§5.3

#1 $f(x) = x^2$.

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} c_k^2 \Delta x_k = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k) \Delta x_k$$

$$= \int_0^2 x^2 dx$$

#2 $f(x) = 2x^3$

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} 2c_k^3 \Delta x_k = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k) \Delta x_k = \int_{-1}^{0} 2x^3 \, dx$$

#3 $f(x) = x^2 - 3x$

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} (c_k^2 - 3c_k) \Delta x_k = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k) \Delta x_k = \int_{-7}^{5} (x^2 - 3x) \, dx$$

#4 $f(x) = \frac{1}{x}$

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} \left(\frac{1}{c_k}\right) \Delta x_k = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k) \Delta x_k = \int_1^4 \frac{1}{x} \, dx$$

#5 $f(x) = \frac{1}{1-x}$

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} \frac{1}{1-c_k} \Delta x_k = \lim_{\|P\|\to 0} f(c_k) \Delta x_k = \int_2^3 \frac{1}{1-x} \, dx$$

#9 (a) By zero-width interval, $\int_2^2 g(x)\,dx = 0$.

(b). $\int_5^1 g(x)\,dx = -\int_1^5 g(x)\,dx$ (by order of integration).

$= -8$

(c) $\int_1^2 3f(x)\,dx = 3\int_1^2 f(x)\,dx$ (by constant multiple)

$= 3\cdot(-4) = -12$

(d) $\int_1^5 f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^5 f(x)\,dx$ (by additivity)

$-4 = 6 + \int_2^5 f(x)\,dx \Rightarrow \int_2^5 f(x)\,dx = -4-6 = -10.$

(e) $\int_1^5 [f(x) - g(x)]\,dx = \int_1^5 f(x)\,dx - \int_1^5 g(x)\,dx$ (by difference)

$= 6 - 8 = -2.$

(f) $\int_1^5 [4f(x)-g(x)]\,dx = \int_1^5 4f(x)\,dx - \int_1^5 g(x)\,dx$ (by difference)

$= 4\int_1^5 f(x)\,dx - \int_1^5 g(x)\,dx$ (by constant multiple)

$= 4(6)-8 = 24-8=16.$

#10

(a) $\int_1^9 -2f(x)\,dx = -2\int_1^9 f(x)\,dx = -2(-1) = 2$

(b) $\int_7^9 [f(x)+h(x)]\,dx = \int_7^9 f(x)\,dx + \int_7^9 h(x)\,dx = 5+4=9$

(c) $\int_7^9 [2f(x)-3h(x)]\,dx = \int_7^9 2f(x)\,dx - \int_7^9 3h(x)\,dx$

$= 2\int_7^9 f(x)\,dx - 3\int_7^9 h(x)\,dx$

$= 2\cdot 5 - 3\cdot 4 = 10-12 = -2$

(d) $\int_9^1 f(x)\,dx = -\int_1^9 f(x)\,dx = -(-1) = 1$

(e) $\int_1^9 f(x)\,dx = \int_1^7 f(x)\,dx + \int_7^9 f(x)\,dx$

$-1 = \int_1^7 f(x)\,dx + 5 \Rightarrow \int_1^7 f(x)\,dx = -6$

(f) $\int_9^7 [h(x)-f(x)]\,dx = -\int_7^9 [h(x)-f(x)]\,dx$

$= -\left(\int_7^9 h(x)\,dx - \int_7^9 f(x)\,dx\right)$

$= -\int_7^9 h(x)\,dx + \int_7^9 f(x)\,dx$

$= -4+5$

$=1$

#12

(a) $\int_0^{-3} g(t)\,dt = -\int_{-3}^{0} g(t)\,dt = -\sqrt{2}$

(b) $\int_{-3}^{0} g(u)\,du = \int_{-3}^{0} g(t)\,dt = \sqrt{2}$ (dummy variable).

(c) $\int_{-3}^{0} [-g(x)]\,dx = -\int_{-3}^{0} g(x)\,dx = -\sqrt{2}$

(d). $\int_{-3}^{0} \frac{g(r)}{\sqrt{2}}\,dr = \frac{1}{\sqrt{2}}\int_{-3}^{0} g(r)\,dr = \frac{1}{\sqrt{2}}\cdot\sqrt{2} = 1$

#14

(a) $\int_{-1}^{3} h(r)\,dr = \int_{-1}^{1} h(r)\,dr + \int_{1}^{3} h(r)\,dr$

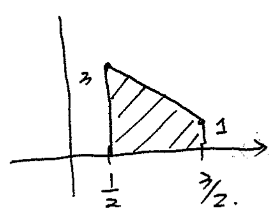
$6 = 0 + \int_{1}^{3} h(r)\,dr$

$\Rightarrow \int_{1}^{3} h(r)\,dr = 6.$

(b) $-\int_{3}^{1} h(u)\,du = \int_{1}^{3} h(u)\,du = 6$

#16 $\int_{\frac{1}{2}}^{3/2} (-2x+4)\,dx$ means the area under $y=f(x)=-2x+4$

over interval $[\frac{1}{2}, \frac{3}{2}]$.

