Using base-change and the fact that $K_X = K_Y = 0$, we have

$$\begin{aligned} K_{X\times_S Y} &= \pi_X^* p_X^* K_S + \pi_X^* K_{X/S} + \pi_Y^* K_{Y/S} \\ &= \pi_X^* K_X + \pi_Y^* K_Y - \pi_X^* p_X^* K_S \\ &= -\pi_X^* p_X^* K_S. \end{aligned}$$

Since K_S is divisible by 2 by assumption, we conclude that $K_{X\times_S Y}$ is divisible by 2 as well, so we're done. □

Corollary 6.7.4. *Consider the elliptic Calabi-Yau threefold X constructed in Example 6.2.3, and let $Y = X^2$ (see Theorem 6.6.2). Then X and Y are not isomorphic to each other, and there exists a Hodge isometry*

$$H^3(X, \mathbf{Z})_{\text{free}} \cong H^3(Y, \mathbf{Z})_{\text{free}}.$$

Proof. Immediate from the previous proposition and the analysis in Example 6.2.3. □

Remark 6.7.5. Unfortunately, because of lack of time, we are unable to prove that X and Y are not birational to each other. To do this, one would have to finalize the analysis in Example 6.2.3, in order to find all minimal models for X. However, in view of the geometry of this example, it seems highly unlikely that X and Y are birational. This should be viewed in the context of [42], where a similar statement to Corollary 6.7.4 is proved when X and Y are Calabi-Yau threefolds which are birationally equivalent, and thus differ by a sequence of flops. (Of course, that result would follow immediately from our analysis, using the result of [4] that states that, at least for some classes of flops, one gets equivalent derived categories when performing these flops.) Our example probably contradicts the conjecture, made in [42], that the flop example is the most general situation when X and Y have Hodge isometric H^3's.

6.8 Relationship to Work of Aspinwall-Morrison and Vafa-Witten

This section is largely speculative, and should provide just an insight into why derived categories of twisted sheaves might prove to be relevant in physics.

Let us first start by reviewing an example originally due to Vafa and Witten ([43]), and later expanded by Aspinwall and Morrison ([1]). We use the notation in [1].

Consider $Z = E_1 \times E_2 \times E_3$, a product of three smooth elliptic curves. The group $G = \mathbf{Z}_2 \times \mathbf{Z}_2$ acts on this space by $(z_1, z_2, z_3) \mapsto (-z_1, -z_2, z_3)$ and $(z_1, z_2, z_3) \mapsto (z_1, -z_2, -z_3)$ (note that these automorphisms leave the canonical class of Z unchanged), and we consider $X^\#$ to be the quotient of Z by G.

One space of interest is Y, which is a crepant resolution of the singularities of $X^{\#}$. (Such a resolution is known to exist, and Y is then a Calabi-Yau threefold). The Hodge numbers of Y are $h^{1,1} = 51$, $h^{2,1} = 3$. These numbers should be interpreted as the number of holomorphic deformations ($h^{2,1} = 3$, coming from three deformations of the complex structure of the three tori we started with) and the number of symplectic structure deformations ($h^{1,1} = 51$). Therefore, up to changes in the complex structure of the three initial tori, the structure is rigid holomorphically, but can be deformed symplectically. The symplectic deformations correspond to the blowing up $Y \to X$.

These deformations can be viewed as smoothings of a physical theory $\mathscr{T}_0$ on $X^{\#}$. This theory is constructed from a physical theory on Z, by taking into account the G-action.

However, there is yet another theory $\mathscr{T}_1$ which can be considered on $X^{\#}$. In fact, the ingredients that go into the construction of a theory on $X^{\#}$ are a theory on Z, the group action of G, and an element α of $H^2(G, \mathbf{C}^*)$. In our case, $H^2(G, \mathbf{C}^*) = \mathbf{Z}_2$, and therefore there exists another theory $\mathscr{T}_1$ on $X^{\#}$.

By physical arguments, one can compute the dimensions of the spaces of holomorphic and symplectic structure deformations of $\mathscr{T}_1$, and one gets $h^{1,1} = 3$ and $h^{2,1} = 51$. Therefore this theory on $X^{\#}$ can be deformed symplectically in 3 ways (which come from varying the sizes of the three original tori, so the problem is rigid in this direction), and holomorphically in 51 directions.

