THEORY SPECTRA AND CLASSES OF THEORIES

URI ANDREWS, MINGZHONG CAI, DAVID DIAMONDSTONE,
STEFFEN LEMPP, AND JOSEPH S. MILLER

Abstract. We analyze the spectra of theories that are $\omega$-stable, theories whose spectra include almost every degree, and theories with uniformly arithmetical $n$-quantifier fragments. We answer a question from Andrews and Miller [2] by showing that there are $\omega$-stable theories whose spectra are not structure spectra. We show that the spectrum created in Andrews and Knight [1] is not the spectrum of an $\omega$-stable theory, but is the minimal spectrum of any theory with uniformly arithmetical $n$-quantifier fragments. In addition, we give examples of theory spectra that contain almost every degree, including ones that are known not to be structure spectra.

1. Introduction

A large body of work in recursive structure theory has been devoted to understanding when a given structure $M$ has a recursive copy, and if it has none, then to classify the set of Turing degrees of copies of $M$, called the spectrum of $M$. A similar notion is the spectrum of a complete first-order theory, which is the set of Turing degrees of models of the theory. In rare special cases, spectra of theories have been characterized. For example, Solovay [12] characterized the spectra of completions of Peano arithmetic (PA).

Andrews and Miller [2] began the study of comparing the two notions. In particular, they showed that there are spectra of theories that are not spectra of structures and vice versa. All of their examples of spectra of theories that are not spectra of structures had the property that they could not be spectra of atomic theories. They asked first whether there are atomic theories whose spectra are not structure spectra, and second whether there are $\omega$-stable theories whose spectra are not structure spectra. Andrews and Knight [1] answered the first question affirmatively by constructing a theory whose spectrum is exactly the set of degrees of nonstandard models of true arithmetic (TA), and showing that this collection is not a structure spectrum.

We show in Theorem 2.13 that the spectrum constructed by Andrews and Knight is not the spectrum of an $\omega$-stable theory and thus cannot be used to answer the second question. However, we then answer the second question affirmatively (see Section 2.2).

Theorem 2.13. There is an $\omega$-stable theory $T$ so that $\text{Spec}(T)$ is not a structure spectrum.

2010 Mathematics Subject Classification. 03D45, 03C57.

The first author’s research was partly supported by NSF grant DMS-1201338. The second author’s research was partly supported by an AMS Simons travel grant and NSF grant DMS-1266214. The fourth author’s research was partially supported by AMS-Simons Foundation Collaboration Grant 209087. The last author’s research was partly supported by NSF grant DMS-1001847.
We also examine theory spectra $\mathcal{S}$ with the property that $\mathcal{S}$ has measure 1. If $\text{Spec}(T)$ has measure 1, we say $T$ is an almost everywhere theory. This is parallel to the development of the theory of measure 1 structure spectra. Kalimullin showed that $\{d \mid d \not\leq b\}$ is a structure spectrum if $b$ is low over a computably enumerable set (see [5] and [6] for preliminary results). On the other hand, he produced a $b \leq 0''$ such that $\{d \mid d \not\leq b\}$ is not a structure spectrum [7]. Andrews, Cai, Kalimullin, Lempp and Miller (unpublished) provided more examples by proving that $\{d \mid d \not\leq b\}$ is not a structure spectrum if $b$ is high or non-GL$_2$ over $0'$, so in particular if $b = 0''$. We show that unlike in the structure spectrum case, $\{d \mid d \not\leq b\}$ is a theory spectrum for any arithmetical $b$. In fact, we prove the following stronger result.

**Theorem 3.7.** Let $f$ be a total recursive function. For each $j$, let $A_j$ be the arithmetical set with arithmetical index $f(j)$. Then there is a theory $T$ so that $\text{Spec}(T) = \{d \mid \forall j (d \not\leq A_j)\}$.

This, in particular, shows that the collection of non-arithmetical sets is a theory spectrum. It is not known if it is a structure spectrum. This is somewhat surprising since the set of non-hyperarithmetical degrees is a structure spectrum by Greenberg, Montalbán and Slaman [4], but is not a theory spectrum by Andrews and Miller [2].

We show in Theorem 3.1 that every almost everywhere theory $T$ has the property that $T \cap \exists_n$ is uniformly $\Sigma^0_n$, and thus there are only countably many almost everywhere theories. We call such theories Solovay theories. This leads us to ask:

**Question 3.4.** Which are the degrees contained in every almost everywhere theory spectrum?

Combining results from the literature, we answer the corresponding question for Solovay theories.

**Theorem 3.5.** A Turing degree $d$ is in $\text{Spec}(T)$ for every Solovay theory $T$ if and only if $d$ computes a presentation of a nonstandard model of TA.

In particular, the theory constructed by Andrews and Knight [1] is a Solovay theory with minimal spectrum.

1.1. Abstracting relations. The following notion of abstracting relations will be necessary below in Sections 2.2 and 3.

In order to answer a question of Hodges and Macintyre about the quantifier complexity of axiomatizations of $\aleph_0$-categorical almost strongly minimal theories, Marker [11] developed a way to add quantifier-complexity to a theory without changing its underlying nature. In particular, he developed what are now called Marker extensions, which preserve categoricity, $(\omega\text{-}super\text{-})$-stability, and almost strong minimality of a theory while adding quantifiers to the definitions of the important relations.

Given an $n$-ary relation symbol $R$ with a theory $T$ (in some language $\mathcal{L}$), we have two types of Marker extensions, $T_{\forall,R}$ and $T_{\exists,R}$, defined as follows:

We let $T_{R}$ be the following theory:

- $\mathcal{L}'$ is the language obtained by adding to $\mathcal{L}$ two new unary relation symbols $U$ and $V$ and a new $(n + 1)$-ary relation symbol $S$.
- $U$ and $V$ partition the universe into two infinite parts.
- Each symbol of $\mathcal{L}$ only holds on tuples from $V$, and $V$ is a model of $T$.  

This, in particular, shows that the collection of non-arithmetical sets is a theory spectrum. It is not known if it is a structure spectrum. This is somewhat surprising since the set of non-hyperarithmetical degrees is a structure spectrum by Greenberg, Montalbán and Slaman [4], but is not a theory spectrum by Andrews and Miller [2].

We show in Theorem 3.1 that every almost everywhere theory $T$ has the property that $T \cap \exists_n$ is uniformly $\Sigma^0_n$, and thus there are only countably many almost everywhere theories. We call such theories Solovay theories. This leads us to ask:

**Question 3.4.** Which are the degrees contained in every almost everywhere theory spectrum?

Combining results from the literature, we answer the corresponding question for Solovay theories.

**Theorem 3.5.** A Turing degree $d$ is in $\text{Spec}(T)$ for every Solovay theory $T$ if and only if $d$ computes a presentation of a nonstandard model of TA.

In particular, the theory constructed by Andrews and Knight [1] is a Solovay theory with minimal spectrum.

1.1. Abstracting relations. The following notion of abstracting relations will be necessary below in Sections 2.2 and 3.

In order to answer a question of Hodges and Macintyre about the quantifier complexity of axiomatizations of $\aleph_0$-categorical almost strongly minimal theories, Marker [11] developed a way to add quantifier-complexity to a theory without changing its underlying nature. In particular, he developed what are now called Marker extensions, which preserve categoricity, $(\omega\text{-}super\text{-})$-stability, and almost strong minimality of a theory while adding quantifiers to the definitions of the important relations.

Given an $n$-ary relation symbol $R$ with a theory $T$ (in some language $\mathcal{L}$), we have two types of Marker extensions, $T_{\forall,R}$ and $T_{\exists,R}$, defined as follows:

We let $T_{R}$ be the following theory:

- $\mathcal{L}'$ is the language obtained by adding to $\mathcal{L}$ two new unary relation symbols $U$ and $V$ and a new $(n + 1)$-ary relation symbol $S$.
- $U$ and $V$ partition the universe into two infinite parts.
- Each symbol of $\mathcal{L}$ only holds on tuples from $V$, and $V$ is a model of $T$.  

This, in particular, shows that the collection of non-arithmetical sets is a theory spectrum. It is not known if it is a structure spectrum. This is somewhat surprising since the set of non-hyperarithmetical degrees is a structure spectrum by Greenberg, Montalbán and Slaman [4], but is not a theory spectrum by Andrews and Miller [2].

We show in Theorem 3.1 that every almost everywhere theory $T$ has the property that $T \cap \exists_n$ is uniformly $\Sigma^0_n$, and thus there are only countably many almost everywhere theories. We call such theories Solovay theories. This leads us to ask:

**Question 3.4.** Which are the degrees contained in every almost everywhere theory spectrum?

Combining results from the literature, we answer the corresponding question for Solovay theories.

**Theorem 3.5.** A Turing degree $d$ is in $\text{Spec}(T)$ for every Solovay theory $T$ if and only if $d$ computes a presentation of a nonstandard model of TA.

