NAME:
(1) (10 points) For each of the following, give an example if one exists. If there is none, state that there is no example. You do not need to give any justification.

(a) A non-empty bounded subset of $\mathbb{Q}$ with no infimum in $\mathbb{Q}$.
(b) A subset of $\mathbb{R}$ containing $\mathbb{N}$ in which $\{1\}$ is open but $\{2\}$ is not.
(c) An unbounded convergent sequence in $\mathbb{R}$.
(d) An infinite metric space where every subset is open.
(e) A continuous function $f$ from $\mathbb{R}$ to $\mathbb{R}$ and a pair of sequences $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ in $\mathbb{R}$ so that $(x_n - y_n)_{n \in \mathbb{N}}$ converges to 0 but $(f(x_n) - f(y_n))_{n \in \mathbb{N}}$ does not converge to 0.

Solution:
(a) $\{x \mid x^2 < 2\}$
(b) $\mathbb{N} \cup (2 - \frac{1}{2}, 2 + \frac{1}{2})$
(c) There is no example.
(d) $\mathbb{N}$ as a subset of $\mathbb{R}$ or $\mathbb{R}$ with the discrete metric.
(e) $f(x) = x^2$, $x_n = n$, and $y_n = n + \frac{1}{n}$. 
(2) (10 points) Recall that a sequence \((p_n)\) is Cauchy if for every \(\epsilon > 0\) there is an integer \(N\) such that for all \(k, n \geq N\), \(d(p_k, p_n) < \epsilon\). We say a metric space is complete if every Cauchy sequence converges. Recall that we proved that \(\mathbb{R}\) is complete.
   a) Show that if \(M\) and \(N\) are complete, then so is \(M \times N\) with the metric \(d_{\text{max}}\).
   b) Use part a to show that if \(M\) and \(N\) are complete, then so is \(M \times N\) with the metric \(d_{E}\) or \(d_{\text{sum}}\).
   c) Conclude that \(\mathbb{R}^m\) is complete for any \(m\).

Solution:
   a) We want to show that \(M \times N\) is complete. Suppose \((p_n, q_n)_{n \in \mathbb{N}}\) is a Cauchy sequence. Then for each \(\epsilon > 0\), there is an \(N\) so that \(n, k \geq N \Rightarrow d_{\text{max}}((p_n, q_n), (p_k, q_k)) < \epsilon\). But then \(n, k \geq N \Rightarrow d_{M}(p_n, p_k) < \epsilon\) and \(d_{N}(q_n, q_k) < \epsilon\). Thus \((p_n)_{n \in \mathbb{N}}\) and \((q_n)_{n \in \mathbb{N}}\) are both Cauchy. Thus they both converge. But we know that if \((p_n)_{n \in \mathbb{N}}\) converges and \((q_n)_{n \in \mathbb{N}}\) converges, then \((p_n, q_n)_{n \in \mathbb{N}}\) converges. Thus we have shown that an arbitrary Cauchy sequence in \(M \times N\) converges. Thus \(M \times N\) is complete.
   b) We know that a sequence in \(M \times N\) converges with respect to the metric \(d_{\text{max}}\) if and only if it converges with respect to the metric \(d_{E}\) if and only if it converges with respect to the metric \(d_{\text{sum}}\). Thus if every Cauchy sequence converges with respect to the metric \(d_{\text{max}}\), then every Cauchy sequence converges with respect to the other two metrics as well.
   c) Induct: Base case: \(m = 1\). We know \(\mathbb{R}\) is complete.
   \(m = k + 1\): By the inductive hypothesis, \(\mathbb{R}^k\) is complete. Also, \(\mathbb{R}\) is complete. So, using part a, \(\mathbb{R}^{k+1} = \mathbb{R}^k \times \mathbb{R}\) is complete.
(3) (10 points) Show that a subset \( S \) of a metric space \( M \) is closed in \( M \) if and only if \( \partial S \subseteq S \); show that \( S \) is open in \( M \) if and only if \( S \cap \partial S = \emptyset \).

Solutions: There were several ways to do this problem.

**Claim 1.** If \( S \) is closed in \( M \), then \( \partial S \subseteq S \).

*Proof.* Since \( S \) is closed, \( \text{cl}(S) = S \). But then \( \partial S = \text{cl}(S) \setminus \text{int}(S) \subseteq \text{cl}(S) = S \). \( \square \)

**Claim 2.** If \( \partial S \subseteq S \), then \( S \) is closed.

*Proof.* \( \partial S = \text{cl}(S) \setminus \text{int}(S) \subseteq S \). Thus \( (\text{cl}(S) \setminus \text{int}(S)) \cup \text{int}(S) \subseteq S \cup \text{int}(S) \). So, \( \text{cl}(S) \subseteq (\text{cl}(S) \setminus \text{int}(S)) \cup \text{int}(S) \subseteq S \cup \text{int}(S) \subseteq S \). The last \( \subseteq \) is because \( \text{int}(S) \subseteq S \). \( \square \)

**Claim 3.** \( S \) is open if and only if \( S \cap \partial S = \emptyset \).

*Proof.* \( S \) is open if and only if \( S^C \) is closed if and only if \( \partial(S^C) \subseteq S^C \) if and only if \( \partial(S^C) \cap S = \emptyset \).

But \( \partial S = \partial S^C \) (this was a homework assignment), so \( S \) is open if and only if \( \partial(S^C) \cap S = \partial(S^C) \cap S = \emptyset \). \( \square \)
(4) (10 points) For which intervals $[a, b]$ in $\mathbb{R}$ is the intersection $[a, b] \cap (\mathbb{R} \smallsetminus \mathbb{Q})$ a clopen subset of the metric space $\mathbb{R} \smallsetminus \mathbb{Q}$.

The answer: $[a, b] \cap (\mathbb{R} \smallsetminus \mathbb{Q})$ is a clopen subset of $\mathbb{R} \smallsetminus \mathbb{Q}$ if and only if both $a$ and $b$ are rational.

Claim 4. If either $a$ or $b$ is irrational, then $[a, b] \cap (\mathbb{R} \smallsetminus \mathbb{Q})$ is not open in $\mathbb{R} \smallsetminus \mathbb{Q}$.

Proof. Without loss of generality, we assume $a$ is irrational. Then $a \in [a, b] \cap (\mathbb{R} \smallsetminus \mathbb{Q})$. But for any $r > 0$, $B_r(a)$ contains irrational numbers less than $a$, thus $B_r(a) \cap (\mathbb{R} \smallsetminus \mathbb{Q}) \not\subseteq [a, b] \cap (\mathbb{R} \smallsetminus \mathbb{Q})$. Thus $[a, b] \cap (\mathbb{R} \smallsetminus \mathbb{Q})$ is not open in $\mathbb{R} \smallsetminus \mathbb{Q}$. $\square$

Claim 5. If $a$ and $b$ are rational, then $[a, b] \cap (\mathbb{R} \smallsetminus \mathbb{Q})$ is a clopen subset of $\mathbb{R} \smallsetminus \mathbb{Q}$.

Proof. By the inheritance principle, $[a, b] \cap (\mathbb{R} \smallsetminus \mathbb{Q})$ is a clopen subset of $\mathbb{R} \smallsetminus \mathbb{Q}$, since it is the intersection of $[a, b]$ with $\mathbb{R} \smallsetminus \mathbb{Q}$. Similarly, it is open, as it the intersection of $(a, b)$ with $\mathbb{R} \smallsetminus \mathbb{Q}$. $\square$