A New Spectrum of Recursive Models Using An Amalgamation Construction

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Abstract

We employ an infinite-signature Hrushovski amalgamation construction to yield two results in Recursive Model Theory. The first result, that there exists a strongly minimal theory whose only recursively presentable models are the prime and saturated models, adds a new spectrum to the list of known possible spectra. The second result, that there exists a strongly minimal theory in a finite language whose only recursively presentable model is saturated, gives the second non-trivial example of a spectrum produced in a finite language.

1 Introduction

Baldwin and Lachlan [2] developed the theory of $\aleph_1$-categoricity in terms of strongly minimal sets. They show, in particular, that for any $\aleph_1$-categorical theory in a countable language which is not $\aleph_0$-categorical, the countable models form a chain of elementary embeddings of length $\omega+1$: $M_0 \prec M_1 \prec \ldots \prec M_\omega$, where $M_0$ is the prime model and $M_\omega$ is the saturated model. Furthermore, there is a strongly minimal formula such that each $M_i$ is characterized by the size of a maximal algebraically independent subset realizing that formula. In particular, the models of a strongly minimal theory are characterized by the size of a maximal algebraically independent subset. We call such a subset a basis for the model.

Let $T$ be an $\aleph_1$-categorical theory in a countable language which is not $\aleph_0$-categorical. We have, by Baldwin-Lachlan, the chain of countable models of $T$: $M_0 \prec M_1 \prec \ldots \prec M_\omega$. We associate to $T$ the Spectrum of Recursive Models of $T$, $SRM(T) = \{i | M_i$ has a recursive presentation}. We call a set $S \subseteq \omega + 1$ a spectrum if there exists an $\aleph_1$-categorical $T$ such that $SRM(T) = S$.

A general problem of recent years has been to characterize which subsets of $\omega + 1$ are spectra. Many results in this direction have been of the form “$S$ is a spectrum” and have been proved by providing a construction of a theory $T$ yielding $S$ as $SRM(T)$. For example, Goncharov [3] showed that $\{0\}$ is a spectrum, and Kudaiibergenov [8] generalized Goncharov’s method to show that $\{0, \ldots, n\}$ is a spectrum for any $n$. In [5], it is shown that $\{\omega\}$ is a spectrum, and in [7], it is shown that $\omega$ and $\omega + 1 \setminus \{0\}$ are both spectra, and Nies [9] shows that $\{1, \ldots, n\}$ is a spectrum for any natural number $n$.

One thing that these examples all have in common is that each spectrum is an interval within $\omega + 1$, leading some to make the conjecture that this is always the case. In Theorem 1, we provide a counterexample to this conjecture.

Theorem 1. There exists a strongly minimal theory $T$ such that $SRM(T) = \{0, \omega\}$. 

There is also the question as to which of the possible spectra can be achieved with a finite language. That is, for which $S \subseteq \omega + 1$ does there exist a theory $T$ in a finite language such that $\text{SRM}(T) = S$? This question has its roots in the paper of Herwig, Lempp, and Ziegler [4], where they show that $\{0\}$ can be achieved as a spectrum of an $\aleph_1$-categorical theory in a finite language. In [1], this result is extended to show that $\{0, \ldots, n\}$ can be achieved as a spectrum of an $\aleph_1$-categorical theory in a finite language for any $n$, and similarly for the set $\omega$. In Theorem 2, we present another such result.

**Theorem 2.** There exists a strongly minimal theory $T$ in a finite language such that $\text{SRM}(T) = \{\omega\}$

In [1], as in this paper, the construction proceeds via an alteration of the Hrushovski construction. Aside from this similarity, the content of the proofs are different. In [1], the proof proceeds via encoding information into the type of a set based on how independent the set is, yielding that a truly independent $n + 1$-tuple codes non-recursive information. In the proof of Theorem 2, we do not directly code information, but rather we infinitely often change our mind about which formulas are algebraic to entice a recursive model to generate an independent sequence of elements.

Both main theorems will proceed via an alteration of the Hrushovski construction (see [6]) to allow for infinite languages. Through section 2 and most of section 3, we will work over a general countable language and allow for any $\mu$ function, to demonstrate the construction in generality. The amalgamation method, as developed by Hrushovski in [6], will be followed closely to ensure that the theory resulting from the construction is in fact strongly minimal. The new content in sections 2 and 3 is confined to the use of an infinite language and to the use of recursion theoretic information to change algebraicity within the Hrushovski construction. Nonetheless, we state the core lemmas and some chosen proofs in the interest of self-containment of this paper.

# 2 The Amalgamation Class

Let $L$ be a countable relationary language. Though the amalgamation construction described here works with any relationary language, we work with a language whose symbols are all ternary. Throughout the construction, we enforce that each relation is symmetric and holds only on distinct triples. For a finite $L$-structure $A$ and relation symbol $R \in L$, we write $|R(A)|$ for the number of triples from the set $A$ on which $R$ holds, counting each triple only once.

**Definition 3.** Set $\delta : \{\text{finite } L\text{-structures}\} \to \mathbb{Z} \cup \{\infty\}$ by $\delta(A) = |A| - \sum_{R \in L} |R(A)|$

From $\delta$, we make the standard definitions for a Hrushovski construction:

**Definition 4.** For any finite $L$-structures $A$ and $B$ and infinite $L$-structure $D$, we define:

- $\delta(B/A) = \delta(A \cup B) - \delta(A)$.
- If $A \subseteq B$, we set $\delta(A, B) = \min\{\delta(C)\mid A \subseteq C \subseteq B\}$.
- If $A \subseteq B$, we say $A$ is strong in $B$ or $A \leq B$ if $\delta(A) = \delta(A, B)$.
  We say $A$ is strong in $D$ if $A \subseteq D$ and $A$ is strong in $C$ for each finite $A \subseteq C \subseteq D$.
- We say $B$ is simply algebraic over $A$ if $B \neq \emptyset$, $A \cap B = \emptyset$, $A \leq A \cup B$, $\delta(B/A) = 0$, and there is no proper subset $B'$ of $B$ such that $\delta(B'/A) = 0$. 


We say that $B$ is minimally simply algebraic over $A$ if $B$ is simply algebraic over $A$ and there is no proper subset $A'$ of $A$ such that $B$ is simply algebraic over $A'$.

The standard necessary property of pre-dimension functions holds here, namely that $\delta(A \cup B) \leq \delta(A) + \delta(B) - \delta(A \cap B)$. We call this property sub-modularity. This is verified by seeing that each relation is counted at least as many times on the left as on the right. Also, note that equality holds if and only if there are no relations holding on tuples from $(A \cup B)$ other than those already holding in $A$ or $B$. In this case, we say $A$ and $B$ are freely joined over $A \cap B$. From the sub-modularity property, the following standard lemmas hold.