In particular, the theory constructed by Andrews and Knight [1] is a Solovay theory with minimal spectrum.
THEORY SPECTRA AND CLASSES OF THEORIES

\begin{itemize}
  \item \(S(x, \bar{y}) \rightarrow x \in U \setminus \bar{y} \in V^n\).
  \item \(\forall a \in V^n (R(\bar{a}) \rightarrow \exists u \in U S(u, \bar{a}))\).
  \item \(\forall a \in V^n (\neg R(\bar{a}) \rightarrow \neg \exists u \in U S(u, \bar{a}))\).
  \item \(\forall u \in U \exists \bar{a} \in V^n (S(u, \bar{a}))\).
\end{itemize}

\(T'_R\) is a complete theory, and \(T_{3,R}\) is the reduct to the language \(L' \setminus \{R\}\). Note that \(R(\bar{x})\) is definable in \(T_{3,R}\) via the formula \(\exists u \in U S(u, \bar{x})\). We let \(T_{\forall, R}\) be \(T_{3, \neg R}\). This makes \(R(\bar{x})\) \(\forall\)-definable in \(T_{\forall, R}\). If \(A \models T_{3,R}\) or \(A \models T_{\forall, R}\), respectively, and \(V^A \cong M\) holds for the expansion to a model of \(T'_{R}\), then we say that the \(T\)-part of \(A\) is isomorphic to \(M\).

The two-quantifier \(\exists_2\)-Marker extension of \(T\) for the relation \(R\) is the theory \((T_{\forall, R})_{\exists_2, S}\). The two-quantifier \(\forall_2\)-Marker extension of \(T\) for the relation \(R\) is the theory \((T_{3, R})_{\forall, S}\). Continuing in this fashion, we can define the \(n\)-quantifier \(\exists_n\)-Marker extension and the \(\forall_n\)-Marker extension of \(T\) for the relation \(R\). Using the natural interpretation, we refer to the \(T\)-part of a model of these theories as well.

**Lemma 1.1** (Folklore). Let \(M\) be a model of \(T\) such that the atomic diagram of \(M\) in the language \(L' \setminus \{R\}\) is recursive in \(d\). Suppose further that \(R^M\) is \(\Sigma^0_n\) in \(d\). Then the model of the \(\exists_n\)-Marker extension of \(T\) with \(T\)-part isomorphic to \(M\) is recursively presentable in \(d\).

Similarly, we can define Marker extensions \(T_{(\exists_n_1, R_1), (\exists_n_2, R_2), \ldots}\) for whole collections of relations by simply abstracting each relation \(R_i\) to \(n_i\) quantifiers as above. The analogous theorem is as follows:

**Lemma 1.2** (Folklore). Let \(R_1, \ldots, R_i, \ldots\) be a recursive list of relation symbols in \(L\), and let \(n_1, \ldots, n_i, \ldots\) be a recursive list of positive integers. Let \(M\) be a model of \(T\) such that the atomic diagram of \(M\) in the language \(L' \setminus \{R_1, \ldots\}\) is recursive in \(d\). Suppose further that each \(R^M_i\) is \(\Sigma^0_{n_i}\) uniformly in \(d\). Then the model of \(T_{(\exists_n_1, R_1), (\exists_n_2, R_2), \ldots}\) with \(T\)-part isomorphic to \(M\) is recursively presentable in \(d\). In fact, such a \(d\)-index can be recursively computed from the index witnessing that each \(R^M_i\) is \(\Sigma^0_{n_i}\) uniformly in \(d\).

2. SPECTRA OF \(\omega\)-STABLE THEORIES

Andrews and Miller asked two questions, whether there is an atomic theory whose spectrum is not a structure spectrum, and whether there is an \(\omega\)-stable theory whose spectrum is not a structure spectrum. Andrews and Knight \([1]\) answered the first question by constructing an atomic theory (in fact a completion of PA) whose spectrum is precisely the degrees of nonstandard models of TA. By results of Solovay \([12]\) and Marker \([10]\) on the complexity of models of PA, we know that this is the set of degrees that compute Scott sets containing all arithmetical sets. One attempt to answer the second question of Andrews and Miller would be to show that the same spectrum is also the spectrum of an \(\omega\)-stable theory. We show that this is not the case, after which we answer the second question via a different construction.


**Theorem 2.1.** The set of degrees of nonstandard models of TA is not the spectrum of an \(\omega\)-stable theory.
Proof. We first recall that Andrews and Knight [1] Lemma 15] showed that there is a representation $R$ of a countable Scott set containing the arithmetical sets and a permutation $G$ of $\omega$, generic over $R$, so that $R \oplus G$ computes no representation of the collection of all arithmetical sets. This is proved via forcing to construct a sufficiently generic pair $R, G$ in a particular forcing partial order $P$. In particular, $R$ is a representation of the Scott set $\{X \mid \exists k (X \not\leq_T 0^{(k)} \oplus H)\}$ for a fixed Cohen generic $H$.

Suppose towards a contradiction that $T$ is $\omega$-stable and $\text{Spec}(T)$ is the set of degrees of nonstandard models of TA.

Lemma 2.2. Let $R$ be as above and let $M$ be a model of $T$ computable from $R$. Then there is an $\exists$-type realized in $M$ that is not arithmetical.

Proof. The proof of Lemma 2.2 follows exactly Andrews and Knight [1, Lemma 13]. Suppose towards a contradiction that all $\exists$-types realized in $M$ are arithmetical. We consider a forcing partial order on pairs of partial permutations of $\omega$ as follows: $(p_1, p_2)$ extends $(q_1, q_2)$ if $p_1$ extends $q_1$ and $p_2$ extends $q_2$. We force in this partial order over $M$. Since $G$ is generic over $R$ and thus over $M$, $G$ can be split into two permutations of $\omega$, $G_1$ and $G_2$, so that $(G_1, G_2)$ is generic in this partial order.

We want to show that $G_1(M) \oplus G_2(M)$ computes a representation of the family of arithmetical sets. Let $(p_1, p_2)$ force that $\varphi_{\omega_1}^{G_1(M)}$ and $\varphi_{\omega_2}^{G_2(M)}$ are total and are representations of Scott sets containing all arithmetical sets. Call these $S_1$ and $S_2$, respectively.

The following is exactly Andrews and Knight [1, Claim 14].

Claim 2.2.1. $S_1 \cap S_2$ is the collection of all arithmetical sets.

The key observation in the proof of Claim 2.2.1 is as follows: Let $(q_1, q_2)$ force that a particular column of $S_1$ is equal to a particular column of $S_2$. If we view $q_i$ as finite partial maps $c_i : \omega \to d_i \in M$, it suffices to know the existential type of $d_1$ or $d_2$ to compute the column. This is because this existential type declares the possible ways that the quantifier-free diagram of $G_1(M)$ and $G_2(M)$ could be extended. Since these existential types are assumed to be arithmetical, the column in both $S_1$ and $S_2$ is arithmetical.

Finally, a computation from $G_1(M) \oplus G_2(M)$ of these two Scott sets gives a computation of a representation of their intersection, thus $G_1(M) \oplus G_2(M)$ computes a representation of the collection of all arithmetical sets. But this contradicts Andrews and Knight [1, Lemma 15]. Thus some $\exists$-type in $M$ is non-arithmetic. \hfill \Box

Lemma 2.3. Let $\{p_i(x) \mid i \in \omega\}$ be a countable collection of non-arithmetical $\exists$-types consistent with $T$. Let $H$ be Cohen generic over each of the types $p_i$. Let $R, G$ be sufficiently generic for the forcing partial order $P$ that if $W^R_e = p_i$, then some condition forces this. Let $M$ be a model of $T$ computable from $R$. Then none of the types $p_i$ are realized in $M$.

Proof. Suppose $tp_M(a) = p_i$. Then some condition forces that $W^R_e = p_i$ for the appropriate index $e$. But then $p_i$ is arithmetical over the condition, which in turn is arithmetical over $H$. This contradicts the fact that $H$ is Cohen generic over each of the $p_i$. \hfill \Box

Suppose, towards a contradiction, that there are only countably many non-arithmetical $\exists$-types. Then Lemma 2.3 gives a model $M$ where none of these
types are realized. By Lemma 2.2, some non-arithmetic $\exists$-type is realized in $M$, which yields the contradiction. Thus there are uncountably many non-arithmetic $\exists$-types, so uncountably many types, contradicting $\omega$-stability. □

2.2. A spectrum of an $\omega$-stable theory that is not a structure spectrum.

Given an infinite set $A$ and a degree $d$, we say that $A$ is the range of a limitwise monotonic function recursive in $d$, if there is a $d$-computable function $f(x, s)$ such that $f(x, s) \geq f(x, t)$ if $s > t$ (limitwise monotonicity), $\lim_s f(x, s)$ exists for each $x$, and $A$ is the range of the limit function $\lim_s f(x, s)$, i.e., $A = \{\lim_s f(x, s) \mid x \in \omega\}$.

