Math 240 practice problems

February, 2005

1. Which one among the following statements are true for an orthogonal ($n \times n$ matrix P?

(I) $P + P^{-1}$ is symmetric.

(II) $P - P^{-1}$ is symmetric.

(III) $||P \cdot \vec{v}|| = ||\vec{v}||$ for every vector $\vec{v} \in \mathbb{R}^n$.

(IV) $\det(P) \geq 0$

(V) If $n = 1$, then either $P = [1]$ or $P = [-1]$.

(VI) P has n distinct eigenvalues.

(VII) There exists an invertible symmetric $n \times n$ matrix Q such that $Q^{-1} \cdot P \cdot Q$ is diagonal.

2. Let R be the region

$$R = \{(x, y) \in \mathbb{R}^2 \,|\, 1 \leq 3x + 2y \leq 5, 0 \leq 2x + 3y \leq 4\},.$$

Compute

$$\iint_R (x^2 + y^2)\, dx\, dy$$

3. Let $\vec{F}$ be the vector field defined by

$$\vec{F}(x, y, z) = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{\sqrt{x^2 + y^2 + z^2}}$$

Compute the oriented surface integral

$$\iint_S \vec{F} \cdot \vec{n}\, dS$$

where S is the surface

$$S = \{(x, y, z) \,|\, x^2 + y^2 + z^2 = 4,\ z \geq 0\},$$

oriented by the unit normal vector field $\vec{n}$ on S such that $\vec{n}(0, 0, 1) = \vec{k}$.

4. Let A be the square matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Find an orthogonal matrix C such that $C^t \cdot A \cdot C$ is diagonal.

5. Let C be the boundary of the square

$$Q := \{(x, y) \mid -1 \le x, y \le 1\}$$

on the plane, oriented counterclockwise. Compute the line integral

$$\oint_C (y^2 + \cos(x^2))\, dx + (x + \sin(y^2))\, dy$$

6. Compute the oriented surface integral

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, dS$$

where

$$\vec{F}(x, y, z) = (y + x)\vec{i} + (y - x)\vec{k} + \sin(xyz)\vec{k}\,,$$

and S is the surface

$$S = \{(x, y, z) \mid z = x^2 + y^2 - 9,\ z \le 0\}$$

oriented by the continuous unit normal vector field $\vec{n}$ on S such that $\vec{n}(0, 0, -9) = -\vec{k}$.

7. Let $\vec{F}(x, y) = xy\,\vec{i} + xe^{-y\cos z}\,\vec{j} + \frac{e^x - e^{-x}}{2x} yz\,\vec{k}$. Compute the triple integral

$$\iiint_V \operatorname{div}(\operatorname{curl}(\vec{F}))\, dx\, dy\, dz$$

where V is the unit sphere $\{x^2 + y^2 + z^2 \le 1\}$.

8. For what values of the parameter λ is

$$\vec{F}(x, y) = -6x \sin y\,\vec{i} + (\lambda^2 - 4)x^2 \cos y\,\vec{j}$$

a conservative vector field? For such values of λ, compute the line integral

$$\oint_C \vec{F} \cdot d\vec{r}\,,$$

where C is the straight line segment from $(1, 0)$ to $(0, 1)$.

9. Let C be the rectangle whose vertices are $(-2, -3)$, $(2, -3)$, $(2, 3)$ and $(-2, 3)$, oriented counterclockwise. Compute the line integral

$$\oint_C x^2 y\, dx + y^3 x^3\, dy\,.$$

10. Let $\vec{F} = \vec{F}(x, y, z)$ be a smooth vector field defined on $V = \{1 \le x^2 + y^2 + z^2 \le 100\}$. Let $\vec{n}(x, y, z) = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{\sqrt{x^2 + y^2 + z^2}}$. Which ones among the following statements are true?

(I) If $\vec{F} = \text{curl}(G)$ for a smooth vector field G defined on V, then $\text{div}(\vec{F}) = 0$.

(II) If $\text{div}(\vec{F}) = 0$, then there exists a smooth vector field G such that $\vec{F} = \text{curl}(G)$.

(III) If $\text{div}(\vec{F}) = 0$, then

$$\iint_{\{x^2+y^2+z^2=100\}} \vec{F} \cdot \vec{n}\, dS = 0\,.$$

(IV) If $\text{div}(\vec{F}) = 0$, then

$$\iint_{\{x^2+y^2+z^2=100\}} \vec{F} \cdot \vec{n}\, dS = \iint_{\{x^2+y^2+z^2=1\}} \vec{F} \cdot \vec{n}\, dS\,.$$

(V) If $\vec{F} = \text{curl}(G)$ for a smooth vector field G, then

$$\iint_{\{x^2+y^2+z^2=100\}} \vec{F} \cdot \vec{n}\, dS = 0\,.$$