**Lemma 5.** Suppose $A \leq N$. Then:

1. $\delta(X \cap A) \leq \delta(X)$ for any $X \subseteq N$
2. $\delta(A', A) = \delta(A', N)$ for any $A' \subseteq A$
3. $A' \leq A \leq N$ implies $A' \leq N$

**Proof.** 2 and 3 follow from 1 trivially, so we will prove 1. $\delta(X \cup A) \leq \delta(X) + \delta(A) - \delta(X \cap A)$. Reordering, we get $\delta(X \cap A) \leq \delta(X) + \delta(A) - \delta(X \cup A) \leq \delta(X)$, where the last inequality holds because $A \leq N$. $\square$

**Lemma 6.** If $X, A$, and $B$ are finite $L$-structures such that $A \subseteq B$ and $X \cap B = \emptyset$, then $\delta(X/A) \geq \delta(X/B)$.

**Proof.** $\delta((X \cup A) \cup B) \leq \delta(X \cup A) + \delta(B) - \delta((X \cup A) \cap B)$, which simplifies to $\delta(X \cup B) - \delta(B) \leq \delta(X \cup A) - \delta(A)$, as needed. $\square$

The following two lemmas are from Lemma 2 and the proof of Lemma 3 in [6].

**Lemma 7.** Let $M$ be a finite $L$-structure. Let $A \subseteq M$ and suppose $B_j$ are simply algebraic over $A$ and $A \leq (A \cup \bigcup_j B_j), (j \in J)$. Then:

1. The $B_j$ are pairwise equal or disjoint.
2. $A \cup \bigcup_j B_j$ is a free join of the $B_j$ over $A$.
3. Suppose $A \subseteq A' \subseteq M$, $A' \leq A' \cup B_j$, and $B_j$ is not a subset of $A'$ (j = 1, 2). Then any isomorphism of $B_1$ with $B_2$ over $A$ extends to an isomorphism over $A'$. In fact, $A' \cup B_j$ is a free join of $A'$ and $B_j$ over $A$.

**Definition 8.** Let $C_0$ be the class of finite $L$-structures $C$ such that if $A \subseteq C \in C_0$, then $\delta(A) \geq 0$.

**Lemma 9.** Suppose $A, B_1, B_2 \in C_0$, $A = B_1 \cap B_2$, and $A \leq B_1$. Let $E$ be the free-join of $B_1$ with $B_2$ over $A$. Suppose $C^1, \ldots, C^r, F$ are disjoint substructures of $E$ such that each $C^i$ is minimally simply algebraic over $F$ and the structures $C^i$ and $C^j$ are isomorphic over $F$ for each $1 \leq i, j \leq r$. Then one of the following holds:

1. One of the $C^i$ is contained in $B_1 \setminus A$ and $F \subseteq A$.
2. The set $F \cup \bigcup_{i=1}^r C^i$ is entirely contained in $B_1$ or $B_2$. Further, in the case that $F \cup \bigcup_{i=1}^r C^i$ is entirely contained in $B_1$, one of the $C^i$ is contained in $B_1 \setminus A$.
3. $r \leq \delta(F)$
4. For one $C^i$, setting $X = (F \cap A) \cup (C^i \cap B_2)$, $\delta(X/X \cap A) < 0$. Further, one of the $C^j$ is contained in $B_1 \setminus A$. (Note that this cannot happen if $A \leq B_2$ by Lemma 5).

Continuing as in the standard Hrushovski construction, we define a bounding function $\mu$. $\mu$ will tell us how many copies of a given minimally simply algebraic extension to allow within structures in our amalgamation class. For a pair of finite structures $A$ and $B$, $\mu(B, A) \in \omega$ will decide this bound. Since $L$ may have infinite signature, it is not first order to bound the number of extensions isomorphic to $Y$ over $X$. In place of this, we will bound the number of extensions of the “form of $Y/X$”.

For any disjoint $L$-structures $\bar{a}, \bar{b} \subseteq C$, we write $tp_{r.q.f.}(\bar{b}/\bar{a})$ for the set $\{R(\bar{x}, \bar{y})| R \in L, (\bar{b} \cup \bar{a})^3 \triangleleft (\bar{a}^3, i \in \omega, \text{ and } R(\bar{b}, \bar{a}_i) \text{ holds}\}$. We call this set the relative quantifier-free type of $\bar{b}$ over $\bar{a}$. We say two relative quantifier-free types are the same if they are equal after a re-ordering of $\bar{b}$ and a re-ordering of $\bar{a}$. Thus we can talk about the relative quantifier-free type of the set $B$ over $A$, and we write $tp_{r.q.f.}(B/A)$.

We write $tp_{q.f.}(X)$ to refer to the quantifier-free type of $X$.

**Definition 10.** Let $Y$ and $X$ be finite $L$-structures such that $Y$ is minimally simply algebraic over $X$. Let $L_{Y/X}$ be the language generated by $\{R| R \in L, \exists \bar{x} \subseteq (B \cup A)^3 \triangleleft (\bar{a})^3, i \in \omega, \text{ and } R(\bar{b}, \bar{a}_i) \text{ holds}\}$. We say the language occurring in $tp_{r.q.f.}(Y/X)$. Suppose $B$ and $A$ are finite $L$-structures such that $tp_{r.q.f.}(B/A)|_{L_{Y/X}} = tp_{r.q.f.}(Y/X)$ and $tp_{q.f.}(X) = tp_{q.f.}(A)$. Then we say the extension $B/A$ is of the form of $Y/X$.

Fix a function $\mu$ from pairs of $L$-structures $(B, A)$ with $B$ minimally simply algebraic over $A$ to $\mathbb{N}$ so that $\mu$ depends only on the atomic type of the pair $(B, A)$ and $\mu(B, A) \geq |A|$. Further, $\mu$ must be such that if $\Gamma$ is a relative quantifier-free type, then there exists a sub-language $L'$ with a finite sub-signature of $L$ so that if $tp_{r.q.f.}(B/A) = \Gamma = tp_{r.q.f.}(B'/A')$ and $tp_{q.f.}(A)|_{L'} = tp_{q.f.}(A')|_{L'}$ then $\mu(B, A) = \mu(B', A')$.

Now we use $\mu$ to bound the number of extensions allowed of the form of a given minimally simply algebraic extension and define our amalgamation class. We may also fix a $b \in \omega$ which will become the dimension of the prime model of the theory we are constructing.

**Definition 11.** Let $C = C_{\mu, b}$ be the class of finite $L$-structures $C$ such that the following hold:

- If $A \subseteq C$ then $\delta(A) \geq \min(|A|, b)$.
- Let $Y/X$ be a minimally simply algebraic extension. Let $B_i$, $i = 1, \ldots, n$, and $A$ be disjoint subsets of $C$ such that $B_i/A$ is an extension of the form $Y/X$ for each $i$. Then $n \leq \mu(Y, X)$.

Note that if we choose $b \geq \text{arity}(R)$ then realizations of $R$ do not appear in members of $C$, thus will not appear in the amalgam we construct. Thus with our choice of a ternary language, only $b < 3$ will yield interesting models. In fact, we will use $b = 2$. This class $C$ is the class of finite $L$-structures which we will amalgamate together to form our generic model. To do so, we must show that $C$ satisfies an amalgamation property. Recall that the free-join of $X$ with $Y$ over $X \cap Y$ is the structure on $X \cup Y$ where a relation holding on $x$ implies that $x$ is contained entirely in $X$ or entirely in $Y$.

