Math 217 Exam #1 Spring 2010

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TA:

Grading Table

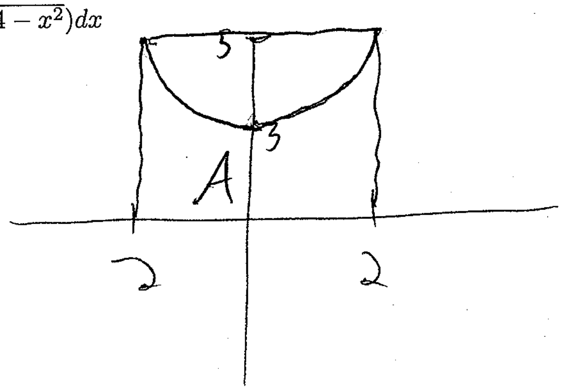
Question	Possible Points	Points Earned
1	10	
2	15	
3	15	
4	15	
5	15	
6	15	
7	15	
EC	**10**	
Total	100	

Please read these instructions carefully before beginning.

1. Do not open the exam until 5:30. You will have 90 minutes to complete the exam.
2. Final answers must be written clearly in the proper space provided. No credit will be given for illegible or ambiguous answers. Answers with no justification will not be given credit.
3. No notes, calculators, or electronic devices are permitted.

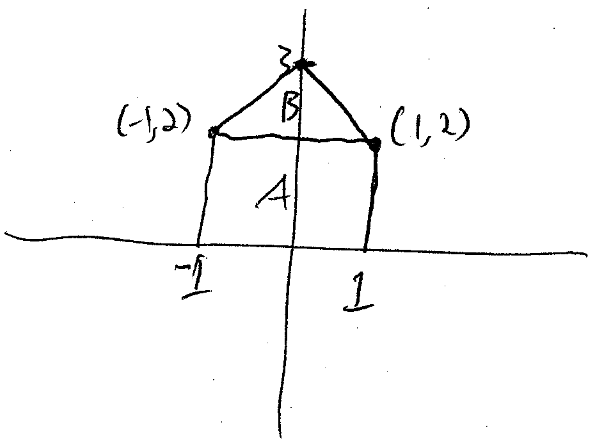
1. (10 points) Compute the following definite integrals by computing the area of the corresponding region of the plane.

(a) $\int_{-2}^{2}(5-\sqrt{4-x^2})dx$



$$A = 5 \cdot 4 - \frac{1}{2}\pi \cdot 2^2 = 20 - 2\pi$$

(b) $\int_{-1}^{1}(3-|x|)dx$



$A = 4$

$B = \frac{2 \cdot 1}{2} = 1$

$$\int_{-1}^{1}(3-|x|)\,dx = 4+1 = 5$$

2. (15 points) Show that the function $f(x) = x^3 + \frac{4}{x^2} + 7$ has exactly one zero on the interval $(-\infty, 0)$. Be clear about which theorems you are using (if you use any).

$$f(-5) = -125 + \frac{4}{25} + 7 < 0$$

$$f(-1) = -1 + \frac{4}{1} + 7 > 0$$

By the intermediate value theorem, since f is continuous on $[-5,-1]$, there is a zero of f in $(-5,-1)$.

Suppose there were 2 zeros in $(-\infty, 0)$: a_1, a_2 then by Rolle's Thm (since f is differentiable on (a_1, a_2) & continuous on $[a_1, a_2]$)

there would be a $c \in (a_1, a_2)$ so that $f'(c) = 0$.

But $f'(c) = 3c^2 - \frac{4}{c^3}$, which is positive for $c < 0$.

This contradiction shows that our supposition is false.

3. (15 points) **Using Riemann Sums**, evaluate the following definite integral

$$\int_1^3 x^2 dx$$

$$\int_1^3 x^2 dx = \lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{2}{n}\left(1+\frac{2}{n}\cdot k\right)^2$$

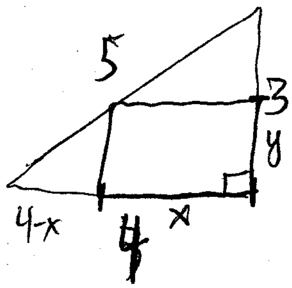
$$= \lim_{n\to\infty} \frac{2}{n}\cdot \sum_{k=0}^{n-1} 1 + \frac{4}{n}k + \frac{4}{n^2}k^2$$

$$= \lim_{n\to\infty} \frac{2}{n}\left(n + \frac{4}{n}\cdot\frac{n(n-1)}{2} + \frac{4}{n^2}\cdot\frac{(n-1)(n)(2n-1)}{6}\right)$$

$$= \lim_{n\to\infty}\left(2 + \frac{4(n-1)}{n} + \frac{8}{6}\cdot\frac{(n-1)(2n-1)}{n^2}\right)$$

$$= 2 + 4 + \frac{8}{3} = 6 + \frac{8}{3} = \frac{26}{3}.$$

4. (15 points) What are the dimensions of the rectangle of maximal area which can be inscribed (entirely contained) in the right triangle with side-lengths 3, 4 and 5.



$$A = xy$$

$$\frac{4-x}{y} = \frac{4}{3}$$

$$12 - 3x = 4y$$

$$y = 3 - \frac{3}{4}x$$

$$A = x\left(3 - \frac{3}{4}x\right)$$

$$\frac{dA}{dx} = \left(3 - \frac{3}{4}x\right) + x\left(-\frac{3}{4}\right).$$

$$= 3 - \frac{6}{4}x$$

$$\frac{dA}{dx} = 0 \Rightarrow 3 - \frac{6}{4}x = 0$$

$$x = \frac{4 \cdot 3}{6} = 2$$

$$y = 3 - \frac{3}{4} \cdot 2 = 3 - \frac{3}{2} = \frac{3}{2}.$$

First-deriv. test:

$x=2$

	$x<2$	$x>2$
dA/dx	+	−
A	↗	↘

So 2 is a maximum.