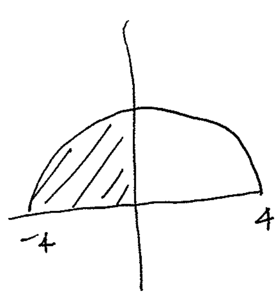


$f(\frac{1}{2}) = -2(\frac{1}{2}) + 4 = 3$

$f(\frac{3}{2}) = -2(\frac{3}{2}) + 4 = 1$

and the area is $\frac{(3+1)\cdot 1}{2} = 2.$

#18



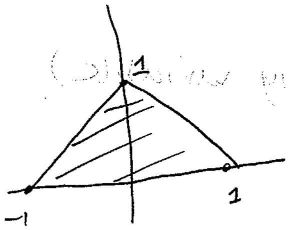
$y = f(x) = \sqrt{16-x^2} \Rightarrow y^2 = 16 - x^2$

$\Rightarrow x^2 + y^2 = 16$: a circle centered at (0.0) with radius 4.

the area is $\frac{1}{4}$ of the circle

$= \frac{1}{4}\cdot(\pi\cdot 4^2) = 4\pi.$

#20 $y = f(x) = 1 - |x|.$



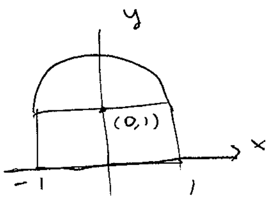
Its area is $\frac{bh}{2}$ with $b = 2,\ h = 1$

$$= \frac{2 \times 1}{2} = 1.$$

#22 $y = f(x) = 1 + \sqrt{1-x^2} \Rightarrow y - 1 = \sqrt{1-x^2},$

$$\Rightarrow (y-1)^2 = 1 - x^2$$

$$\Rightarrow x^2 + (y-1)^2 = 1,$$ which is a circle centered at $(0,1)$ of radius 1.



$$\text{Area} = (1-(-1))(1) + \frac{\pi(1)^2}{2}$$
$$= 2 + \frac{\pi}{2}$$

#52

$$\int_0^b \pi x^2\, dx = \pi \int_0^b x^2\, dx$$

$$= \pi \cdot \frac{1}{3} b^3 - \pi \cdot \frac{1}{3} 0^3$$

$$= \frac{1}{3} \pi b^3$$

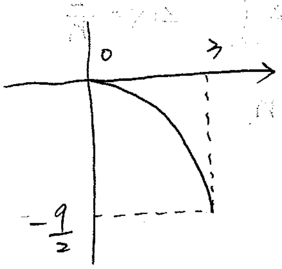
#54

$$\int_0^b \left(\frac{x}{2} + 1\right) dx = \int_0^b \frac{x}{2}\, dx + \int_0^b 1\, dx$$

$$= \frac{1}{2} \int_0^b x\, dx + \int_0^b 1\, dx$$

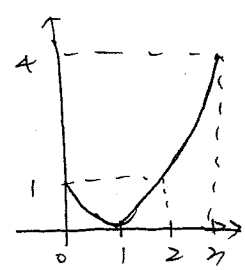
$$= \frac{1}{4} b^2 + b.$$

#56



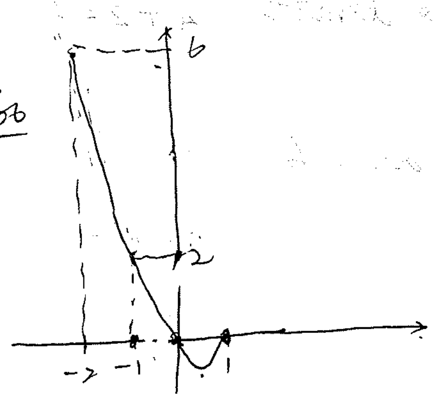
$$av(f) = \frac{1}{3-0}\int_0^3 f(x)\,dx$$
$$= \frac{1}{3}\int_0^3 -\frac{x^2}{2}\,dx$$
$$= -\frac{1}{6}\int_0^3 x^2\,dx$$
$$= -\frac{1}{6}\cdot\frac{1}{3}x^3\Big|_0^3$$
$$= -\frac{1}{6}\cdot\frac{1}{3}(27-0)$$
$$= -\frac{27}{18} = -\frac{3}{2}.$$