**Lemma 12.** (Algebraic Amalgamation Lemma) Suppose $A, B_1, B_2 \in C$, $B_1 \cap B_2 = A$, and $B_1 \setminus A$ is simply algebraic over $A$. Let $E$ be the free-join of $B_1$ and $B_2$ over $A$. Then $E \in C$ unless one of the following holds:
• $B_1 \triangleleft A$ is minimally simply algebraic over $F \subseteq A$, and $B_2$ contains $\mu(B_1 \triangleleft A, F)$ many disjoint extensions of $F$ of the form $(B_1 \triangleleft A)/F$.

• There exist a set $X \subseteq B_2$ and a subset $\hat{L}$ of $L_{B_1 \triangleleft A/A}$ such that $(A \cap X)|_{\hat{L}} \not\leq X|_{\hat{L}}$. Further, $B_1|_{\hat{L}}$ contains an isomorphic copy of $X|_{\hat{L}}$.

Proof. We focus on the last condition for $E$ to be in $C$. Suppose $Y$ is minimally simply algebraic over $X$ and $E$ contains disjoint sets $C^1, \ldots, C^r$, $F$ where each of the $C^j$ over $F$ is of the form of $Y$ over $X$ and we will look at the structure $E|_{L_{Y/X}}$. Call this structure $E'$. Our focus will be on using the structure $E'$ to count the $C^j$'s, so we abuse notation and write $C^j$ also for the subset of $E'$. Here, each of the $C^j$ over $F$ are minimally simply algebraic extensions. Note further that $E'$ is the free-join of $B_1|_{L_{Y/X}}$ with $B_2|_{L_{Y/X}}$ over $A|_{L_{Y/X}}$; both $B_1|_{L_{Y/X}}$ and $B_2|_{L_{Y/X}}$ are in $C_0$, each of the $C^j$ are simply algebraic over $F$, and $A|_{L_{Y/X}} \leq B_1|_{L_{Y/X}}$. By Lemma 9, there are 4 cases to consider:

1. One of the $C^i$ is contained in $B_1 \triangleleft A$ and $F \subseteq A$. Since $B_1 \triangleleft A$ is simply algebraic over $A$ in $E$, $C^i = B_1 \triangleleft A$. As the $C^j$ and $F$ are disjoint and one is $B_1 \triangleleft A$, each of the other $C^j$ as well as $F$ are contained in $B_2$. If $r > \mu(Y, X)$ then there must be $\mu(Y, X)$ of them contained in $B_2$ putting us in the case of the first exception to this lemma.

2. $F \cup \bigcup_{i=1}^{r} C^i$ is entirely contained in either $B_1$ or $B_2$. Then $r \leq \mu(Y, X)$ as $B_1, B_2 \in C$.

3. $r \leq \delta(F)$. Then $r \leq \delta(F) \leq |F| = |X| \leq \mu(Y, X)$.

4. For one $C^i$, setting $X = (F \cap A) \cup (C^i \cap B_2)$, we see that $\delta(X/X \cap A) < 0$. Further, one of the $C^j$ is contained in $B_1 \triangleleft A$. Letting $\hat{L} = L_{Y/X}$ yields the second exception in our lemma.

Lemma 13. (Strong Amalgamation Lemma) Suppose $A, B, C \in C$, $A = B \cap C$, $A \subseteq B$, and $A \leq C$. Then there exist $E \in C$ and embeddings $f : B \to E$, $g : C \to E$ so that $f|_A = g|_A$, $f(B) \leq E$, and $g(C) \leq E$.

Proof. This is a consequence of inductively applying the previous lemma as in Lemma 4 of [6].

Repeatedly applying the strong amalgamation lemma yields a model $M$. $M$ is strongly minimal, saturated, and recursive if $\mu$ is recursively approximable from below, as will be shown in the next section.

3 The Amalgam

Using the Strong Amalgamation Lemma, we get a model $M$ which satisfies the following:

1. $M$ is countable
2. Every finite substructure of $M$ is an element of $C$
3. Suppose $B \leq M$, $B \leq C$, and $C \in C$. Then there exists an embedding $f : C \to M$ such that $f|_B = id_B$, and $f(C) \leq M$. 

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Claim 14. \((1, 2, 3)\) and \((1, 2, 3', 3'')\) are equivalent.

Proof. \(\rightarrow\): \(3'\) follows from \(3\) directly and \(3''\) follows from the Algebraic Amalgamation Lemma.

\(\leftarrow\): \(3\) follows as in the proof of the Strong Amalgamation Lemma. \(\Box\)

Corollary 15. \(\mathcal{M}\) is saturated.

Proof. Any countable elementary extension of \(\mathcal{M}\) satisfies \((1, 2, 3', 3'')\), hence is isomorphic to \(\mathcal{M}\). It follows that there are only countably many types realized in elementary extensions of \(\mathcal{M}\). Hence, there is a saturated countable elementary extension of \(\mathcal{M}\), which \(\mathcal{M}\) must be isomorphic to. \(\Box\)

We want to characterize algebraicity in \(\mathcal{M}\). We define \(d(A) = \min\{\delta(C)|A \subseteq C \subseteq \mathcal{M}, C\ \text{finite}\}\). Clearly for any \(A\) and \(x\), either \(d(xA) = d(A)\) or \(d(xA) = d(A) + 1\).

Lemma 16. If \(d(xA) = d(A) + 1\) and \(d(yA) = d(A) + 1\), then \((\mathcal{M}, Ax) \cong (\mathcal{M}, Ay)\).

Proof. Let \(B\) be such that \(A \subseteq B, \delta(B) = d(A)\). Then \(B \subseteq \mathcal{M}, d(xB) = d(xA) = d(A)+1\). Thus \(xB \subseteq \mathcal{M}\), and similarly \(yB \subseteq \mathcal{M}\). Using property 3 and a standard back-and-forth argument, we see that \((\mathcal{M}, xB)\) and \((\mathcal{M}, yB)\) are isomorphic. \(\Box\)

We have shown that there is a unique 1-type over \(A\) of an element \(x\) such that \(d(xA) > d(A)\). Next we show that \(d(xA) = d(A)\) implies that \(x \in acl_{\mathcal{M}}(A)\).

Lemma 17. If \(d(xA) = d(A)\), then \(x \in acl_{\mathcal{M}}(A)\).

Proof. Suppose \(d(xA) = d(A)\). First, let \(B\) be a minimal set such that \(A \subseteq B\) and \(\delta(B) = d(A)\). We show that \(B\) is algebraic over \(A\) in \(\mathcal{M}\). Suppose there were two realizations of the type of \(B\) over \(A\). Call the second realization \(B'\). Then \(\delta(B \cup B') \leq \delta(B) + \delta(B') - \delta(B \cap B') < \delta(B') = d(A)\). The strict inequality is due to \(B\) being a minimal set with the properties that \(A \subseteq B\) and \(\delta(B) = d(A)\). This inequality contradicts the definition of \(d(A)\).