One is led to believe then that these holomorphic deformations should come from deformations of the complex structure on $X^{\#}$. These are well-understood, and it is known that there are 115 such deformations. The problem is, now, that there seem to be "too many" deformations.

The solution suggested by Vafa and Witten was that, for some yet unexplained reason, not all deformations of the complex structure of $X^{\#}$ can be extended to deformations of $\mathscr{T}_1$. Since $X^{\#}$ has 64 points where the singularities get worse, they suggested that any deformation of $X^{\#}$ on which $\mathscr{T}_1$ can be extended must preserve some singularity at these 64 points, and they required these singularities to be ordinary double points (ODP's).

One more insight was given by the work of Aspinwall-Morrison-Gross: they observed that when one deforms $X^{\#}$ through a general deformation (which completely removes the singularities of $X^{\#}$), the Brauer group of the resulting space X is zero. However, they were able to prove that if the deformation was done in such a way as to preserve ODP's at the special 64 points, $\mathrm{Br}(X) = \mathbf{Z}_2$ for the resulting space. They suggested that there should be some strong relationship between this phenomenon and the fact that $X^{\#}$ should not be deformed so as to remove all singularities.

The first suggestion that we make towards a solution to this puzzle is this: derived categories of sheaves on spaces are known to be related to mirror symmetry through Kontsevich's Homological Mirror Symmetry Conjecture ([26]). Until now, the only categories taken into account were those of coherent sheaves. Our suggestion is that one could construct physical theories (and, thus, mirrors), by taking into account also derived categories of twisted sheaves. This is rather vague, but it seems to coincide with physicists' belief that when the Brauer group of a space is non-trivial, one can construct more than one physical theory on that space, by introducing into the game the elements of the Brauer group.

One can even attempt to pursue this one step further: by work of Bridgeland-King-Reid ([8]), it is known that one has

$$\mathbf{D}^G_{\mathrm{coh}}(Z) \cong \mathbf{D}^b_{\mathrm{coh}}(Y),$$

where Y is the crepant resolution we constructed before. Here, $\mathbf{D}^G_{\mathrm{coh}}(Z)$ denotes the bounded derived category of coherent, G-equivariant sheaves on Z.

The group $H^2(G, \mathbf{C}^*)$ classifies central extensions of the form

$$0 \to \mathbf{C}^* \to K \to G \to 0.$$

If one considers the trivial extension, $K = G \oplus \mathbf{C}^*$, one obtains that the K-equivariant sheaves on Z are just the G-equivariant sheaves on Z. However, when one considers the non-trivial extension K (which corresponds to the non-trivial element of $H^2(G, \mathbf{C}^*)$), one obtains a new derived category, $\mathbf{D}^K_{\mathrm{coh}}(Z)$. This seems to suggest that one should expect an equivalence of the sort

$$\mathbf{D}^K_{\mathrm{coh}}(Z) \cong \mathbf{D}^b_{\mathrm{coh}}(X, \alpha),$$

where X is any deformation of Z having ODP's at the 64 prescribed points, and α is the unique non-trivial element of $\mathrm{Br}(X)$.

Of course, this cannot be expected to hold as such. Probably, we would have to take a small, analytic resolution of the singularities of X, as we did in the case of the relative Jacobian. Also, another problem with this equivalence is that it seems to be independent of the particular deformation taken. The fact that X and Y are mirrors seems to suggest that $\mathbf{D}^b_{\mathrm{coh}}(X, \alpha)$ should in fact be equivalent to some Fukaya category on Z.

The relationship of this situation to the one we studied before is the following: if X is the elliptic Calabi-Yau X that we considered in the previous sections, let J be its relative Jacobian. On $\bar{J}$, a small resolution of J, we have two interesting, and entirely different, derived categories: one, $\mathbf{D}^b_{\mathrm{coh}}(\bar{J}, 0)$ is the usual derived category of $\bar{J}$. The other one, $\mathbf{D}^b_{\mathrm{coh}}(\bar{J}, \alpha)$, is the same as $\mathbf{D}^b_{\mathrm{coh}}(X)$. If derived categories are thought of as physical theories, this means that on $\bar{J}$ we can construct two different physical theories, both of them "smooth" (since they correspond to $\mathbf{D}^b_{\mathrm{coh}}(X)$ and $\mathbf{D}^b_{\mathrm{coh}}(\bar{J})$, and both spaces are smooth). However, if one attempts to deform J so as to remove its singularities, one observes that its Brauer group becomes zero. Thus, from a deformed J, one cannot hope to recover more than one physical theory ($\mathbf{D}^b_{\mathrm{coh}}(J)$), a situation very similar to the one studied by Vafa and Witten.