#59



$$av(f) = \frac{1}{3-0}\int_0^3 f(t)\,dt$$
$$= \frac{1}{3}\int_0^3 (t-1)^2\,dt$$
$$= \frac{1}{3}\cdot\frac{1}{3}(t-1)^3\Big|_0^3 = \frac{1}{9}\left(2^3-(-1)^3\right)$$
$$= \frac{1}{9}(2^3+1^3)$$
$$= 1$$

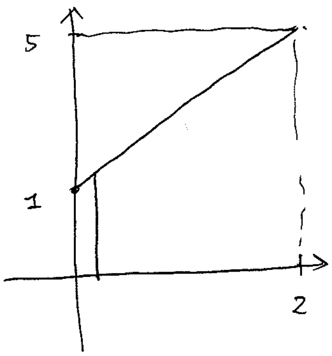
#60



$$av(f) = \frac{1}{1-(-2)}\int_{-2}^1 f(t)\,dt$$
$$= \frac{1}{3}\int_{-2}^1 (t^2-t)\,dt$$
$$= \frac{1}{3}\int_{-2}^1 t^2\,dt - \int_{-2}^1 t\,dt$$
$$= \frac{1}{3}\left(\frac{1}{3}t^3\Big|_{-2}^1 - \frac{1}{2}t^2\Big|_{-2}^1\right)$$
$$= \frac{1}{3}\left[\frac{1}{3}(1^3-(-2)^3) - \frac{1}{2}(1^2-(-2)^2)\right]$$

$$= \frac{1}{3}\left(\frac{1}{3}\cdot 9 - \frac{1}{2}(-3)\right)$$
$$= \frac{3}{2}$$

#64 $\int_0^2 (2x+1)\,dx$



$f(x) = 2x+1$

$P = \left\{ 0, \frac{2}{n}, \frac{4}{n}, \cdots, \frac{2n-2}{n}, 2 \right\}, \quad \Delta x = \frac{2}{n}$

$c_k = \frac{2k}{n}, \quad k = 1, \cdots, n.$

$$\sum_{k=1}^{n} f(c_k)\cdot \Delta x$$

$$= \sum_{k=1}^{n} \left(2\cdot \frac{2k}{n} + 1\right)\cdot \frac{2}{n}$$

$$= \sum_{k=1}^{n} \left(\frac{4k}{n} + 1\right) \frac{2}{n}$$

$$= \sum_{k=1}^{n} \frac{8k}{n^2} + \frac{2}{n}$$

$$= \sum_{k=1}^{n} \frac{8k}{n^2} + \sum_{k=1}^{n} \frac{2}{n}$$

$$= \frac{8}{n^2} \sum_{k=1}^{n} k + \frac{2}{n} \sum_{k=1}^{n} 1$$

$$= \frac{\overset{4}{\cancel{8}}}{{}_{n}\cancel{n^2}} \cdot \frac{\cancel{n}(n+1)}{\cancel{2}} + \frac{2}{n}\cdot n$$

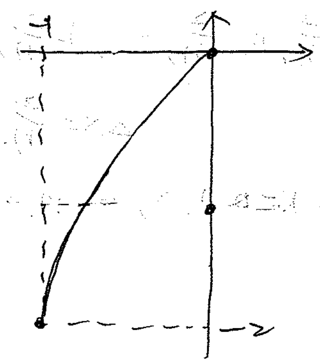
$$= \frac{4(n+1)}{n} + 2$$

As $n \to \infty$, $\|P\| \to 0$ and the last expression on the right has the limit $4+2=6$.