Fix \(E\) to be a set such that \(xA \subseteq E\) and \(\delta(E) = d(A)\). Then \(\delta(E \cup B) \leq \delta(E) + \delta(B) - \delta(E \cap B)\). If \(E\) does not contain \(B\), then \(\delta(E \cap B) > d(A)\) by minimality of \(B\). Then
\( \delta(E \cup B) \leq d(A) + d(A) - \delta(E \cap B) < d(A) \), again a contradiction. Thus, \( E \) contains \( B \) and \( d(xB) = d(B) \).

Take a sequence of extensions \( B_0, B_1, B_2, \ldots B_n \) such that \( B_0 = B, B_n = E \), and \( B_{i+1} \) is a minimal set such that \( B_i \subsetneq B_{i+1} \subseteq E \) and \( \delta(B_{i+1}) = d(A) \). Then \( B_{i+1} \) is a simply algebraic extension of \( B_i \). Thus \( B_{i+1} \) is algebraic over \( B_i \). We conclude that \( E \) is algebraic over \( A \). In particular, \( x \in acl_M(A) \). \( \square \)

**Corollary 18.** \( \text{Th}(M) \) is strongly minimal.

**Proof.** In the previous lemma, we showed that over any set there is a unique non-algebraic 1-type realized in \( M \). Since \( M \) is saturated, we see that \( \text{Th}(M) \) is strongly minimal. \( \square \)

**Corollary 19.** Let \( A \subset M \) be such that \( |A| < b \). Then \( acl_M(A) = A \). In particular, the dimension of the prime model of \( \text{Th}(M) \) is \( b \).

**Proof.** Suppose \( c \in acl_M(A) \setminus A \). By the characterization of algebraicity above, this means that there exists a set \( D \) so that \( cA \subseteq D \) and \( \delta(D) = d(A) \leq \delta(A) \leq |A| \), but \( |A| < |D| \) and \( |A| < b \). Thus \( \delta(D) < \min(|D|, b) \) yielding a contradiction.

Let \( B \leq M \) be of size at least \( b \) so that \( \delta(B) = |B| \). Suppose towards a contradiction that \( acl_M(B) \) is finite. By the characterization of algebraicity above, \( \delta(acl_M(B)) = |B| \), so \( acl_M(B) \leq M \). Let \( C \in C \) be any minimally simply algebraic extension over \( acl_M(B) \), which exists since \( |acl_M(B)| \geq b \). Since \( acl_M(B) \leq M \), property 3 yields that \( C \) embeds in \( M \) over \( acl_M(B) \). But then the embedded image of \( C \) is algebraic over \( B \) but not in \( acl_M(B) \), a contradiction. Therefore \( acl_M(B) \) is infinite, yielding a model of dimension \( |B| \). Thus the prime model has dimension exactly \( b \). \( \square \)

Thus far we have worked generally, constructing a strongly minimal model from any countable relational language \( L \), bounding function \( \mu \), and \( b \in \omega \). In what follows, we will fix \( L \) and \( b \), as well as describe a function \( \mu \) relative to any \( S \subseteq \mathbb{N} \). We will then fix the set \( S \) in section 5 to yield the theorems.

Let \( S \) be a subset of \( \mathbb{N} \). We view \( S \) as a set of pairs of natural numbers \((j,k)\) using a standard pairing function (a recursive bijection from \( \mathbb{N} \) to \( \mathbb{N} \times \mathbb{N} \)). We refer to \( \{m \in S \mid \exists k \ (m = \langle j, k \rangle) \} \) as the \( j^{th} \) column of \( S \) and will write \( S^{j} \) to denote this set. From the set \( S \), we define the set \( T \) to consist of the first two elements of each column not contained in \( S \), ie: \( T = \{ \langle j, k \rangle \mid (j, k) \notin S \} \) and \( \neg \exists 3k' (k' < k \land (j, k') \notin S) \).

We define \( L \) to be the language with signature \( \{ R \} \cup \{ R_i | i \in \omega \} \) where each relation symbol is ternary, and we fix \( b = 2 \).

We enumerate recursively the relative quantifier-free types of extensions \( Y \) over \( X \) such that \( Y \) is minimally simply algebraic over \( X \), \( |X| = 3 \), and the only relation symbol occurring in \( tp_{r,q,f}(Y/X) \) is \( R \). We refer to the \( i^{th} \) such enumerated relative quantifier-free type as \( \Lambda_i \).

We say \( B/A \) is a \( \Lambda_i \)-extension if \( tp_{r,q,f}(B/A)|_\{R\} = \Lambda_i \). Note that a minimally simply algebraic extension can be a \( \Lambda_i \)-extension and have a relation hold on the base. We now define the bounding function \( \mu \).

**Definition 20.**

\[
\mu(Y, X) = \begin{cases} 
4 & \text{if } Y/X \text{ is a } \Lambda_{(i,k)} \text{-extension, } \langle i, k \rangle \in T, \text{ and } R_i(X) \text{ holds} \\
4 & \text{if } Y/X \text{ is a } \Lambda_{(i,k)} \text{-extension and } \langle i, k \rangle \in S \\
|X| & \text{otherwise}
\end{cases}
\]

\(^1\)For a ternary language, a set \( \{a_1, \ldots a_n\} \) has a minimally simply algebraic extension \( C = \{c_1, \ldots c_n\} \) formed by setting \( R(a_i, c_i, c_{i+1}) \) to hold where \( i + 1 \equiv \text{mod } n \). In any relational language it is also true that sets of size \( \geq b \) have minimally simply algebraic extensions.
Note that in the first two cases $|X| = 3$ as the $\Lambda$’s are relative quantifier-free types over 3 element sets. Any integer greater than 3 could be used in place of 4 in the above definition.

**Lemma 21.** Suppose $S$ is $\Sigma_1$. Then $\mu$ recursively approximable from below, and thus $\mathcal{M}$ is a recursively presentable structure.

**Proof.** We fix recursive approximations $S_s$ to $S$ such that $S_s \subseteq S_{s+1}$. We will use these recursive approximations to $S$ to build recursive approximations to the amalgamation class and will be able to amalgamate to build $\mathcal{M}$. We define $T_s$ to be the set comprised of the least 2 elements in each of the first $s$ columns of $\omega \setminus S_s$.

At stage $s$, define a recursive approximation to $\mu$ by

$$
\mu_s(Y, X) = \begin{cases} 
4 & \text{if } Y/X \text{ is a } \Lambda_{(i,k)}\text{-extension, } (i, k) \in T_s, \text{ and } R_i(X) \text{ holds} \\
4 & \text{if } Y/X \text{ is a } \Lambda_{(i,k)}\text{-extension and } (i, k) \in S_s |X| \text{ otherwise}
\end{cases}
$$

Note that in the first two cases, $|X| = 3$, as the $\Lambda$’s are over 3 element sets. The 4 here corresponds to the same number in the definition of the true $\mu$. We define $C_s$, the amalgamation class allowed at stage $s$, from $\mu_s$.

