Math 521 **Midterm 2** **Spring 2013**

NAME:

Grading Table

Question	Possible Points	Points Earned
1	10	
2	10	
3	15	
4	10	
Total	45	

(1) (10 points) For each of the following, give an example if one exists. If there is none, state that there is no example. You do not need to give any justification.

(a) A continous map from C (the cantor set) onto $\mathbb{R}$.
Solution: No example. C is compact, $\mathbb{R}$ is not.

(b) A continous function and a compact set A so that $f^{\text{pre}}(A)$ is not compact.
Solution: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 0$ for every $x \in \mathbb{R}$. Then $A = \{0\}$ is an example.

(c) A nested sequence $A_0 \supseteq A_1 \supseteq A_2 \dots$ of non-empty closed and bounded subsets of $\mathbb{R}^2$ with empty intersection.
Solution: No example. Closed and bounded subsets of $\mathbb{R}^2$ are compact.

(d) A closed and bounded set A in a complete metric space M so that A is not compact.
Solution: $\mathbb{R}_{\text{disc}}$ ($\mathbb{R}$ with the discrete metric) as a subset of itself.

(e) A finite collection of compact sets A_i so that $A = \bigcup_i A_i$ is not compact.
Solution: No example. A finite union of compact sets is compact. (Think about why - you can prove it fairly easily using either definition of compactness.)

(2) (10 points)

(a) Let A be compact, $x \in A$. Let (x_n) be a sequence in A such that every convergent subsequence of (x_n) converges to x. Show that (x_n) converges to x.

(b) Show that every compact set is separable (A is separable if there is a countable set $X \subseteq A$ so $A \subseteq \mathrm{cl}(X)$).

Solution: These are homework problems 94 and Prelim 13 respectively. See the homework solutions for solutions.

(3) (10 points)

(a) Give the definition of *uniform continuity*.

(b) Let $f : M \to N$ be a function. Suppose $M = A \cup B$ and $B_1(x)$ is contained in A or contained in B for every point $x \in M$. Suppose $f|_A$ is uniformly continuous ($f|_A : A \to N$ is given by $f|_A(a) = f(a)$) and $f|_B$ is uniformly continuous. Show that f is uniformly continuous.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose $f(x) = 0$ for all $x \geq 100$ and $f(x) = 0$ for all $x \leq -100$. Show that f is uniformly continuous. (Hint: Use part (b).)

Solutions:

(a) $f : M \to N$ is uniformly continuous if

$$\forall \epsilon > 0 \exists \delta > 0 \forall x \in M \forall y \in M (d_M(x,y) < \delta \to d_N(f(x), f(y)) < \epsilon$$

(b) Suppose f is uniformly continuous on A and B. Fix $\epsilon > 0$. There are δ_A and δ_B so that for any $x, y \in A$ where $d_M(x,y) < \delta_A$ or $x, y \in B$ where $d_M(x,y) < \delta_B$, the distance $d_N(f(x), f(y)) < \epsilon$. Let $\delta = \min\{\delta_A, \delta_b, 1\}$. Then if $d_M(x,y) < \delta \leq 1$ implies that either $x, y \in A$ or $x, y \in B$. If $x, y \in A$, then $d_M(x,y) < \delta \leq \delta_A$, so $d_N(f(x), f(y)) < \epsilon$. If $x, y \in B$, then $d_M(x,y) < \delta \leq \delta_B$, so $d_N(f(x), f(y)) < \epsilon$. So, we see that $d_M(x,y) < \delta$ implies $d_N(f(x), f(y)) < \epsilon$.

(c) Consider the two sets $A = [-102, 102]$ and $B = (-\infty, -100] \cup [100, \infty)$. Then f is uniformly continuous on A, since A is compact, and f is uniformly continuous on B, since f is constant on B. By part (b), we see that f is uniformly continuous on all of $\mathbb{R}$.

(4) (15 points) Show that any connected metric space M containing at least two points is uncountable. (Hint: Let a and b be two points in M. Try to show that for any $\alpha \in [0,1]$, there is a point x so $d(a,x) = \alpha d(a,b)$.)

Solution: Let $\alpha \in (0,1)$ be given. Suppose, towards a contradiction that there is no x so that $d(a,x) = \alpha d(a,b)$. Then $B_{\alpha d(a,b)}(a) = C_{\alpha d(a,b)}$. Thus, this is a clopen subset of M. Also, a is in this set and b is not. So, it is a proper clopen subset of M. This is a contradiction. So, for each $\alpha \in (0,1)$, there is an x so that $d(a,x) = \alpha d(a,b)$. These must all be distinct x's, so we have found Card$((0,1))$ different x in M. Thus M is uncountable.

Alternatively, and easier, apply the IVT to the function $d(a,-)$. It hits 0 and it hits $d(a,b)$, so it must hit every value in between.