For any two $\Delta_2^0$-sets $A$ and $B$, we construct an $\omega$-stable theory whose spectrum is the set of degrees $d$ so that either $d \geq_T B$ or $A \in \mathrm{lwm}(d)$. We do this by first constructing a pair of prime structures $M_A$ and $M_B$. The countable saturated model of $\mathrm{Th}(M_A)$ is recursively presentable, while the other models of $\mathrm{Th}(M_A)$ are presentable in any degree $d$ so that $A$ is the range of a limitwise monotonic $d$-computable function. The model $M_B$ is recursively presentable, while any presentation of a non-prime model of $\mathrm{Th}(M_B)$ computes $B$. We then construct a structure $M$ by “gluing” together $M_A$ and $M_B$ so that models of $\mathrm{Th}(M)$ either represent a pair of prime models ($M_A, M_B$) or a pair of non-prime models ($N_A, N_B$). In the first case, the presentation computes a presentation of $M_A$, so it makes $A$ the range of a limitwise monotonic function. In the second case, the presentation presents a nonstandard model of $\mathrm{Th}(M_B)$, so it computes $B$.

We believe that structure spectra should not have a dual nature such as this, but are unable to prove a theorem to this effect. Rather, we choose $A$ and $B$ carefully to ensure that $\mathrm{Spec}(\mathrm{Th}(M))$ is not a structure spectrum.

Lemma 2.4. Given infinite $\Delta_2^0$-sets $A$ and $B$, there is an $\omega$-stable structure $M_{A,B}$ such that

$$\mathrm{Spec}(\mathrm{Th}(M_{A,B})) = \{d \mid d \geq_T B\} \cup \{d \mid A \in \mathrm{lwm}(d)\}.$$ 

Proof. Let $L_1 = \{E\}$ where $E$ is a binary relation symbol. For $n \in \omega + 1$, we will write $K_n$ for the $n$-clique. Let $M_A$ be the union of one copy of $K_n$ for each $n \in A$. Then $M_A$ is a prime structure, and the other models of $\mathrm{Th}(M_A)$ are of the form $M_A \cup \bigcup_{i \in I} K_n$. Then the countable saturated model of $\mathrm{Th}(M_A)$ is recursively presentable and the other models are presentable in $d$ if and only if $A$ is the range of a limitwise monotonic function recursive in $d$. (This is essentially in Khoussainov, Nies and Shore [8].)

Fix a recursive function $g$ so that $B(x) = \lim_s g(x, s)$. Let

$$L_2 = \{U_i \mid i \in \omega\} \cup \{c_j \mid j \in \omega\},$$

where the $U_i$ are unary relations and the $c_j$ are constant symbols. Let $M_B$ be the structure with universe $\omega$, where $c_j$ is interpreted as the element $j$ and $U_i(j)$ holds if and only if $g(i, j) = 1$. Note that $M_B$ is a prime recursively presentable structure.

In any non-prime model $N \equiv M_B$, there is an element $x$ so that $x \neq c_j$ for all $c_j$. Then $N \models U_i(x)$ if and only if $B(x) = 1$ (as if $\lim_s g(x, s) = k$, then in $M_B$, $U_j(x) = k$ for all $x$ except for all but finitely many $c_j$). Thus, any presentation of $N$ must compute $B$.

Now, we define $L = \{S, V, W\} \cup L_1 \cup L_2$ where $S$ is a binary relation, and $V$ and $W$ are unary relations. We define the structure $M$ as follows:

- $V$ and $W$ form a partition of $M$. 

THEORY SPECTRA AND CLASSES OF THEORIES 5
• Relations in $\mathcal{L}_1$ hold only on tuples from $V$ and relations in $\mathcal{L}_2$ hold only on elements from $W$.
• $V^M \cong M_A$ in the language $\mathcal{L}_1$.
• $W^M \cong M_B$ in the language $\mathcal{L}_2$.
• $S$ is the graph of a function from $V$ to $W$.
• For $n \in A$, the $n$th element of $A$, $S$ sends $K_n$ in $V$ to the element $c_n$.

We claim that $M$ is bi-interpretable with $M_A$. It follows from the $\omega$-stability of $M_A$ that $M$ is also $\omega$-stable. We interpret $M_B$ as $M_A/E$ (using the fact that $E$ is an equivalence relation, because $M_A$ is a union of cliques). For each symbol $U_i$, the pre-image in $M_A$ is either a finite union of finite cliques or is a complement of a finite union of finite cliques. Thus they are all definable in $M_A$. This gives the interpretation of $M_B$ in $M_A$ as $M_A/E$. The disjoint union of $M_A$ with $M_A/E$ with $S$ as the projection map gives an interpretation $I$ of $M$ in $M_A$. The obvious interpretation of $M_A$ in $M$ along with $I$ gives a bi-interpretation between $M$ and $M_A$.

Lastly, we define $M'$ as the $\exists_2$-Marker extension of $M$ for the relation $S$. Thus $M'$ is bi-interpretable with $M$. For any $N \equiv M$, we write $N'$ for the bi-interpretable structure that is elementarily equivalent to $M'$. Since $M'$ is bi-interpretable with $M$, $\text{Th}(M')$ is also $\omega$-stable, and we will show that $\text{Spec}(\text{Th}(M')) = \{d \mid d \not\geq T B \} \cup \{d \mid A \in \text{lwm}(d)\}$. We claim that $\text{lwm}(d)$ is bi-interpretable with $M_A$. We define $\text{lwm}(d)$ as the projection map gives an interpretation $I$ of $M$ in $M_A$. The obvious interpretation of $M_A$ in $M$ along with $I$ gives a bi-interpretation between $M$ and $M_A$.

Given any degree $d$ so that $A$ is the range of a limitwise monotonic function recursive in $d$, the structure $M'$ is recursive in $d$. This can be seen since $d$ presents $M_A$ and $M_B$ and $d'$ presents $S$, which, by Lemma 1.2, suffices.

Let $N$ be the structure with
\[
V^N \cong M_A \cup \bigcup_{i \in \omega} K_i, \quad \text{and}
\]
\[
W^N \cong M_B \cup \{x_i \mid i \in \omega\},
\]

where $S$ sends $V^N$ to $W^N$ as in $M$ and sends all of $K_i$ to $x_i$. Note that $N$ is the result of the interpretation $I$ in the structure $M_A \cup \bigcup_{i \in \omega} K_i$. Since $M_A \cup \bigcup_{i \in \omega} K_i \cong M_A$, we have $N \cong M$. $N'$ is recursively presented in $B$ since $V^N$ is recursive. $B$ presents $W^N$, and the map $S$ is recursive in $B'$, which suffices by Lemma 1.2. Thus $\{d \mid d \not\geq T B\} \cup \{d \mid A \in \text{lwm}(d)\} \subseteq \text{Spec}(\text{Th}(M'))$.

Now, suppose $N'$ is a model of $\text{Th}(M')$ of degree $d$. If $N'$ is isomorphic to the model $M'$, then $N'$ presents $M_A$, thus $A$ is the range of a limitwise monotonic function recursive in $d$. If $N'$ is not isomorphic to the model $M'$, then following the bi-interpretation, we see that $V^{N'} \not\cong M_A$, and therefore there is a copy of $K_\omega$ in $V^{N'}$. As $M \models \forall x, y(S(x) = S(y) \rightarrow E(x, y))$, we see that the $S$-image of an element of $K_\omega$ is not equal to $c_j$ for any $j$. Thus $W^{N'}$, and therefore also $d'$, computes $B$. Thus $\{d \mid d \not\geq T B\} \cup \{d \mid A \in \text{lwm}(d)\} \supseteq \text{Spec}(\text{Th}(M'))$.  

We now wish to construct a particular pair of $\Delta_2^0$-sets $D$ and $L$, where $L$ is in fact low, such that $\text{Spec}(\text{Th}(M_D, L))$ is not a structure spectrum. In addition, we will construct a third set $C$ so that $c = \text{deg}(C)$ witnesses that $\text{Spec}(\text{Th}(M_D, L))$ is not a structure spectrum. In particular, we will show that $c \not\in \text{Spec}(\text{Th}(M_D, L))$, but $c \in S$ for every structure spectrum $S$ containing $\text{Spec}(\text{Th}(M_D, L))$.

The technique we use here for showing that a particular set of Turing degrees is not a structure spectrum involves a property of structures known as the “c.e.
extension property”, or “c.e.e.p.” If one knows that a structure has this property, it becomes much easier to compute copies of the structure. This can be used to show that a given collection of degrees cannot be a structure spectrum. For example, Andrews and Miller [2] proved that if a structure $M$ has two presentations that form a $\Sigma^0_1$-minimal pair, then $M$ has the c.e. extension property, and then used this result to prove that the PA degrees are not a degree spectrum of a structure.