Let $C_s$ be the class of finite $L$-structures $C$ such that the following hold:

- If $A \subseteq C$ then $\delta(A) \geq \min(|A|, 2)$.
- Let $Y/X$ be a minimally simply algebraic extension. Let $B_i$, $i = 1, \ldots, n$, and $A$ be disjoint subsets of $C$ such that $B_i/A$ is an extension of the form $Y/X$ for each $i$. Then $n \leq \mu_s(Y, X)$.

Since $\mu_s(Y, X) \leq \mu_{s+1}(Y, X)$, we see that $C_s \subseteq C_{s+1}$. As $\lim_s \mu_s = \mu$, we see that $C = \bigcup_s C_s$. To construct $\mathcal{M}$, we work in stages. At the $s^{th}$ stage, we amalgamate the first $s$ possible amalgamations allowed in $C_s$. As $C = \bigcup_s C_s$, every possible amalgamation in $C$ is amalgamated at a finite stage, and since $C_s \subseteq C$, we never leave the amalgamation class $C$. This constructs a generic model for $C$ which is therefore isomorphic to $\mathcal{M}$. 

From here forward we assume $S$ is a $\Sigma_1$ set, and thus the result of the lemma holds. We fix a recursive presentation of $\mathcal{M}$, and we refer to this particular presentation as $\mathcal{M}$ from here on.

# 4 The Restricted Language

To obscure the recursion theoretic content of the construction from the presentation of the model, we will restrict to the language generated by the single relation symbol $R$. Also, to force the prime model to be recursive in Theorem 1, we will name constants which will identify the prime model.

We fix a non-algebraic pair of elements $x$ and $y$ from $\mathcal{M}$. By the characterization above, $acl_{\mathcal{M}}(\{x, y\})$ is a $\Sigma_1$ set (ie: $z \in acl_{\mathcal{M}}(\{x, y\})$ if and only if $d(\{x, y, z\}) = 2$ if and only if $\exists A \supseteq \{x, y, z\} (\delta(A) = 2)$, which is a $\Sigma_1$ condition). Using this observation, we fix a recursive enumeration of $acl_{\mathcal{M}}(\{x, y\})$, $i \mapsto z_i$.

**Definition 22.** Let $\mathcal{M}'$ be the model obtained by restricting $\mathcal{M}$ to the language generated by $\{R\}$.

Let $\mathcal{M}''$ be the model constructed by adding constant symbols $\{c_i | i \in \omega\}$ to $\mathcal{M}'$ where $c_i$ names the element $z_i$. 

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Our next goal is to understand algebraicity in the model \( M' \). In particular, we will see that the relations that count are \( R \) and the \( R_i \) such that \( S^{[i]} \neq \omega^{[i]} \). From here forth, we call the language generated by \( \{ R \} \cup \{ R_i | S^{[i]} \neq \omega^{[i]} \} \) by the name \( L' \). We will see that the relations in \( L' \) are precisely the ones determining algebraicity in \( M' \). We will construct \( S \) so that \( S^{[i]} \) is either finite or equals \( \omega^{[i]} \), though the results of this section hold even where \( T^{[i]} \) contains exactly 1 element.

Recall that \( \Lambda^{(i,k)} \) is a relative quantifier-free type of an extension of a 3-element set and involves only the relation symbol \( R \). In the context of a first order formula, we write \( \Lambda^{(i,k)}(\vec{y},\vec{x}) \) to represent the formula which states that \( \vec{y} \) over \( \vec{x} \) is a \( \Lambda^{(i,k)} \)-extension. Note that \( \Lambda^{(i,k)}(\vec{y},\vec{x}) \) is a first order formula involving only the relation symbol \( R \).

**Lemma 23.** Let \( i \) be an integer such that \( S^{[i]} \neq \omega^{[i]} \), and let \( \langle i, k \rangle \) be an element of \( T^{[i]} \). Then \( M \models \forall \vec{x}(R_i(\vec{x}) \leftrightarrow \exists \vec{y}\Lambda^{(i,k)}(\vec{y}, \vec{x})) \)

**Proof.** \( \leftarrow \): If there are 4 disjoint \( \Lambda^{(i,k)} \)-extensions over \( \vec{x} \) and \( R_i(\vec{x}) \) does not hold, then taking the finite set \( A \) comprised of the 4 extensions and \( \vec{x} \), we see that \( A \notin \mathcal{C} \) as this explicitly violates the \( \mu \)-bound. This contradicts property (2) of \( \mathcal{M} \).

\( \rightarrow \): Suppose \( R_i(\vec{x}) \) holds. Then \( \delta(\vec{x}) = 2 \), which shows \( \vec{x} \subseteq \mathcal{M} \). By (3”), we see that there are 4 disjoint \( \Lambda^{(i,k)} \)-extensions over \( \vec{x} \). \( \square \)

Since each of the relations \( R_i \in L' \) are definable in \( M' \), we will abuse notation and say \( R_i(\vec{x}) \) holds in \( \mathcal{M}' \) to mean that the equivalent statement involving only \( R \) holds in \( M' \). Similarly for \( \mathcal{M}'' \). This shows that these relations still count in the reduct \( \mathcal{M}' \). Lemma 27 shows that these are the only relations that still count.

**Lemma 24.** Suppose \( \text{tp}_{\text{r.q.f.}}(B/A) = \text{tp}_{\text{r.q.f.}}(Y/X) \) and \( B/A \) is a minimally simply algebraic extension. Then \( Y/X \) is a minimally simply algebraic extension.

**Proof.** It is easy to check that \( \delta(Y/X) = 0 \) and \( \delta(Y_0/X) > 0 \) for any \( 0 \neq Y_0 \subseteq Y \), as these are true for \( B/A \). Thus \( Y \) is simply algebraic over \( X \). Also, each \( x \in X \) satisfies some relation in \( Y \cup X \) with an element in \( Y \), as it must be so for \( B/A \). Thus if \( Z \subseteq X \), then \( \delta(Y/Z) > \delta(Y/X) = 0 \), so \( Y \) is minimally simply algebraic over \( X \). \( \square \)

**Definition 25.** For finite \( A \subseteq M' \), let \( \delta'(A) = \delta(A|L') = |A| - |R(A)| - \sum_{R_j \in L'}|R_j(A)| \).

Let \( d'(A) = \min\{\delta'(B)|A \subseteq B \subseteq M', B \ finite\} \).

Let \( \mathcal{C}_{L'} \) be the class of finite \( L' \)-structures in \( \mathcal{C} \).

**Lemma 26.** If \( A \in \mathcal{C} \), then \( A|L' \in \mathcal{C}_{L'} \).

**Proof.** For any \( X' \subseteq A|L' \), let \( X \) be the expansion to \( L \). Then \( \delta(X') \geq \delta(X) \geq \min(|X|, 2) = \min(|X'|, 2) \).