Open Questions and Further Directions

In this chapter we collect everything that was left loose-ended in the other chapters. Its motto should be "a conclusion is the place where you stopped because you were tired of thinking."

First, there are technical issues whose solution is quite unsatisfactory. In this category I would list the following:

1. One would like to be able to deal confidently with derived categories of twisted sheaves even when the twisting is not in the Brauer group. In particular one would like to understand the boundedness of derived functors in this context.

2. There seems to be no particular reason why duality for proper morphisms should not hold in general for twisted sheaves.

3. Condition (TD) in Section 3.1 seems unnecessary. A correct definition of the Mukai vector and of the bilinear pairing should eliminate this restriction.

4. The requirement that there exist X and X' in Theorem 5.5.2 appears superfluous. Conjecture 5.5.3 should hold in general.

5. All the results in Chapter 6 should hold for arbitrary elliptic Calabi-Yau threefolds (no restrictions on the singularities of the fibers).

6. In particular, one should have $\mathbf{D}^b_{\mathrm{coh}}(X, \alpha) \cong \mathbf{D}^b_{\mathrm{coh}}(X, \alpha^k)$ for any elliptic Calabi-Yau threefold X, and any $\alpha \in \mathrm{Br}(X)$ with $\mathrm{ord}(\alpha)$ and k coprime.

In the second category I would list open questions whose solution I would have liked to be able to find, but I was not:

1. What is the relationship of the group $\mathrm{Br}(X)/\,\mathrm{Aut}(X)$ to the group of Morita equivalence classes of sheaves of Azumaya algebras on a scheme X? (Here we have considered the action of the automorphism group of X, $\mathrm{Aut}(X)$, on $\mathrm{Br}(X)$.) Are they the same?

2. If one has $\mathbf{D}^b_{\mathrm{coh}}(X,\alpha) \cong \mathbf{D}^b_{\mathrm{coh}}(Y)$ for smooth spaces X and Y, and for $\alpha \in \mathrm{Br}'(X)$, does it imply that $\alpha \in \mathrm{Br}(X)$? This would have implications for the conjecture that $\mathrm{Br}(X) = \mathrm{Br}'(X)$ for a smooth space X.

3. What are the conditions for being able to find a smooth space Y with $\mathbf{D}^b_{\mathrm{coh}}(X,\alpha) \cong \mathbf{D}^b_{\mathrm{coh}}(Y)$, given X and α?

4. What is the most general setup in which one has

$$\mathbf{D}^b_{\mathrm{coh}}(X,\alpha) \cong \mathbf{D}^b_{\mathrm{coh}}(X,\alpha^k)$$

for smooth X, $\alpha \in \mathrm{Br}(X)$, k coprime to the order of α?

5. Is there more geometry to the "virtual" K3 surfaces from Remark 5.5.4?

6. In Example 6.2.3, is X^2 deformation equivalent to X? This would give a complete counterexample to the Torelli problem (and it would also prove that X and Y are not birational to each other). Note that in order to prove this, it would be enough to solve the following problem: let C be a smooth curve of genus 1, let r and d be coprime positive integers, and let $\mathscr{V}$ be an irreducible vector bundle of rank r and degree d. Then one would need to find a *natural* isomorphism

$$H^0(C,\mathscr{V}) \cong H^0(C,\det \mathscr{V}).$$

7. What kind of singularities can arise through processes similar to the one that generate the singularities of the relative Jacobian? In all the "generic" examples I know of these singularities are ordinary double points. Are there other cases, more degenerate?

8. How should one understand the Vafa-Witten, Aspinwall-Morrison example? What is the relationship of twisted sheaves with physics?

The main cause we cannot solve more of these issues is our lack of examples. These would likely provide us with a better understanding of the theory of twisted sheaves, which would bring forth answers to some of these questions.

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