Since graphs are universal for degree spectra, we will phrase the c.e.e.p. in terms of graphs. We use the term “subgraph” to mean “induced subgraph”, i.e., “sub-model” in the model-theoretic sense. (So, for example, all subgraphs of $K_\omega$ are isomorphic to $K_\omega$ or $K_n$ for some $n$.)

**Definition 2.5.** Fix a computable copy $(\omega, E)$ of the random graph. A code for a graph $G$ is a subset $A \subseteq \omega$ such that $\Gamma(A) := (A, E \restriction A) \cong G$.

**Notation 2.6.** We identify subsets of $\omega$ with elements of $2^\omega$. So, for $\alpha \in 2^\omega$ giving the characteristic function for $A$, $\Gamma(\alpha) := \Gamma(A)$. Given a string $\sigma \in 2^{<\omega}$, we write $\Gamma(\sigma)$ as shorthand for $\Gamma(\sigma^{0\infty})$.

**Definition 2.7.** Given a graph $G$, we say that $\sigma \in 2^{<\omega}$ is consistent with $G$ if $\Gamma(\sigma)$ is isomorphic to a subgraph of $G$. Given $\sigma$ consistent with $G$ and an embedding $f : \Gamma(\sigma) \to G$ (or, equivalently, an isomorphism between $\Gamma(\sigma)$ and a subgraph of $G$), we say that $\tau$ is consistent with $\sigma, f, G$ if $\tau \succ \sigma$ and $f$ extends to an isomorphism between $\Gamma(\tau)$ and a subgraph of $G$.

**Definition 2.8.** We say that a graph $G$ has the c.e. extension property (c.e.e.p.) if given any $\sigma$ consistent with $G$ and embedding $f : \Gamma(\sigma) \to G$, the set of all $\tau \in 2^{<\omega}$ such that $\tau$ is consistent with $\sigma, f, G$ is a c.e. set.

The following notion of “c.e.e.p. cover” was introduced by Kalimullin [7]. It is central to our proof, being the way in which we exploit the c.e. extension property.

**Definition 2.9.** Given $X, C \subseteq \omega$, we say that $C \subseteq 2^\omega \omega$ is a c.e.e.p. cover for $X$ if for every $e \in \omega$, if $\Phi_e^X$ is a code for a graph $G$ with the c.e.e.p., then there is a code $A$ for $G$ such that $A = ^* C[e]$ (the $e$th column of $C$).

**Lemma 2.10.** Let $C$ be a c.e.e.p. cover for $X$, then every c.e.e.p. graph $G$ presentable by $X$ is also presentable by $C$.

*Proof.* Using the extension property of the random graph, any presentation of a graph $G$ can compute a set $A$ so that $\Gamma(A) \cong G$. Thus if $G$ is presentable by $X$, then $X$ computes a code for $G$. If $G$ has the c.e.e.p., then there is some $e$ so that $C[e] = ^* A$. Thus $A \equiv_T C$, so $G$ is presentable by $C$ as well. □

We are now ready to construct $L$ and $D$ so that $\text{Spec}(\text{Th}(M_{D,L}))$ is not a structure spectrum. The construction is rather technical, but we start with an arbitrary noncomputable low set $L$, and use $L$ to construct $D$ and $C$ in such a way that $D$ is $\Delta^0_2$ and $C$ is a c.e.e.p. cover for $L$, but $C$ does not compute $L$, nor present $D$ in a limitwise monotonic fashion (i.e., $D \notin \text{lwm}(\text{deg}(C))$). The entire construction is performed by $0'$, so using the fact that $L$ is low, we may ask $L'$-questions during the construction.

One unusual feature of our construction is that it is a tree construction, but there is no approximation to the true path. We have a tree $T$ of nodes, which all have different guesses about whether various functions $\Phi_e^L$ give codes for graphs with
the c.e.e.p., and, if so, about the corresponding indices of c.e. sets giving consistent
finite extensions. However, rather than approximating a true path at stage \( s \), and acting
for nodes along this stage \( s \) approximation, we instead act for all nodes on \( T \)
that have been initialized by stage \( s \), and that might possibly end up along the
true path. The true path is only defined after the construction is complete, and is
then used to define \( C \); both the true path and \( C \) will be recursive in \( \emptyset'' \). In
fact, \( D \) is constructed over the course of the “construction”, but \( C \) is not. Instead,
we construct a tree of possible finite approximations to \( C \). Once the construction
is complete, the true path through \( T \) induces a path through the tree of finite
approximations to \( C \), which then defines \( C \).

This is not a priority construction, and the nodes on the tree never injure each
other. Instead, each node acts entirely independently from all the others to define
a finite approximation to \( C \), which it then passes to its children when they are
activated. At the end of the construction, when the true path is defined, we ignore
all of the approximations produced by nodes not along the true path, and integrate
the approximations produced by true path nodes, at the stages when they are first
activated, to obtain \( C \).

The strategy to make \( C \) a c.e.e.p. cover for \( L \) acts as a passive global requirement:
We are constantly extending a finite approximation to \( C \) (or rather, extending the
tree of finite approximations), and if \( \Phi^C \) is a code for a graph with the c.e.e.p., then
every time we extend the \( n \)th column of \( C \), it must be consistent with being a code
for that graph (modulo some garbage which never changes). We must also take an
active role by constructing the isomorphism, and every so often we must extend
the isomorphism we construct.

The strategy to make \( L \neq \Phi^C \) involves asking \( L \)-questions: When we extend \( C \),
we can ask \( L \) whether it is possible to extend in such a way that \( \Phi^C \neq L \). We
must be careful not to violate our passive condition on \( C \) when we do this. This is
where we take advantage of the c.e.e.p.: We can assume we have a c.e. index for the
collection of valid extensions of \( C \), and use that index to phrase the \( L \)-question.

Finally, the strategy to make \( D \) not limitwise monotonic in \( C \) via \( f^C \) involves
choosing a follower \( x \), and at each stage \( s \) extending \( C \) in such a way to force
\( f^C(x,t) \) to be large, then choosing a new element of \( D \) which is not \( f^C(x,t) \). In so
doing, either we cause \( \lambda_t f^C(x,t) \) to increase infinitely often, whence \( \lim_t f^C(x,t) \)
does not exist, or else we cause \( f^C(x,t) \) to become stuck at some largest possible
value that is not in \( D \).

Lemma 2.11. Given a noncomputable low set \( L \), there are sets \( D \) and \( C \) such that

1. \( D \) is \( \Delta^0_2 \),
2. \( C \) is \( \Delta^0_4 \),
3. \( C \) is a c.e.e.p. cover for \( L \),
4. \( L \notin_T C \), and
5. \( D \) is not the range of any limitwise monotonic function \( f \leq_T C \).

Proof. We perform a \( \emptyset' \)-oracle construction using the tree
\[
T = \{(e_i^j)_{i+j<n} \mid n \in \omega, e_i^j \in \omega \cup \{\ast\}\}.
\]
That is, nodes on this tree consist of triangular arrays of elements of \( \omega \cup \{\ast\} \),
indexed by \( i \) and \( j \), where \( i+j < n \). The entries in the \( j \)th column of the array give
a succession of guesses for c.e. indices witnessing the c.e.e.p. for \( \Gamma(\Phi^L_j) \). A * entry
indicates a guess that \( \Gamma(\Phi^L_j) \) does not have the c.e.e.p. The order is just extension
strings of graphs coming with the specifications of what is the garbage.

Given the data Definition 2.12. We need a notion of "node, and then activate the least inactive child of each active node. During the
is empty).

A note on terminology: Both the "good" and "non-bad" strings with subscript $e$ will be initial segments of codes for graphs isomorphic to $\Gamma(\Phi_e)$. The difference is that "good" strings come together with partial isomorphisms $\delta_{\alpha,e}$, and that "non-bad" strings do not come with any particular partial isomorphism attached.

Construction:

At stage 0, we activate the root node with trivial data (each set/function/string is empty). At stage $s + 1$, given the stage $s$ data for all active nodes, we act for each active node, and then activate the least inactive child of each active node. During the action, we need a notion of "$\alpha$-consistency."

Definition 2.12. Given the data $C_{\alpha,s}$ attached to $\alpha$ at stage $s$ (which we think of as coming with the specifications of what is the garbage $g_\alpha$, what are the good strings $\sigma_{\alpha,e}$ with associated partial isomorphisms $\delta_{\alpha,e}$, and what are the non-bad strings $\tau_{\alpha,e,s}$), we say that an extension $C_{\alpha,s} \supseteq C_{\alpha,s}$ (with associated $\hat{g}_\alpha \supseteq g_\alpha$, $\hat{\delta}_{\alpha,e} = \sigma_{\alpha,e}$, and $\hat{\tau}_{\alpha,e,s} \supseteq \tau_{\alpha,e,s}$) is $\alpha$-consistent if the following hold:

- $\text{dom}(\hat{g}_\alpha) \setminus \text{dom}(g_\alpha)$ is disjoint from $E_\alpha \times \omega$, and
for each \( e \in E_\alpha \), there is an isomorphism of \( \Gamma(\tilde{\tau}_{\alpha,j,s}) \) with a subgraph of \( \Gamma(\Phi^{L^1}_{e,t}) \) (for some \( t \)) that extends \( \delta_{\alpha,e} \).