Let \( Y' \) be a minimally simply algebraic extension over \( X' \). Suppose \( C'_1, \ldots, C'_n, F' \) are disjoint subsets of \( A|L' \) where each \( C'_i \) over \( F' \) is an extension of the form of \( Y' \) over \( X' \). Note that \( L_{Y'/X'} \subseteq L' \). Let \( C_1, \ldots, C_n, F \) be the expansions to \( L \). Let \( Y \) over \( X \) be an extension so that \( X \cong F \) and \( \text{tp}_{\text{r.q.f.}}(Y/X) = \text{tp}_{\text{r.q.f.}}(Y'/X') \). By Lemma 24 \( Y \) is minimally simply algebraic over \( X \). Each \( C_i \) over \( F \) is of the form of \( Y \) over \( X \) and \( \text{tp}_{\text{r.q.f.}}(X)|_{L'} = \text{tp}_{\text{r.q.f.}}(X') \). Since \( A \in \mathcal{C} \), \( n \leq \mu(Y,X) \). Observe from the definition of \( \mu \) that \( \mu(Y,X) \) depends only on the quantifier-free \( L' \)-type of \( Y' \cup X \). Thus \( n \leq \mu(Y,X) = \mu(Y',X') \). \( \square \)

**Lemma 27.** \( M|L' \) is generic for the class \( \mathcal{C}_{L'} \).
Proof. We need to show that $\mathcal{M}|_{L'}$ satisfies the conditions to be a generic model of $\mathcal{C}_{L'}$. We use the versions of $(1, 2, 3, 3')$ for $\mathcal{C}_{L'}$:

1: $\mathcal{M}|_{L'}$ is countable
2: For any finite $A \subseteq \mathcal{M}|_{L'}, A \in \mathcal{C}_{L'}$
3': $\mathcal{M}|_{L'}$ contains an infinite set $I$ such that there are no relations holding on $I$, and any finite $A \subseteq I$ has the property that $A \leq \mathcal{M}|_{L'}$.
3'': Suppose $B \subseteq \mathcal{M}|_{L'}, B \leq C, C \in \mathcal{C}_{L'}$, and $C \setminus B$ is simply algebraic over $B$, say minimally simply algebraic over $F \subseteq B$. Suppose also that for any subset $L$ of $L_{C \setminus B/F}$ and any $X \subseteq C$, there is no set $X'$ such that $X|_L \cong X'|_L$ and $(B \cap X')|_L \not\cong X'|_L$. Then there are $\mu(C \setminus B, F)$ many disjoint sets $A$ in $\mathcal{M}$ such that $A/F$ is an extension of the form $(C \setminus B)/F$.

1 is equivalent to 1 above. 2 follows immediately from 2 above and the previous lemma.
3' follows from 3' above since $B \leq A \leq \mathcal{M}$ implies that $B|_L \leq \mathcal{M}|_{L'}$.

Given $A \subseteq \mathcal{M}$ such that $F \subseteq B = A|_{L'}$ and $C \in \mathcal{C}_{L'}$ are as in 3'', we set $D$ to be the $L$-structure so that $tp_{r,q,f}(D/A) = tp_{r,q,f}(C/B)$. Let $E$ be the extension of $F$ to $L$. By Lemma 24, $D \setminus A$ is simply algebraic over $A$ and minimally simply algebraic over $E$. Since $L_{D,A}/A = L_{C,B}/B \subseteq L'$ and $(D, A, E)|_{L'} \cong (C, B, F)$, we see that $(D, A, E)$ satisfies the hypothesis of 3'' for $\mathcal{M}$. Applying 3'' for $\mathcal{M}$ to the extension of $D$ over $A$, we have $\mu(D \setminus A, E)$ disjoint extensions of the form of $(D \setminus A)/E$ over $E$ in $\mathcal{M}$. Since $\mu$ depends only on the quantifier-free $L'$-type, in $\mathcal{M}|_{L'}$ this gives $\mu(C \setminus B, F)$ disjoint extensions of the form of $(C \setminus B)/F$ over $F$. \hfill $\square$

**Lemma 28.** $x \in acl(\mathcal{M}')(A)$ if and only if $d'(xA) = d'(A)$.

**Proof.** Above we showed that for $\mathcal{M}$ the generic model of $C$, algebraicity meant $d(xA) = d(A)$. By the analogous argument for $\mathcal{M}|_{L'}$, we see that algebraicity here means $d'(xA) = d'(A)$. Since $\mathcal{M}|_{L'}$ is a definitional expansion of $\mathcal{M}'$, algebraicity is the same for $\mathcal{M}'$. \hfill $\square$

**Lemma 29.** $\mathcal{M}'$ and $\mathcal{M}''$ are both recursive, saturated, and strongly minimal.

**Proof.** $\mathcal{M}'$ is recursive, saturated, and strongly minimal, as it is a reduct to a recursive language of a model with all of these properties. $\mathcal{M}''$ is recursive since the assignment of the constants is recursive. It is strongly minimal, as adding constants to a strongly minimal theory retains strong minimality. Take $I$ an infinite algebraically independent sequence in $\mathcal{M}$ beginning with $\{x, y\}$. $I - \{x, y\}$ is algebraically independent over the algebraic closure of $\{x, y\}$ in $\mathcal{M}$. Thus $I - \{x, y\}$ is algebraically independent in $\mathcal{M}''$. This shows that $\mathcal{M}''$ has infinite algebraic dimension, thus is saturated. \hfill $\square$

## 5 Defining $S$

Thus far we have constructed the two models $\mathcal{M}'$ and $\mathcal{M}''$ relative to any given $\Sigma_1$ set $S$. We aim for a construction where $SRM(Th(\mathcal{M}')) = \{\omega\}$ and $SRM(Th(\mathcal{M}'')) = \{0, \omega\}$. To ensure this, we need to diagonalize against the possible finite-dimensional models of each theory. In this section, we construct the $\Sigma_1$ set $S$ to ensure these results.

We want to ensure that the finite dimensional models of $Th(\mathcal{M}')$ and $Th(\mathcal{M}'')$ are not recursive. There is no 0-dimensional model of $Th(\mathcal{M}')$ (i.e., $acl(\emptyset) = \emptyset$), so we will diagonalize only against positive dimensional models. We fix a recursive enumeration of all pairs $(f, U)$ where $f$ is a partial recursive function from the set of quantifier-free formulas in the language $\{R\} \cup \{c_i | i \in \omega\} \cup \mathbb{N}$ to $\{true, false\}$ and $U$ is a non-empty finite
subset of \( \mathbb{N} \). Note that if \( f \) gives the quantifier-free diagram of a model, \( \text{dom}(f) \) contains symbols for the elements of the model, which is why we include the symbols from \( \mathbb{N} \). This is to be interpreted as \( f \) giving the quantifier-free diagram of a model \( N \) with universe \( \mathbb{N} \) and \( U \) representing a basis of the model. Note that even though we work with functions \( f \) describing structures in the language including constants, the construction allows for functions which give a model of \( \text{Th}(M') \), ie: where each \( c_i \) is not interpreted. Thus our construction of \( S \) will simultaneously diagonalize against finite-dimensional models of \( \text{Th}(M') \) and of \( \text{Th}(M'') \).