Thus $\int_0^2 (2x+1)\,dx = 6.$

#66

$\int_{-1}^{0} (x-x^2)\,dx$



$f(x) = x - x^2$

$P = \{-1, -1+\frac{1}{n}, -1+\frac{2}{n}, \cdots, -\frac{1}{n}, 0\}, \ \Delta x = \frac{1}{n}$

$c_k = -1 + \frac{k}{n}, \quad k = 1, 2, \cdots, n$

$$\sum_{k=1}^{n} f(c_k)\,\Delta x$$

$$= \sum_{k=1}^{n}\left[\left(-1+\frac{k}{n}\right) - \left(-1+\frac{k}{n}\right)^2\right]\frac{1}{n}$$

$$= \frac{1}{n}\sum_{k=1}^{n}\left[-1+\frac{k}{n} - \left(1 - \frac{2k}{n} + \frac{k^2}{n^2}\right)\right]$$

$$= \frac{1}{n}\sum_{k=1}^{n} -2 + \frac{3k}{n} - \frac{k^2}{n^2}$$

$$= \frac{1}{n}\left(-2\sum_{k=1}^{n} 1 + \frac{3}{n}\sum_{k=1}^{n} k - \frac{1}{n^2}\sum_{k=1}^{n} k^2\right)$$

$$= \frac{1}{n}\left(-2n + \frac{3}{\cancel{n}}\cdot\frac{\cancel{n}(n+1)}{2} - \frac{1}{n^2}\frac{\cancel{n}(n+1)(2n+1)}{6}\right)$$

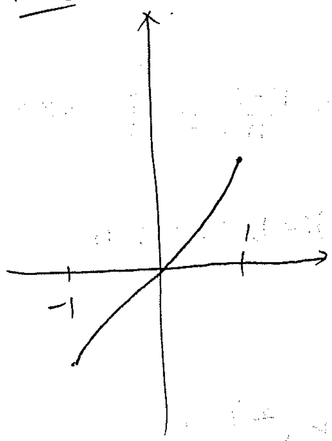
$$= -2 + \frac{3(n+1)}{2n} - \frac{(n+1)(2n+1)}{6n^2}$$

As $n \to \infty$, $\|P\| \to 0$ and the last expression on the right has the limit $-2 + \frac{3}{2} - \frac{1}{3}$

$= -\frac{5}{6}$

Thus, $\int_{-1}^{0} (x-x^2)\,dx = -\frac{5}{6}$

#68



$f(x) = x^3$

$P = \{-1, -1+\frac{2}{n}, \cdots, -\frac{2}{n}, 0, \frac{2}{n}, \cdots, 1-\frac{2}{n}, 1\}$

$\Delta x = \frac{2}{n}$

$c_k = -1 + \frac{2k}{n}, \quad k = \cancel{0}, 1, 2, \cdots, n$

$$\sum_{k=1}^{n} f(c_k)\Delta x$$

$$= \sum_{k=1}^{n} \left(-1+\frac{2k}{n}\right)^3 \cdot \frac{2}{n}$$

$$= \sum_{k=1}^{n} \left(-1 + 3\cdot\frac{2k}{n} - 3\cdot\left(\frac{2k}{n}\right)^2 + \left(\frac{2k}{n}\right)^3\right)\cdot\frac{2}{n}$$

$$= \frac{2}{n}\left(\sum_{k=1}^{n} -1 + \sum_{k=1}^{n}\frac{6k}{n} - \sum_{k=1}^{n}\frac{12k^2}{n^2} + \sum_{k=1}^{n}\frac{8k^3}{n^3}\right)$$

$$= \frac{2}{n}\cdot\left[-n + \frac{\cancel{6}^{3}}{n}\cdot\frac{n(n+1)}{\cancel{2}} - \frac{\cancel{12}^{2}}{n^2}\cdot\frac{n(n+1)(2n+1)}{\cancel{6}} + \frac{8}{n^3}\left(\frac{n(n+1)}{2}\right)^2\right]$$

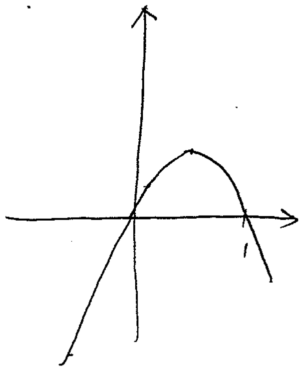
$$= -2 + \frac{6(n+1)}{n} - \frac{4(n+1)(2n+1)}{n^2} + \frac{4(n+1)^2}{n^2}$$

the limit is $-2+6-8+4$ as $n\to\infty$.

$= 0$

Thus, $\int_{-1}^{1} x^3\,dx = 0$

#71



Between $(0,1)$, $f(x) = x - x^2$ is positive

$\therefore \int_0^1 (x - x^2)\,dx$ is the maximal value.