Note that since \( L \) is low, we can \( \Theta^t \)-computably check, given \( C_{\alpha,s} \), whether \( \tilde{C}_{\alpha,s} \) is a consistent extension.

When we act, we take five steps. Steps 1–4 are designed to ensure that \( D \) is not limitwise monotonic relative to \( C \), and Step 5 is designed to ensure that \( C \) does not compute \( L \). Step 5 also acts to ensure that \( C \) is a c.e.e.p cover for \( L \). However, we also take care when we define \( C_{\alpha,s+1} \) in Step 3 that this is consistent with making \( C \) a c.e.e.p. cover for \( L \) (at least, under the hypothesis that \( \alpha \) is on the true path of the construction).

1. Pick the new threshold \( T_{s+1} \) to be larger than \( T_s + k \), where \( k \) is the number of active followers \( x_{\alpha,f} \), plus the number of active nodes \( \alpha \) (which is the number of followers \( x_{\alpha,f} \) that will be defined in the next step).

2. For each active \( \alpha \), define a follower \( x_{\alpha,f} \), where \( f_s \) is the \( s \)th limitwise monotonic oracle function. This follower should be such that \( f_s^X(x_{\alpha,f}, t) > T_s \) for some \( \alpha \)-consistent extension \( X \) of \( C_{\alpha,s} \) and some \( t \). (We can \( \Theta^t \)-computably find such a follower, or determine that no such follower exists. If no such follower exists, we do not choose a follower \( x_{\alpha,f} \), as we will not need one.)

3. For each active \( \alpha \), we will define \( C_{\alpha,s+1} \) by extending \( C_{\alpha,s} \) a total of \( s + 1 \) times, once for each of the first \( s + 1 \) limitwise monotonic functions:
   - Set \( C^0_{\alpha,s+1} = C_{\alpha,s} \).
   - For each \( i \leq s \), use the \( \Theta^t \)-oracle to find an \( \alpha \)-consistent extension \( C^i_{\alpha,s+1} \) of \( C^i_{\alpha,s+1} \) and a stage \( t \) where \( f_i^C(x_{\alpha,f}, t) > T_{s+1} \), if possible, or where \( f_i^C(x_{\alpha,f}, t) \) attains the maximum possible value, if \( f_i^C(x_{\alpha,f}, t) > T_{s+1} \) is impossible. (It may be that “undefined” is the largest possible value, but only if it is impossible to find an \( \alpha \)-consistent extension which makes \( f_i^C(x_{\alpha,f}, t) \) defined.)
   - Set \( C_{\alpha,s+1} = C^i_{\alpha,s+1} \).

4. Choose \( S_{s+1} \) to be the least value larger than \( T_s \) which is not any of the largest possible values found in Step 2. (Note that by choice of \( T_{s+1} \), we have \( T_s < S_{s+1} < T_{s+1} \).)

5. For each active \( \alpha \), let \( \beta \) be the least inactive child of \( \alpha \). We activate \( \beta \). We define the data attached to \( \beta \) as follows:
   - \( g_{\beta,s+1} = g_{\alpha,s+1} \).
   - \( E_\beta \) is determined by \( \beta = (e^i_{j})_{i+j<n} \).
   - Use \( \Theta^t \equiv T L' \) to test if there is some finite partial function \( \tilde{C} : \omega^2 \to 2 \) (with downwards-closed domain) such that \( \tilde{C} \supseteq C_{\alpha,s+1} \), and, for each \( j \in E_\beta \), if \( e^j_{a-j} \neq * \), then the string \( \tau_j \) (such that \( \sigma_{\alpha,j} \vdash \tau_j \) extends \( \tau_{\alpha,j,s+1} \) and agrees with the \( j \)th column of \( \tilde{C} \) off of \( \text{dom}(g_{\alpha,s+1}) \)) is in \( W_{e^j_{a-j}} \), and for some \( x \),
     \[
     \Phi_n^C(x) \downarrow \neq L(x).
     \]

This is not quite the same as a \( \beta \)-consistent extension \( \tilde{C} \) where \( \Phi_n^C \neq L \), because we use the c.e. indices we guessed to measure consistency, rather than directly looking at \( \Gamma(\Phi^L_e) \). If there is some such \( \tilde{C} \), define
$g_{\beta,s+1}$ to be the union of $g_{\alpha,s+1}$ with the part of $\check{C}$ in the non-$E_{\beta}$ columns, and define $\delta_{\beta,j} = \sigma_{\alpha,j} \sim \tau_j$. Otherwise, define $g_{\beta,s+1} = g_{\alpha,s+1}$ and $\delta_{\beta,j} = \tau_{\alpha,j,s+1}$. (Every $j \in E_{\beta}$ is also in $E_{\alpha}$, so this makes sense, with the exception of $j = n - 1$. One should interpret $\tau_{\alpha,n-1,s+1}$ as the empty string, and $\delta_{\alpha,n-1,s}$ as the empty function.)

- For each $e \in E_{\beta}$, we search for a $\sigma \succ \delta_{\beta,e}$ and an isomorphism $\delta$ from $\Gamma(\sigma)$ to some subgraph of $\Gamma(\Phi_{e}^{L})$ extending $\delta_{\alpha,e}$ and containing the least element of $\Gamma(\Phi_{e}^{L})$ not in the range of $\delta_{\alpha,e}$. (Using $L' \leq_{T} 0'$, we can check whether such an extension exists.) If such an extension is found, then define $\sigma_{\beta,e} = \sigma$ and define $\delta_{\beta,e} = \delta$. If no such extension exists, then cancel $\beta$.

- For each $e \in E_{\beta}$, we define $\tau_{\beta,e,s+1} = \sigma_{\beta,e}$.

- For each of the first $s + 1$ limitwise monotonic oracle functions $f^{C}$, we define $x_{\beta,f} = x_{\alpha,f}$ (provided that $\alpha$ had some follower $x_{\alpha,f}$).

Notice that the way we have defined the stage $s + 1$ data attached to $\beta$, we have $C_{\beta,s+1} \supseteq C_{\alpha,s+1}$. (It may or may not be the case that $C_{\beta,t} \supseteq C_{\alpha,t}$ for $t > s + 1$, but this will not matter.)

For each $\alpha$, let $s_{\alpha}$ be the stage when $\alpha$ is activated. We define the true path $\Lambda$ inductively by letting the empty string $\lambda \subset \Lambda$, and, given $\alpha = (e_{i}^{j})_{i+j<n} \subset \Lambda$, the true child of $\alpha$ is also on $\Lambda$, where the true child is the node $(e_{i}^{j})_{i+j<n+1}$ for the least c.e. index $e_{n+1-j}$ for the set of $\tau$ such that $\sigma_{\alpha,j} \sim \tau$ is consistent with $\sigma_{\alpha,j}, \delta_{\alpha,j}, \Phi_{j}^{L}$ (if $\Phi_{j}^{L}$ is total and the set of such $\tau$ is c.e.), and is * otherwise. Note that $0''$ can determine whether $\Phi_{j}^{L}$ is total (using the fact that $L$ is low). If it is total, then $L$ can enumerate the set of $\alpha$-consistent extensions, and hence $L'' \equiv_{T} 0''$ can determine, for each $e$, whether $e$ is a c.e. index for the set of such extensions. Therefore, $0''$ can determine whether an index exists, and, if so, what the least index is, so can determine the true child of $\alpha$ (uniformly in $\alpha \in \Lambda$), which means $\Lambda$ is $\Delta_{0}$. We define $C = \bigcup_{\alpha \in \Lambda} C_{\alpha,s_{\alpha}}$. By the note at the end of Step 5 of the construction, this union is well defined. The way we have defined $C$, it is a function from $\omega^{2} \to 2$, but we can regard it as a subset of $\omega$ in the usual way (via a standard pairing function). Since the true path $\Lambda$ is $\Delta_{0}$ and since $C_{\alpha,s_{\alpha}}$ is $\Delta_{0}$ uniformly in $\alpha$, the set $C$ is also $\Delta_{0}$.

We define $D = \{S_{0}, S_{1}, S_{2}, \ldots\}$. Since $D$ was enumerated in increasing order during the $0'$-computable construction, $D$ is $\Delta_{0}^{2}$.

**Verification:**

When we defined $C$ and $D$ above, we verified that they were $\Delta_{0}^{1}$ and $\Delta_{2}^{0}$, respectively. We must check that $C$ is a c.e.e.p. cover for $L$, that $C$ does not compute $L$, and that $D$ is not limitwise monotonic in $C$. In order to prove these three things, we will first need two preliminary lemmas.