We will describe a routine for enumerating \( S \). For the \( i^{\text{th}} \) pair \((f, U)\), we will have an \( i^{\text{th}} \) subroutine \( \text{Routine}_i \) whose job it is to ensure that this pair does not represent a model \( N \) with a basis \( U \) satisfying either of the theories of \( \mathcal{M}' \) or \( \mathcal{M}'' \).

Given a pair \((f, U)\), at stages we read off information about the model it describes from \( f_s \), the computation of \( f \) at stage \( s \). We let \( N_0 \) be the empty model, and let \( N_s \) be comprised of all \( n \leq s \) such that for each \( m < n \), \( f_s(n = m) \models \text{false} \). In \( N_s \), we say \( R(\bar{x}) \) holds if \( f_s(R(\bar{x})) \models \text{true} \). We say for \( R_i \in L \) with \( i < s \), \( R_i(\bar{x}) \) holds if there is an \( (i, k) \in T_s[i] \) such that \( N_s \models \exists \vec{y} \Lambda_{(i,k)}(\vec{y}, \vec{x}) \), where \( \Lambda_{(i,k)}(\vec{y}, \vec{x}) \) has already been defined in \( N_s \) as a conjunction of \( R \)-statements. For a set \( A \) of natural numbers, we write \( \delta(A) \) for \( \delta(A) \) as \( A \) is seen in the structure \( N_s \). Finally, we set \( K_s \subseteq N_s \) to be the set of elements \( x \in N_s \) such that \( f_s(x = c_i) \models \text{true} \) for some \( i \leq s \).

\( \text{Routine}_i \) is the only part of our program allowed to enumerate anything into \( S[i] \). When \( \text{Routine}_i \) is initialized, \( S[i] = \emptyset \). The routine runs in parts as follows:

Part 1) Wait until a stage \( s \) where there is some set \( X \subseteq N_s \) and a set \( K \subseteq K_s \) such that \((X \cup U \cup K)|_{R_s} \) is a minimally simply algebraic extension over \((U \cup K)|_{R_i} \). Once found, for the duration of its run \( \text{Routine}_i \) refers to these sets as \( X \) and \( K \).

Part 2) The first thing \( \text{Routine}_i \) does when it reaches part 2 is to define the set of obstructions to moving to part 3. Suppose we reach part 2 on stage \( t \). A set \( Y \subseteq N_t \) is an obstruction to moving to part 3 if \( \delta_t(Y/K_t) < |U| \) and \( U \subseteq Y \). For each \( k \in \omega \), if during a stage \( s > t \) an element is enumerated into \( S[k] \), then we say \( R_k \) is removed. If at a stage \( s \) enough \( R_k \) are removed so that counting only the non-removed \( R_k \), \( \delta_s(Y/K_t) \geq |U| \), then we say the obstruction \( Y \) has been removed. That is, if

\[
\left| Y \cup K_t \right| - |R(Y \cup K_t)| - \sum_{R_j \text{ not removed}} |R_j(Y \cup K_t)| - \left| K_t \right| - |R(K_t)| - \sum_{R_j \text{ not removed}} |R_j(K_t)| \geq |U|,
\]

then the obstruction \( Y \) is removed.

If for each tuple \( \bar{x} \in X \cup U \cup K \), \( N_s \models \exists \bar{y} \Lambda_{(i,l)}(\bar{y}, \bar{x}) \leftrightarrow \exists \bar{y} \Lambda_{(i,m)}(\bar{y}, \bar{x}) \) where \( \{(i, l), (i, m)\} = T_s[i] \), then we say \( \text{Routine}_i \) is ready for part 3. If \( \text{Routine}_i \) is ready for part 3 and all obstructions have been removed, \( \text{Routine}_i \) moves to part 3.

Part 3) \( \text{Routine}_i \) takes the least element of \( \omega[i] \) which has not yet been enumerated into \( S \) and enumerates it into \( S \). \( \text{Routine}_i \) then goes back to part 2.

The possible outcomes of a run of \( \text{Routine}_i \) are that it gets stuck in part 1, it gets stuck in part 2, or it cycles between part 2 and part 3 infinitely often. In the first case, \( S[i] = \emptyset \). In the second case, \( S[i] \) is a finite initial segment of \( \omega[i] \), and in the third case \( S[i] = \omega[i] \). In any case, we will show that either \( N \) does not satisfy the right theory or \( U \) is not its basis.
6 The Main Theorems

In the previous section we defined a \(\Sigma_1\) set \(S\), and in the section before we gave a construction of two models \(\mathcal{M}'\) and \(\mathcal{M}''\) from any fixed \(\Sigma_1\) set. We fix \(\mathcal{M}'\) and \(\mathcal{M}''\) to be those models obtained by applying the construction to the set \(S\) defined in the previous section.

It is clear that \(\omega \in SRM(Th(\mathcal{M}'))\) and \(0, \omega \in SRM(Th(\mathcal{M}''))\). The first is because \(\mathcal{M}'\) has a recursive presentation and is saturated. The second is because \(\mathcal{M}''\) has a recursive presentation and is saturated and the set of constants in \(\mathcal{M}''\) is algebraically closed and infinite, hence also a model of the same theory. Since the set of constants is \(\Sigma_1\) in the recursive presentation of \(\mathcal{M}''\), they form a recursive prime model. It remains to show that for any other \(n \in \omega + 1\), \(n\) is not in \(SRM(Th(\mathcal{M}'))\) or \(SRM(Th(\mathcal{M}''))\).

**Theorem 30.** \(SRM(Th(\mathcal{M}')) = \{\omega\}\)

*Proof.* Suppose \(N\) is a recursive model of \(Th(\mathcal{M}')\), and \(N\) has a finite basis \(U\). Let \(i\) be the index of the pair \((f, U)\) where \(f\) is the recursive function describing the quantifier-free diagram of \(N\). Note that we let the domain of \(f\) include symbols for constants, but \(f(c_i = n) \downarrow \text{false}^*\) for each \(n \in \omega\).

Case 1: \(Routine_i\) gets stuck in part 1. \(R_i \in L'\), as \(Routine_i\) is never in stage 3. Note that \(Th(\mathcal{M}')\) has no model with a basis of size \(< 2\). So, we may assume \(|U| \geq 2\). We are guaranteed by 3\(''\) that if \(N\) satisfies \(Th(\mathcal{M}')\) then \(N\) must contain minimally simply algebraic extensions involving only the relation \(R_i\) over \(U\). Thus \(N \not\models Th(\mathcal{M}')\).

Case 2: \(Routine_i\) gets stuck in part 2.

Case 2a: \(Routine_i\) gets stuck in part 2 because it is never ready for part 3. This means that for some \(\bar{x} \in X \cup U\) and \(\langle i, l \rangle, \langle i, m \rangle \in T^{[i]}_s\), \(N \not\models \exists^2 \bar{y} \Lambda_{\langle i, l \rangle}(\bar{y}, \bar{x}) \leftrightarrow \exists^3 \bar{y} \Lambda_{\langle i, m \rangle}(\bar{y}, \bar{x})\). Since \(Routine_i\) never gets to part 3 again, \(T^{[i]}_s = T^{[i]}_s\). By Lemma 23, \(\mathcal{M}' \models \exists^2 \bar{y} \Lambda_{\langle i, l \rangle}(\bar{y}, \bar{x}) \leftrightarrow R_i(\bar{x}) \leftrightarrow \exists^3 \bar{y} \Lambda_{\langle i, m \rangle}(\bar{y}, \bar{x})\). Thus \(N \not\models Th(\mathcal{M}')\).

