MATH 240PRACTICE PROBLEMS FOR THE FINAL EXAM

APRIL 23, 2005

1. Suppose that $y(t)$ is a function defined on $(0, \infty)$ such that

$$y'(t) + 2y(t) = \mathcal{U}(t-3)\, e^t \,,$$

$y(0) = y'(0) = 0$, where $\mathcal{U}(t)$ denotes the standard Heaviside function. Find the Laplace transform $\mathcal{L}\{y(t)\}(s)$ of $y(t)$, and determine the function $y(t)$.

2. Let A be the matrix

$$A = \begin{bmatrix} -5 & 26 \\ -1 & 5 \end{bmatrix}$$

Find A^9, A^{10}, and e^A.

3. Find the solution of the differential equation

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} - 12x(t) = 0$$

subject to the condition that $x(0) = 3$, $\frac{dx}{dt}(0) = -2$

4. Let $\vec{F}$ be the vector field

$$\vec{F}(x,y,z) = \frac{(x+y-z)\,\vec{i} + (y+z-x)\vec{j} + (z+x-y)\vec{k}}{(x^2+y^2+z^2)^{3/2}} \,.$$

Let S_1 be the surface $\{4x^2 + 4y^2 + 4z^2 = 1\}$ and let S_2 be the surface $\{x^2 + 4y^2 + 4z^2 = 1\}$, and orient both S_1 and S_2 by the unit normal vector field pointing away from the origin. Show that

$$\iint_{S_1} \vec{F} \cdot \vec{n} dS = \iint_{S_2} \vec{F} \cdot \vec{n} dS \,,$$

and evaluate the integral.

5. Find the general solution of the differential equation

$$t^2 \frac{d^2x}{dt} + 3t\frac{dx}{dt} + 5x = 0$$

6. Suppose that $\vec{x}(t)$ satisfies the system of differential equation

$$\vec{x}'(t) - A \cdot \vec{x}(t) = 0, \quad \text{where } A = \begin{bmatrix} 1 & 6 \\ 1 & 2 \end{bmatrix}$$

and $\vec{x}(0) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Determine the value of $\vec{x}(\ln(2))$.

7. Compute the surface integral

$$\iint_S \frac{x^2\vec{i} + y^2\vec{j} + z^2\vec{k}}{x^2 + y^2 + z^2 + 5} \cdot \vec{n}\, dS\,,$$

where S is the sphere $\{x^2+y^2+z^2 = 25\}$, and $\vec{n}$ is the unit normal vector field on S pointing away from the origin.

8. Suppose that $y(t)$ satisfies the differential equation

$$y''(t) + 2y'(t) + y(t) = f(t)$$

$y(0) = y'(0) = 0$, and $\mathcal{L}\{t^{-1}f(t)\}(s) = \frac{1}{(s^2+1)^2}$. What is the value of $\mathcal{L}\{y(t)\}(s)$ at $s = 1$.

A. 1

B. $\frac{1}{2}$

C. $\frac{2}{3}$

D. $\frac{1}{8}$

E. 3

F. -2

G. None of the above.

9. Suppose that $y(t)$ is a function on $\mathbb{R}$ which satisfies the differential equation

$$y''(t) + 4y'(t) + 5y(t) = -10t^2 - 21t - 1\,.$$

What is the value of the limit

$$\lim_{t\to\infty} \frac{y(t)}{t^2} =?$$

A. 1

B. 0

C. -2

D. 5

E. -4

F. ∞

G. None of the above.

10. Let $y_s(t)$ be the steady state solution of the differential equation

$$y''(t) + 2y'(t) + 2y(t) = 5\cos t$$

In other words, $y_s(t)$ is the unique solution which is a periodic function in t. Then $y_s(\pi)$= ?

A. 1

B. -1

C. 5

D. -2

E. 2

F. 0

G. None of the above.

11. Find the Laplace transform of the function $f(t) = te^t$

12. Let $y(t)$ be a solution of the differential equation

$$t^2\frac{d^2y}{dt^2} + 3t\frac{dy}{dt} - 3y(t) = 0$$

and suppose that $y(1) = 1$, $y'(1) = 5$. Then $y(2)$=?

A. 3

B. $\frac{15}{4}$

C. $\frac{15}{32}$

D. $\frac{57}{25}$

E. $-\frac{17}{4}$

F. $\frac{31}{8}$

G. None of the above.

13. Consider the following differential equation

$$y''(t) + 4y'(t) + \lambda y(t) = 2e^{-2t}\cos(3t) + 5e^{-2t}\sin(3t)\,.$$

For which of the following values of the parameter λ will the differential equation have a solution $y_p(t)$ such that $e^{2t}y_p(t)$ is *not* bounded as $t \to \infty$? [A function $f(t)$ on $\mathbb{R}$ is bounded as $t \to \infty$ if there exists a constant N such that $f(t) \le N$ for all $t \ge 0$.] A. 4

B. 8

C. 10

D. 12

E. 13

F. 15

G. None of the above.

14. Let $x(t)$, $y(t)$ be two functions on $\mathbb{R}$ such that

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

Suppose that $x(0) = 1$ and $\lim_{t\to\infty} e^{3t}x(t) = 3$. What is the value of $y(1)$?

A. $3e - e^3$

B. $2e + 3e^3$

C. $-2e + e^3$

D. $4e^3$

E. $-3e$

F. 0

G. None of the above.

15. For which values of the parameter a will the following matrix fail to have an inverse?

$$\begin{bmatrix} 1 & 2 & 3 \\ 5 & 2 & 6 \\ 1 & 4 & a \end{bmatrix}$$

16. Compute the flux of the vector field $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$ through the upper half sphere

$$\left\{x^2 + y^2 + z^2 = 1,\ z \geq 0\right\} ,$$

oriented by the unit normal vector field pointing away from the origin.

17. Let S be the intersection of the infinite solid pillar V bounded by the following four planes $\{x = 1\}$, $\{x = -1\}$, $\{y = 1\}\{y = -2\}$ and the plane $E = \{x + 2y + 3z = 6\}$. Let C be the boundary of S, oriented counter-clockwise as seen from above the plane $E = \{x+2y+3z = 6\}$. Compute the line integral

$$\oint_C x^2 dx + xydy + z^3 dz\,.$$

18. (True or False) Let A, B be two 5×5 matrices with real entries such that $AB = BA$. Then $\det(A^2 + B^2) \geq 0$.