5. (15 points) Sketch the curve of the function $f(x) = 8x^2 - x^4$, labeling all local maxima, minima, and inflection points. Also, determine on what regions the function is increasing, decreasing, concave up, and concave down.

$f'(x) = 16x - 4x^3$

$f''(x) = 16 - 12x^2$

Finding Critical pts:

$16x - 4x^3 = 0$

$x(16 - 4x^2) = 0$

$4x(4 - x^2) = 0$

$-4x(x-2)(x+2) = 0$

$x = 0, 2, \text{or } -2$

First Deriv. Test

	$x<-2$	$(-2,0)$	$(0,2)$	$x>2$
f'	+	−	+	−
f	↗	↘	↗	↘

Finding Inflection Pts:

$16 - 12x^2 = 0$

$x^2 = \frac{16}{12} = \frac{4}{3}$

$x = \pm \frac{2}{\sqrt{3}}$

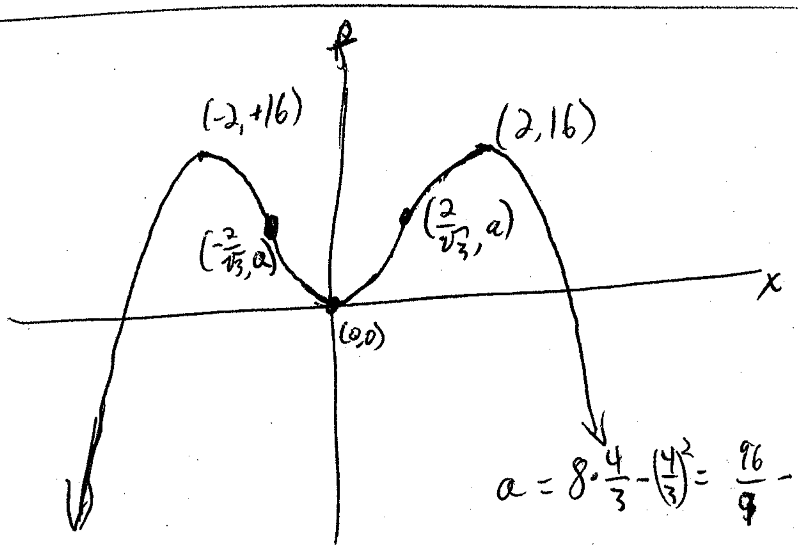
	$(-\infty, -\frac{2}{\sqrt{3}})$	$(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}})$	$(\frac{2}{\sqrt{3}}, \infty)$
f''	−	+	−
f	⌒	∪	⌒

Increasing: $(-\infty, -2) \cup (0, 2)$

Dec: $(-2, 0) \cup (2, \infty)$

Conc. Up: $(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}})$

Conc. Down: $(-\infty, -\frac{2}{\sqrt{3}}) \cup (\frac{2}{\sqrt{3}}, \infty)$



$a = 8 \cdot \frac{4}{3} - \left(\frac{4}{3}\right)^2 = \frac{96}{9} - \frac{16}{9} = \frac{80}{9}$

6. (15 points) Determine the following anti-derivatives.

(a) $\int \sin(4x)dx$

$$\frac{-\cos(4x)}{4} + C$$

(b) $\int (x^3 + \frac{3}{x^2} + \sqrt{x})dx$

$$\frac{x^4}{4} - \frac{3}{x} + \frac{2}{3}x^{3/2} + C$$

(c) $\int \frac{t\sqrt{t} + \sqrt{t}}{t^2}dt$

$$= \int \frac{t^{3/2} + t^{1/2}}{t^2} dt$$

$$= \int (t^{-1/2} + t^{-3/2}) dt = 2t^{1/2} + \left(-\frac{1}{2} t^{-1/2}\right) + C.$$

7. (15 points) Find the absolute maximum and minimum of $f(x) = x(x-4)^2$ on the closed interval $[2, 5]$.

$$f'(x) = (x-4)^2 + x \cdot 2(x-4)$$
$$= (x-4)(x-4+2x) = (x-4)(3x-4)$$

Critical pts: $x = 4$ & $4/3$ ← not in interval

Testing critical pts & endpoints:

$f(2) = 2 \cdot (-2)^2 = 8$ ← absolute max

$f(4) = 0$ ← absolute min.

$f(5) = 5 \cdot 1 = 5$

Extra Credit (10 points) Prove that proposition P_n is true for every natural number n.

$$P_n := 3 \text{ divides the number } n^3 + 2n$$

By induction:

Base: P_1 says 3 divides $1^3+2\cdot 1 = 1+2=3$ ✓

Step: Assume P_n. We have to show P_{n+1}.

ie: we need to show that

3 divides $(n+1)^3+2(n+1)$

ie: 3 divides $(n^3+2n)+(3n^2+3n+1+2)$

$= (n^3+2n)+(3n^2+3n+3)$

3 divides (n^3+2n), since P_n is assumed true.

3 divides $3n^2+3n+3$,

So 3 divides the sum

$(n^3+2n)+(3n^2+3n+3) = (n+1)^3+2(n+1)$.

Having assumed P_n, we showed P_{n+1} to be true.

By induction, we showed P_n for every n.