**Claim 2.11.1.** Define $\alpha_{s}$ to be the longest node along the true path which is active at stage $s$, and let $C_{s} = C_{\alpha_{s},s}$. Then $C = \bigcup_{s} C_{s}$.

**Proof.** We simply observe that if $\beta$ immediately follows $\alpha$ along the true path, and $s_{\alpha} < s < s_{\beta}$, then $C_{\alpha,s_{\alpha}} \subseteq C_{s} = C_{\alpha_{s},s} \subseteq C_{\beta,s_{\beta}}$. 

**Claim 2.11.2.** Suppose $\Phi_{e}^{L}$ is a code for a graph with the c.e. extension property. Then for all $\alpha \in \Lambda$ with $|\alpha| > e$, we have $e \in E_{\alpha}$.
We argue by induction on the length of $\alpha$ that if $\alpha \in \Lambda$ and $|\alpha| > e$, where $\Phi^L_e$ is a code for a graph with the c.e.e.p., then $e \in E_\alpha$. This is trivially true for $|\alpha| \leq e$. Suppose it is true for $\alpha$ of length at least $e$, and $\beta$ is the true child of $\alpha$. By construction, $\sigma_{\alpha,e}$ is a code for a finite graph isomorphic (via $\delta_{\alpha,e}$) with a subgraph of $\Phi^L_e$, and so the set of all $\tau$ such that $\sigma_{\alpha,e} \supseteq \tau$ is consistent with $\sigma_{\alpha,e}, \delta_{\alpha,e}, \Phi^L_e$ is c.e. This in turn implies that $\beta$ extends $\alpha$ with a non-$*$ term in the $e$th column. Since either $e \in E_\alpha$ (which means that no $*$ terms appeared in the $e$th column of $\alpha$) or $|\alpha| \leq e$ (which means $\alpha$ had no $e$th column), it is also true that no $*$ terms appear in the $e$th column of $\beta$. This means $e \in E_\beta$. □

Claim 2.11.3. $C$ is a c.e.e.p. cover for $L$.

Proof. Suppose $\Phi^L_e$ is a code for a graph with the c.e. extension property. By Claim 2.11.2 $e \in E_\alpha$ for all $\alpha \in \Lambda$ with $|\alpha| > e$, so for all but finitely many $\alpha \in \Lambda$ we have defined a string $\sigma_{\alpha,e}$ and a partial isomorphism $\delta_{\alpha,e} : \Gamma(\sigma_{\alpha,e}) \to \Gamma(\Phi^L_e)$, which extend as $\alpha$ moves down the true path. Let $A = \bigcup_{\alpha \in \Lambda} \sigma_{\alpha,e}$, and let $\delta = \bigcup_{\alpha \in \Lambda} \delta_{\alpha,e}$.

By definition, $\sigma_{\alpha,e}$ is the $e$th column of $C_{\alpha,s_\alpha}$, modulo a finite initial segment determined by $g_{\alpha,s_\alpha}$. Moreover, for $\alpha \in \Lambda$ with $e \in E_\alpha$, $g_{\alpha,s_\alpha}$ is never extended in the $e$th column, and when a child $\beta$ of $\alpha$ is activated, it inherits $g_{\alpha,s_\beta}$ as its $g_\beta$, so the finite initial segments in column $e$ determined by $g_{\alpha,s_\alpha}$ are all the same, for all $\alpha \in \Lambda$ with $|\alpha| > e$. Thus $A$ is finitely different from the $e$th column of $C$, and $\delta$ is an isomorphism from $\Gamma(A)$ to a subgraph of $\Gamma(\Phi^L_e)$. Since the range of $\delta_{\beta,e}$ contains the least element of $\Gamma(\Phi^L_e)$ not in the range of $\delta_{\alpha,e}$ (where $\alpha$ is the parent of $\beta$), this subgraph is, in fact, all of $\Gamma(\Phi^L_e)$. This shows that $C$ is a c.e.e.p. cover. □

Claim 2.11.4. $C$ does not compute $L$.

Proof. We will show that $L \neq \Phi^C_n$. Let $\beta \in \Lambda$ be of length $n$, and let $\alpha$ be $\beta$’s parent node. During the construction, when $\beta$ was activated (at stage $s_\beta$), we defined $C_{\beta,s_\beta}$ to be an extension of $C_{\alpha,s_\beta}$ such that for each $j \in E_\beta$, if $\epsilon^j_{n-j} \neq *$, then the string $\tau_j$ giving the extension in column $j$ from $C_{\alpha,s_\beta}$ to $C_{\beta,s_\beta}$ is in $W_{\epsilon^j_{n-j}}$. Moreover, $\epsilon^j_{n-j}$ is the least c.e. index for the set of $\tau$ such that $\sigma_{\alpha,j} \supseteq \tau$ is consistent with $\sigma_{\alpha,j}, \delta_{\alpha,j}, \Phi^L_e$, so all future finite extensions $\hat{C}$ where $C_{\beta,s_\beta} \subseteq \hat{C} \subset C$ obey this same constraint.

While satisfying these constraints, we extended $C_{\alpha,s_\beta}$ in such a way that for some $x$, we have $L(x) \neq \Phi^C_n(x)$ if that was possible, giving us two cases to consider. If it was possible, then $L \neq \Phi^C_n$ and we are done. If it was not possible, then for every $x$ and every finite extension $\hat{C}$ of $C_{\alpha,s_\beta}$ obeying the constraint, if $\Phi^L_n(x) \downarrow$, then $\Phi^C_n(x) = L(x)$. There must be some $x_0$ such that for all constraint-obeying finite extensions $\hat{C}$, we have $\Phi^C_n(x) \uparrow$, or else this would give an algorithm for computing $L$. (Since whether $\hat{C}$ obeys the constraint is determined by the c.e. indices $\epsilon^j_{n-i}$, the set of all finite constraint-obeying extensions is c.e.) Thus in this case, $\Phi^C_n(x_0) \uparrow$. So, either way, we obtain $\Phi^C_n \neq L$. □

Claim 2.11.5. $D$ is not limitwise monotonic in $C$.

Proof. We will show that $D$ is not the range of $\lim_s f^C_e(\cdot, s)$. Let $\alpha$ be the longest node along the true path $\Lambda$ that was active at the beginning of stage $e$ of the construction. There are two possible cases to consider.
Case 1: We did not define any follower \( x_{\alpha,f_e} \) at stage \( e \). Because of how we choose followers, that means that for every \( t > e \), every \( X \supseteq C_{\alpha,e} \) which is \( \alpha \)-consistent at stage \( e \), and every \( x \), we have \( f^X_e(x,t) \leq T_x \). Hence the range of \( \lim_s f^C_e(-,s) \) is bounded by \( T_x \). Since \( D \) is infinite, it cannot be the range of \( \lim_s f^C_e(-,s) \).

Case 2: We defined a follower \( x_{\alpha,f_e} \) at stage \( e \). Because of how followers are inherited when new nodes are activated, we have \( x_{\beta,f_e} = x_{\alpha,f_e} \) for every \( \beta \subset \Lambda \) extending \( \alpha \). Using the approximation \( C = \bigcup_s C_s \), when we extended \( C_s \) to \( C_{s+1} \) at stage \( s > e \), we defined \( C_{s+1} \) to either force \( f^{C_{s+1}}_e(x,t) > T_{s+1} \) for some \( t \) (if that was possible), or else to force \( f^{C_s}_e(x,t) \) to take the largest possible value otherwise (which must be in the interval \( (T_s,T_{s+1}) \) at the first stage \( s \) when we were unable to force it above \( T_s \)). If the latter ever occurs, the unique value of \( D \) in that interval, namely, \( S_{s+1} \), was chosen different from that largest possible value, so \( \lim_s f^C_e(x,t) \notin D \). Otherwise, the former occurs at every stage, and hence \( \lim_s f^C_e(x,t) = \infty \).

This completes the verification of Lemma 2.11.

We can now prove the main result of this section.

**Theorem 2.13.** There is an \( \omega \)-stable theory whose spectrum is not the spectrum of any structure.

**Proof.** Fix \( L \) low noncomputable, and let \( C \) and \( D \) be as in Lemma 2.11. Let \( M_{D,L} \) be as in Lemma 2.3 i.e.,

\[
\text{Spec}(\text{Th}(M_{D,L})) = \{ e \mid e \supseteq_T L \} \cup \{ e \mid D \in \text{lwm}(e) \}.
\]

This is an \( \omega \)-stable theory, and since \( L \notin_T C \) and \( D \) is not the range of a limitwise monotonic \( C \)-computable function, \( e = \text{deg} C \) is not in \( \text{Spec}(\text{Th}(M_{D,L})) \).