Case 2b: There is an obstruction \(Y\) which is never removed.

As \(N\) is assumed to be a model of \(Th(\mathcal{M}')\), for each \(n \in \omega\) and constant symbol \(c_i\), \(N \not\models n = c_i\). Thus \(K_t = \emptyset\) for each stage \(t\), and we can simplify notation and write \(\delta_t(Y)\) for \(\delta_s(Y/K_t)\). When counting the non-removed relations, \(\delta_t(Y) = \delta_s(Y/K_t) < |U|\). Since the obstruction is never removed, \(\delta'(Y) < |U|\), contradicting \(U\) being an independent set in \(N\).

Case 3: \(Routine_i\) loops through part 2 and part 3 infinitely often.

The construction is built so that in this case, \(U\) will not be a basis for the model \(N\). We will derive a contradiction from the assumption that \(U\) is a basis for \(N\). From this assumption, we see that \(X\) is algebraic over \(U\), which means that there is a set \(Y\) such that \(\delta'(Y) = |U|\), and \(X \cup U \subseteq Y\). Let \(t\) be a stage when \(Routine_i\) enters part 2 which is large enough that \(Y \subseteq N_t\) for each relation \(R_j\) in \(L'\) occurring on \(Y\), \(S^{[i]}_t = S^{[i]}\). Such a \(t\) exists since by construction \(S^{[i]}\) is finite for each \(R_j \in L'\). We will show that \(Routine_i\) never enters part 3 after stage \(t\), leading to a contradiction.

As \(Routine_i\) is ready for part 3 each time it leaves part 2, we see that at all stages \(s\) after \(Routine_i\) completes part 1, \(N_s\) realizes occurrences of \(R_i\) on \(X\). Let \(s > t\) be a stage when \(Routine_i\) is in part 2. Then \(\delta_s(Y) < \delta'(Y)\) since \(R_i\) occurs on \(X\) but does not count in \(\delta'\). Then \(R_i\) is a relation which has not been removed since entering part 2 and neither have any of the relations counted in \(\delta'\), so counting only the non-removed relations, \(\delta_s(Y) < |U|\). Thus, \(Y\) is an obstruction which is never removed after stage \(t\), contradicting our being in case 3.
In any case, we get a contradiction to the assumption that $N$ is a recursive model of $Th(M')$ with finite basis $U$. \qed

**Lemma 31.** Let $U$ be a finite subset of $M''$. Then $x \in acl_{M''}(U)$ if and only if there is a finite set $C$ of elements named by constants and $x \in acl_{M'}(U \cup C)$.

**Proof.** The right-to-left direction is trivial. To prove the left-to-right direction, take an $C$ and a finite set of constants $C$.

The construction is built so that in this case, $U$ is not an independent set over the constants $C$. We will derive a contradiction from the assumption that $U$.

**Theorem 32.** $SRM(T h(M'')) = \{0, \omega\}$

**Proof.** Suppose $N$ is a recursive model of $Th(M'')$ and $N$ has a finite basis $U$. Let $i$ be the index of the pair $(f, U)$ where $f$ is the function describing the quantifier-free diagram of $N$.

Case 1: Routine$_i$ gets stuck in part 1. $R_i \in L'$, as Routine$_i$ is never in stage 3. $M'$ has minimally simply algebraic extensions involving only the relation $R_i$. Thus $N$ does not satisfy $Th(M')$. In particular, if $|U| \geq 2$ then $3''$ guarantees that there are minimally simply algebraic extensions of $U$ in $N$ involving only the relation symbol $R_i$. If $|U| = 1$, then $3''$ guarantees the same for $U \cup \{c_k\}$ for any constant $c_k$.

Case 2: Routine$_i$ gets stuck in part 2.

Case 2a: Routine$_i$ gets stuck in part 2 because it is not ready for part 3. This means that for some $\bar{x} \in X \cup U \cup K$, $(i, l), (i, m) \in T_s^{[i]}$, $N \not\models \exists^4 \bar{y} \Lambda_{(i,l)}(\bar{y}, \bar{x}) \leftrightarrow \exists^3 \bar{y} \Lambda_{(i,m)}(\bar{y}, \bar{x})$. But since Routine$_i$ never gets to part 3 again, $T^{[i]} = T^{[i]}$. By Lemma 23, $M' \models \exists^4 \bar{y} \Lambda_{(i,l)}(\bar{y}, \bar{x}) \leftrightarrow R_0(\bar{x}) \leftrightarrow \exists^3 \bar{y} \Lambda_{(i,m)}(\bar{y}, \bar{x})$. Thus $N \not\models Th(M')$.

Case 2b: There is an obstruction $Y$ that is never removed. There is a finite set of constants $C$ in $N$ such that counting only the non-removed relations on $Y$, $\delta_s(Y/C) < |U|$. As the obstruction is never removed, $\delta'(Y/C) < |U|$, implying that $U$ is not an independent set over the constants $C$. Hence $U$ is not algebraically independent over $\emptyset$.

Case 3: Routine$_i$ loops through part 2 and part 3 infinitely often. The construction is built so that in this case, $U$ will not be a basis for the model $N$. We will derive a contradiction from the assumption that $U$ is a basis for $N$. From this assumption, we see that $X$ is algebraic over $U$, which means that there is a finite set $Y$ and a finite set of constants $C \supseteq K$ such that $\delta'(Y/C) = |U|$ and $X \cup U \subseteq Y$. Let $t$ be a stage when Routine$_i$ enters part 2 which is large enough that $Y \cup C \subseteq N_t$ and for each relation $R_j$ in $L'$ occurring on $Y \cup C$, $S_t^{[j]} = S^{[j]}$. Such a $t$ exists as $S^{[j]}$ is finite for each $R_j \in L'$. We will show that Routine$_i$ never enters part 3 after stage $t$, leading to a contradiction.

As Routine$_i$ is ready for part 3 each time it leaves part 2, we see that at all stages $s$ after Routine$_i$ completes part 1, $N_s$ realizes occurrences of $R_i$ on $X$. Let $s > t$ be a stage when Routine$_i$ is in part 2. Then $\delta_s(Y/C) < \delta'(Y/C)$ since $R_i$ occurs on $X$ but does not count in $\delta'$. Then $R_i$ is a relation which has not been removed since entering part 2 and neither have any of the relations counted in $\delta'$, so counting only the non-removed relations, $\delta_s(Y/C) < |U|$. Thus, $Y$ is an obstruction which is never removed after stage $t$, contradicting our being in case 3.

In any case, we get a contradiction to the assumption of $N$ being a recursive model of $Th(M'')$ with finite basis $U$. \qed
References