#73

Since

$\frac{1}{2} \le \frac{1}{1+x^2} \le 1$ when $0 \le x \le 1$,

$\therefore \Rightarrow \int_0^1 \frac{1}{2}\,dx \le \int_0^1 \frac{1}{1+x^2}\,dx \le \int_0^1 1\,dx$

and $\int_0^1 \frac{1}{2}\,dx = \frac{1}{2} x \Big|_0^1 = \frac{1}{2}$,

$\int_0^1 1\,dx = x \Big|_0^1 = 1$

Thus, lower bound is $\frac{1}{2}$, and upper bound is 1.

#74

when $0 \le x \le 0.5$, $\frac{1}{1+(0.5)^2} \le \frac{1}{1+x^2} \le 1$

i.e. $\frac{4}{5} \le \frac{1}{1+x^2} \le 1$

Thus $\frac{2}{5} = \int_0^{0.5} \frac{4}{5}\,dx \le \int_0^{0.5} \frac{1}{1+x^2}\,dx \le \int_0^{0.5} 1\,dx = \frac{1}{2}$.

when $0.5 \le x \le 1$, $\frac{1}{2} \le \frac{1}{1+x^2} \le \frac{1}{1+(0.5)^2}$

i.e. $\frac{1}{2} \le \frac{1}{1+x^2} \le \frac{4}{5}$

Thus, $\frac{1}{4} = \int_{0.5}^1 \frac{1}{2}\,dx \le \int_{0.5}^1 \frac{1}{1+x^2}\,dx \le \int_{0.5}^1 \frac{4}{5}\,dx = \frac{2}{5}$

Hence, $\therefore\ \frac{13}{20} = \frac{2}{5} + \frac{1}{4} \le \int_0^{0.5} \frac{1}{1+x^2}\,dx + \int_{0.5}^1 \frac{1}{1+x^2}\,dx \le \frac{1}{2} + \frac{2}{5} = \frac{9}{10}$.

#78 Let $g(x) = -f(x)$ on $[a,b]$.

$g(x) = -f(x) \geq 0$ on $[a,b]$.

and g is integrable on $[a,b]$.

By #77, $\int_a^b g(x)\,dx \geq 0$.

In fact, $\int_a^b -f(x)\,dx \geq 0$

~~and~~ $\Rightarrow -\int_a^b f(x)\,dx \geq 0$

Thus, $\int_a^b f(x)\,dx \leq 0$.

#79

$\sin x \leq x$ for $x \geq 0$

$\int_0^1 \sin x\,dx \leq \int_0^1 x\,dx = \frac{1}{2}x^2\Big|_0^1 = \frac{1}{2}$.

#81 If $b > a$, and f is integrable on $[a,b]$, then

$$\int_a^b av(f)\,dx = av(f)\int_a^b 1\,dx \quad \text{since } av(f) \text{ is a constant}$$

$$= av(f)(b-a) \quad \text{by def of } av(f)$$

$$= \frac{1}{b-a}\int_a^b f(x)\,dx \cdot (b-a)$$

$$= \int_a^b f(x)\,dx.$$

#82

(a) $$av(f+g) = \frac{1}{b-a}\int_a^b (f(x)+g(x))\,dx$$
$$= \frac{1}{b-a}\left[\int_a^b f(x)\,dx + \int_a^b g(x)\,dx\right]$$
$$= \frac{1}{b-a}\int_a^b f(x)\,dx + \frac{1}{b-a}\int_a^b g(x)\,dx$$
$$= av(f) + av(g).$$

(b). $$av(kf) = \frac{1}{b-a}\int_a^b kf(x)\,dx$$
$$= \frac{k}{b-a}\int_a^b f(x)\,dx$$
$$= k\cdot av(f)$$

(c). If $f(x) \le g(x)$ on $[a,b]$, then
$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$
$$\Rightarrow \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{1}{b-a}\int_a^b g(x)\,dx$$
i.e. $av(f) \le av(g)$.

#86

(a) The area of the shaded regions in the first part of the figure ~~are~~ is ~~also~~ the lower sum

$$L = m_1 \Delta x_1 + m_2 \Delta x_2 + \cdots + m_n \Delta x_n$$

(b) The area of the shaded regions in the second part is the upper sum

$$U = M_1 \Delta x_1 + M_2 \Delta x_2 + \cdots + M_n \Delta x_n$$

(c) The rectangles in the third part are the difference of first and second part. Thus, the area of the third part is $U - L = (M_1 - m_1)\Delta x_1 + (M_2 - m_2)\Delta x_2 + \cdots + (M_n - m_n)\Delta x_n$