Suppose \( G \) is a graph so that \( \mathcal{S} = \text{Spec}(G) \) is a structure spectrum containing \( \text{Spec}(\text{Th}(M_{D,L})) \). Then \( \text{deg}(L) \in \mathcal{S} \). It is nontrivial but standard to show that if \( q \) has sufficient escape power (relative to \( D \)), then \( D \) is the range of a limitwise monotonic function computable from \( q \). In particular, if \( d(n) \) is defined to be the least stage \( s \) such that \( D_s \) is stable on an initial segment containing an element of \( D \) greater than \( n \), then it is enough that \( q \) is infinitely often greater than the function \( d \). Hence if \( X \) and \( Y \) are sufficiently mutually generic, then \( \text{deg}(X) \) and \( \text{deg}(Y) \) are both in \( \mathcal{S} \) and form a \( \Sigma^0_1 \)-minimal pair, i.e., any set that is both \( \Sigma^0_1 \) in \( X \) and \( \Sigma^0_1 \) in \( Y \) is already \( \Sigma^0_1 \), which is ensured by the mutual genericity \( X \) and \( Y \). By Andrews and Miller [2 Prop. 3.6], any structure whose spectrum contains two such degrees has the c.e. extension property. In particular, \( G \) has the c.e.e.p. Hence \( C \), being a c.e.e.p. cover for \( L \), can compute a copy of \( G \), so \( c \in \mathcal{S} \). Therefore, no structure spectrum can equal \( \text{Spec}(\text{Th}(M_{D,L})) \). \( \square \)

3. Almost Everywhere Theories

We now show that there are only countably many almost everywhere theories (i.e., theories \( T \) such that \( \text{Spec}(T) \) has measure 1), and that they are all Solovay theories (i.e., that \( T \cap \exists_n \) is uniformly \( \Sigma^0_n \)). We give several examples of almost everywhere theory spectra, including some known not to be structure spectra. We believe that the following theorem and the lemma in its proof are known, but we include proofs for completeness.

**Theorem 3.1.** If \( T \) is an almost everywhere theory, then \( T \) is a Solovay theory.
Proof. For an almost everywhere theory $T$, we fix a recursive function $\varphi_e$ so that \( \{ S \mid \varphi^S_e = T \} \) has positive measure. By the Lebesgue Density Theorem, there is a string $\sigma$ so that the relative measure of \( \{ S \mid \varphi^S_e = T \} \) above $\sigma$ is greater than $\frac{1}{2}$, i.e., \( \mu(\{ S \succ \sigma \mid \varphi^S_e = T \}) / \mu(\{ S \succ \sigma \}) > \frac{1}{2} \). By changing $e$ to an index $e'$ where $\Phi^S_{e'} = \Phi^S_{e}^{-S}$, we may assume $\mu(\{ S \mid \varphi^S_e = T \}) > \frac{1}{2}$. Thus, for any $\exists_n$-sentence $\psi := \exists \bar{x}_1 \ldots Q_n \bar{x}_n \psi_0(\bar{x})$,

$$\psi \in T \text{ iff } \mu(\{ S \mid N \models \exists \bar{x}_1 \ldots Q_n \bar{x}_n (\varphi^S_e = \psi_0(\bar{x})) \}) > \frac{1}{2}.$$  

This can be uniformly translated into a $\Sigma^0_n$-question.

**Lemma 3.2.** For any arithmetical formula $\varphi(\bar{x}, S)$, there is a formula $\psi(\bar{x})$ of the same arithmetical complexity so that for every $\bar{x}$,

$$N \models \psi(\bar{x}) \text{ iff } \mu(\{ S \mid N \models \varphi(\bar{x}, S) \}) > \frac{1}{2}.$$  

Moreover, there is a recursive function mapping $\varphi$ to $\psi$.

**Proof.** We prove the result by induction on the arithmetical complexity of the formula $\varphi$. If $\varphi$ is $\Sigma^0_1$, then $\psi(\bar{x})$ is simply

$$\exists m \exists \text{ pairwise incomparable } \sigma_1, \ldots, \sigma_m \left( \sum_{i \leq m} 2^{-|\sigma_i|} > \frac{1}{2} \right) \land \varphi(\bar{x}, \sigma_i).$$

Taking negations gives the result for $\Pi^0_1$-formulas.

Now, we prove the result for $\Sigma^0_n$-formulas, assuming it holds for $\Pi^0_{n-1}$-formulas. Let $\varphi$ be a $\Sigma^0_n$-formula $\exists y_1 \forall y_2 \ldots Q y_n (\varphi_0(\bar{x}, \bar{y}, S))$. We rewrite $\varphi(\bar{x}, S)$ as $\exists m \chi(\bar{x}, m, S)$ where

$$\chi(\bar{x}, m, S) := \bigvee_{y_1 < m} \forall y_2 \ldots Q y_n (\varphi_0(\bar{x}, \bar{y}, S)).$$

Then $\varphi(\bar{x}, S)$ holds for a collection of $S$ of measure $> \frac{1}{2}$ if and only if there is some $m$ so that $\chi(\bar{x}, m, S)$ holds for a collection of $S$ of measure $> \frac{1}{2}$. Using the inductive hypothesis, let $\chi(\bar{x}, m, S)$ holding for a collection of $S$ of measure $> \frac{1}{2}$ be equivalent to a $\Pi_{n-1}^0$-formula $\psi_\chi(\bar{x}, m)$. Then $\varphi(\bar{x}, S)$ holds for most $S$ if and only if $\exists m \psi_\chi(\bar{x}, m)$, which is a $\Sigma^0_n$-formula. \qed

This completes the proof of Theorem 3.1.

It follows from the previous theorem that every almost everywhere theory is computable from $0^{(\omega)}$, so there can only be countably many such theories:

**Corollary 3.3.** There are only countably many almost everywhere theories. \qed

Thus there is a set of degrees of measure 1 all of which are contained in every almost everywhere theory spectrum. Without even daring a conjecture, we ask:

**Question 3.4.** Which are the degrees contained in every almost everywhere theory spectrum?

Though we do not know how to characterize the degrees in every almost everywhere theory spectrum, we can characterize the degrees in the spectrum of every Solovay theory. We say that a theory is *uniformly arithmetical* if there is an arithmetical function $f$ such that for each $n$, if $f(n) = (m, k)$, then $T \cap \exists_n \Phi^0_m^{(k)}$.
proof shows that the degrees that compute models of every Solovay theory are the same as the degrees that computes models of every uniformly arithmetical theory.

Theorem 3.5. The following are equivalent for any Turing degree $d$:

1. $d$ is contained in $\text{Spec}(T)$ for every uniformly arithmetical theory;
2. $d$ is contained in $\text{Spec}(T)$ for every Solovay theory;
3. $d$ computes a presentation of a nonstandard model of $\text{Th}(N, +, \cdot)$.

Proof. (1) implies (2) is trivial. (2) implies (3) follows by the construction of Andrews and Knight [1]. In particular, they construct a single Solovay theory $T$ so that $\text{Spec}(T)$ is precisely the degrees computing presentations of nonstandard models of $\text{Th}(N, +, \cdot)$. (3) implies (1) follows by a theorem of Knight [9, Theorem 2.4]. In particular, a theorem of Solovay refined by Marker [9, Corollary 3.3] shows that the degrees that compute presentations of nonstandard models of $\text{Th}(N, +, \cdot)$ are exactly those that compute enumerations of Scott sets containing every arithmetical degree. If $X$ is such an enumeration and $T$ is uniformly arithmetical, then $X''$ uniformly computes the set of $X$-indices for $T \cap \exists_n$. Thus, by [9, Theorem 2.4], $X$ computes a model of $T$. □

We will now show that, unlike in the case of spectra of structures (cf. Kalimullin [7]), for every arithmetical degree $a$, $\{d \mid d \not\leq a\}$ is the spectrum of a theory. The proof can be extended to show that for any arithmetical sequence of arithmetical sets $(A_i)_{i \in \omega}$, $\{d \mid \forall i (d \not\leq \deg(A_i))\}$ is the spectrum of a theory. Note that any such spectrum has measure 1.

Theorem 3.6. Let $a$ be an arithmetical degree. Then $\{d \mid d \not\leq a\}$ is the spectrum of a theory.

Proof. For a (set $A$ of) degree $a$, we create a theory $T$ below so that any arithmetical degree $d$ computes a model of $T$ if and only if $d$ computes a dense set of elements in $2^\omega$ that are each $\not\leq_T A$. The role of the $U$ relations is to code $2^{\leq \omega}$ into the 1-types of our theory. The role of $(\mathbb{N}, +, \cdot)$ is to be able to define when an element of $2^\omega$ (as represented by the 1-types in the $U$ relations) is $\leq_T A$. The $V$ relations enforce that in any model with $(\mathbb{N}, +, \cdot)$ standard, there is no realized element of $2^\omega$ (as represented by the 1-types in the $U$ relations) that is $\leq_T A$.

Fix the language $L' := \{U_\sigma(x) \mid \sigma \in 2^{\leq \omega}\} \cup \{V_i(n, x) \mid i \in \omega\} \cup \{\mathbb{N}(x), +(x, y, z), \cdot(x, y, z)\}$.

Fix the theory $T'$ generated by the following axioms:

1. $U_\emptyset$ and $\mathbb{N}$ form a partition of the universe into two infinite pieces, and $+$ and $\cdot$ hold only on triples from $\mathbb{N}$.
2. $\forall n \forall x (V_i(n, x) \rightarrow (\mathbb{N}(n) \land U_\emptyset(x)))$.
3. $(\mathbb{N}, +, \cdot) \models \text{TA}$.
4. For all $\sigma \in 2^{\leq \omega}$, we have the axioms:
   \[ \forall x [U_\sigma(x) \leftrightarrow [U_{\sigma 0}(x) \lor U_{\sigma 1}(x)]] , \]
   \[ \forall x \neg [U_{\sigma 0}(x) \land U_{\sigma 1}(x)] , \]
   \[ \exists x U_\sigma(x) . \]
5. For all $i$ so that $\Phi^A_{i}$ is a total $\{0, 1\}$-valued function and for each $n$, we have an axiom of the form:
   \[ \forall x [V_i(n, x) \leftrightarrow U_\sigma(x)] . \]
where $|\sigma| = n$ and $\sigma \prec \Phi_i^A$.

(6) For all $i$ so that $\Phi_i^A$ is a total $\{0, 1\}$-valued function, we have the axiom
$$\forall n \exists x[V_i(n, x) \land \neg V_i(n + 1, x)].$$

(7) For all $i$ so that $\Phi_i^A$ is a total $\{0, 1\}$-valued function, we have the axiom
$$\forall n \forall x [V_i(n + 1, x) \rightarrow V_i(n, x)].$$

(8) If $i$ and $j$ are so that $\Phi_i^A = \Phi_j^A$ are total $\{0, 1\}$-valued functions, then we have the axiom
$$\forall x \forall n [V_i(n, x) \leftrightarrow V_j(n, x)].$$

(9) For all $i$ so that $\Phi_i^A$ is not a total $\{0, 1\}$-valued function, we have
$$\forall n \forall x \neg V_i(n, x).$$

(10) For each $i$, we have the axiom
$$\forall x \exists n \neg V_i(n, x).$$

First of all, note that Claim 3.6.3 below implies that $T'$ is consistent. Our proof of Theorem 3.6 now proceeds in three claims.

Claim 3.6.1. $T'$ is a complete theory.

Proof. Consider two saturated models $M$ and $N$ of $T$ of the same size $\kappa$. Certainly $(\mathbb{N}, +, \cdot)^M \cong (\mathbb{N}, +, \cdot)^N$, as they are both saturated models of the complete theory $\text{TA}$ by axiom 4. We need only extend this isomorphism to $U_0$ as well. By saturation and axiom 4 in both $M$ and $N$, for each $\rho \in 2^\omega$, there must be $\kappa$ many elements in $\bigcap_{\sigma \leq \rho} U_\sigma$. If $\rho \not\models_T A$, then by axioms 7 and 8 $V_i(n, -)$ holds only for all $n$ below some fixed standard integer, for any element of $\bigcap_{\sigma \leq \rho} U_\sigma$. Otherwise, fix $i$ such that $\rho = \Phi_i^A$. By axiom 8 the only atomic formulas defining sets nontrivially intersecting $\bigcap_{\sigma \leq \rho} U_\sigma$ are equivalent to $V_i(n, -)$ for nonstandard $n$. By axiom 6 there are $\kappa$ many realizations in both $M$ and $N$ of $V_i(n, x) \land \neg V_i(n + 1, x)$ for each nonstandard $n \in (\mathbb{N}, +, \cdot)^M$ or nonstandard $n \in (\mathbb{N}, +, \cdot)^N$, respectively, so we can extend the isomorphism to these. By axioms 7 and 10 these possibilities exhaust all elements of $M$ and $N$. \qed

Now fix $k$ so that $a \leq 0^{(k-3)}$. Let $T$ be the theory obtained by abstracting each of the $V_i$ relations by $k$ quantifiers. $T$ is complete, as $T'$ is complete. We claim that $\text{Spec}(T) = \{d \mid d \not\equiv a\}$.

Claim 3.6.2. $\text{Spec}(T) \subseteq \{d \mid d \not\equiv a\}$.

Proof. Suppose towards a contradiction that $M \models T$ and $M \not\models_T A$. Choose an element $b \in U_0^M$, to which we associate the unique $X \in 2^\omega$ such that for each $k$, $M \models U_{X^j_0}(b)$. Note that $M$ computes $X$. Since $M$ is arithmetical, $(\mathbb{N}, +, \cdot)^M$ must be the standard model of $\text{TA}$ by Feferman [3]. Thus, axioms 5 and 10 imply that $X \not\models_T A$. Thus $M$ cannot be computable from $a$. \qed

Claim 3.6.3. $\text{Spec}(T) \supseteq \{d \mid d \not\equiv a\}$.

Proof. Let $d$ be a degree not below $a$. Then $d$ uniformly computes a countable dense collection $S$ of reals so that $X \in S$ implies $X \not\models_T A$. Using this, $d$ computes a structure in the language $\{U_\sigma \mid \sigma \in 2^{<\omega}\} \cup \{\mathbb{N}, +, \cdot\}$ where the elements of $U_0$ are the elements of $S$, and for all $X \in S$, $U_\sigma(X)$ iff $\sigma \prec X$. By Lemma 1.2 we
only need the correct interpretations of each $V_i$ to be uniformly $\Sigma^0_k$ in $d$ for $d$ to compute the associated model of $T$. By axioms 5 and 9, $V_i(n, X)$ holds iff $\Phi^A_i$ is a total $\{0, 1\}$-valued function and, if it is, $U_\sigma(X)$ holds, where $\sigma = \Phi^A_i | n$. Using the fact that $A \leq \emptyset^{(k-3)}$, each $V_i$ is, in fact, uniformly $\Pi^0_{k-1}$ in $d$. □

This completes the proof of Theorem 3.6.

**Theorem 3.7.** Let $f$ be a total recursive function. For each $j$, let $A_j$ be the arithmetical set with arithmetical index $f(j)$. Then there is a theory $T$ so that $\text{Spec}(T) = \{d \mid \forall j (d \not\leq \deg(A_j))\}$.

**Proof.** The proof follows the above construction. The minor required changes are as follows: For each $A_j$, we again have a sequence of $V_{i,j}$ to enforce that each element is not equal to $\Phi^A_j$. The function $f$ gives us uniformly an index $k_j$ so that $0^{(k_j-3)}$ computes $A_j$. Again, we abstract each of these $V_{i,j}$ relations using a $k_j$-quantifier abstraction. The rest of the proof remains the same. □

Whether the set of non-arithmetical degrees is the spectrum of a structure is a long-standing open question. The non-hyperarithmetical analog was recently shown to be a structure spectrum by Greenberg, Montalbán and Slaman [4], while it is not a theory spectrum by Andrews and Miller [2]. In contrast to this, the following is an immediate consequence of Theorem 3.7.

**Corollary 3.8.** The set of non-arithmetical degrees is a theory spectrum. □

However, we also have the following:

**Corollary 3.9.** A countable intersection of theory spectra need not be a theory spectrum

**Proof.** Let $\{a_i\}_{i \in \omega}$ be a countable anti-chain of arithmetical degrees. Then for each $S \subseteq \omega$, $D_S := \bigcap_{i \in S} \{d \mid d \not\leq a_i\}$ is a countable intersection of theory spectra. But these form an uncountable family of measure-1 subsets of the Turing degrees. By Corollary 3.3 uncountably many of these are not theory spectra. □

## 4. Open Questions

We finish with two questions, both left over from Section 3. The first we have already stated explicitly:

**Question 3.4.** Which are the degrees contained in every almost everywhere theory spectrum?

In Theorem 3.6 we showed that every arithmetical degree $a$ has the property that $\{d \mid d \not\leq a\}$ is a theory spectrum. In relation to this, we ask:

**Question 4.1.** What are the degrees $a$ so that $\{d \mid d \not\leq a\}$ is a theory spectrum? Are there any non-arithmetical such degrees? Are all such $a$ at least hyperarithmetical? Note that all such $a$ are $\Pi^1_1$-definable as collections of sets.

## References


(Andrews, Lempp, Miller) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WI 53706-1388, USA

(Cai) DEPARTMENT OF MATHEMATICS, DARTMOUTH COLLEGE, HANOVER, NH 03755, USA

(Diamondstone) GOOGLE, 1600 AMPHITHEATRE PARKWAY, MOUNTAIN VIEW, CA 94043, USA

*E-mail address*, Andrews: andrews@math.wisc.edu

*E-mail address*, Cai: mingzhong.cai@dartmouth.edu

*E-mail address*, Diamondstone: ddiamondstone@gmail.com

*E-mail address*, Lempp: lempp@math.wisc.edu

*E-mail address*, Miller: jmiller@math.wisc.edu