VC-MINIMALITY: EXAMPLES AND OBSERVATIONS

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Abstract. VC-minimality is a recent notion in model theory which generalizes strong minimality, o-minimality, weak o-minimality and c-minimality, but is strong enough to imply various other properties of interest such as NIP and dp-minimality. In this paper, we answer several questions posed by Adler about the relationship of VC-minimality and older stability-theoretic notions in model theory. We also give a proof that Presburger arithmetic is not VC-minimal and an example which separates local VC-minimality from convex orderability, answering a question posed by Guingona.

1. Introduction

VC-minimality is a notion of minimality which generalizes strong minimality, o-minimality, weak o-minimality, and C-minimality; VC-minimality is also strong enough to imply various other properties of interest such as NIP and dp-minimality. The notion is based on VC dimension, due to Vapnik and Chervonenkis [12] and independently around the same time to Shelah [11], and was introduced by Adler [1]. In applications, VC-minimality offers various nice properties; for instance, in [3] forking is characterized in VC-minimal theories. There are also several related weaker notions which have been the subject of recent model theoretic investigation. Shelah introduced the notion of dp-minimality [10], whose investigation was taken up by various authors [4] for additional references.

VC-minimality is extremely language dependent (for instance, the notion is not preserved under reducts [11] or section [7], which gives the notion one inherent drawback: showing that a theory is not VC-minimal requires (a priori) one to think about the specific language rather than some other more invariant notion related to the structure in question. Other notions like dp-minimality which have characterizations which do not inherently depend on the language offer a distinct

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advantage with respect to showing whether the property holds in various theories or classes of theories. In part to deal with this drawback, several intermediate notions have been developed \[7\].

After introducing the preliminary definitions \[2\] we turn in section \[3\] to a question of Adler \[1\]: is every VC-minimal stable theory superstable? We answer the question negatively with a counterexample. We also consider a revised version of Adler’s question in section \[4\] is every stable finitely VC-minimal theory superstable? In this case, we answer the question affirmatively; in fact we show that finite VC-minimality and stability implies \(\omega\)-stability.

In section \[5\] we turn towards developing some examples from the dp-minimal context in the VC-minimal setting. Work from \[4, 6\] indicated that dp-minimal theories expanding a divisible ordered abelian group have some properties similar to weak o-minimality (for instance a monotonicity theorem). Following those developments, it was natural to ask whether such a dp-minimal structure was in fact weakly o-minimal. An example of \[4\] shows that this is not the case by investigating a structure which expands the additive group of the valued field \(\mathbb{R}[[x^\mathbb{R}]]\). Of course, since VC-minimality is a stronger property than dp-minimality, one might again ask the question for VC-minimal structures or theories. We prove that the example of \[4\] is a VC-minimal structure, answering the question negatively. The proof as stated clearly depends on the particular structure, and not simply its theory. This is an interesting indication of one of the inherent drawbacks of VC-minimality - establishing non-VC-minimality of a theory in a direct manner seems challenging.

VC-minimality is a stronger notion than dp-minimality, though until this work, the only known example of a dp-minimal but non-VC-minimal theory was a somewhat contrived example in a language of size \(\omega_1\) \[4\] Prop. 3.6. We show that \(\text{Th}(\mathbb{Z}, +, <)\) is not VC-minimal. It is already known to be dp-minimal, yielding a natural example of a theory distinguishing the different minimality notions. We note that Flenner and Guingona \[5\] have independently found a different proof using the fact that \(\text{Th}(\mathbb{Z}, +, <)\) is not convexly orderable.

Convex orderability is a notion which fits in between VC-minimality and dp-minimality, and it was conjectured that convex orderability may be equivalent to local VC-minimality. In section \[7\] we disprove this conjecture by showing that local VC-minimality is not preserved under reducts. As convex orderability is preserved under reducts, this gives an example of a convexly orderable theory which is not locally VC-minimal.
It has been said in several sources (e.g., [7],[4]) that establishing the negation of VC-minimality is quite troublesome, and that VC-minimality is sensitive to language. We give here a concise conjecture which would explain the difficulty of establishing the negation of VC-minimality. Establishing the truth of this conjecture would show that there is no easier path to showing non-VC-minimality for a general theory other than verifying there is no directed family witnessing VC-minimality. The focus here on recursive theories is coincidental. We do so only to isolate the complexity of the notion of VC-minimality from the complexity of the theory itself.

**Conjecture 1.1.** The index set of recursive VC-minimal theories is \( \Sigma^1_1 \)-complete.

i.e. The set of recursive theories (in recursive languages) which are VC-minimal can be described by a formula with one existential set quantifier (there exists a directed family), but cannot be described in any simpler way.

Index set calculations have been done for many model theoretic notions. For example, the local characterization of stability via the order property shows that stability is \( \Pi^0_2 \). Marker showed [8] that \( \omega \)-stability and superstability are \( \Pi^1_1 \)-complete. Andrews and Makuluni [2] showed that uncountable categoricity is complete at the level of \( d - \Sigma^0_2 \) (a difference of two \( \Sigma^0_2 \)-sets).

2. Preliminaries

A complete theory \( T \) is VC-minimal if there is a family \( \Psi = \{ \psi_i(x, \bar{y}_i) \mid i \in I \} \) of formulae such that in any model \( M \) of \( T \):

1. \( \{ \psi_i(x, \bar{b}) \mid \psi \in \Psi, \bar{b} \in M \} \) is a directed family.
2. Any definable subset of \( M \) is a boolean combination of instances of formulas from \( \Psi \) with parameters in \( M \).

When the above conditions hold for a particular structure \( M \) for some \( \Psi \), then we will say that \( M \) is VC-minimal. Such a family \( \Psi \) will be called a directed generating family for \( T \). One can assume that \( x = x \) is in the generating family. We remark that if \( \Psi \) is finite then by using dummy variables and basic coding, one can assume that \( \Psi \) consists of a single formula. In this case, we will call \( T \) finitely VC-minimal.

VC-minimality is rather sensitive to the specific language that one considers (see section [7] for instance). Particularly, in certain theories, adding constants may make a non-VC-minimal theory actually become VC-minimal. Theories which become VC-minimal after adding constants to the language will be called *locally VC-minimal.*
We call instances of formulae from $\Psi$ balls. If $A, B_1, \ldots, B_n$ are balls, then we call $A \setminus (B_1 \cup \ldots \cup B_n)$ a Swiss cheese.

We say that a structure $M$ is convexly orderable if there is a $<$, a linear order on $M$ such that for any $L$ formula $\phi(x; y)$, there is some natural number $n_\phi$ so that for all $b \in M$, $\phi(M; b)$ is a union of at most $n_\phi$ convex subsets of $M$. Convex orderability was introduced by [7]; it is a useful notion for proving the non-VC-minimality of a theory in certain cases, since it fits between VC-minimality and dp-minimality.

**Remark 2.1.** In this setting, the theory of a cyclic order given by a ternary relation $R$ is locally VC-minimal, which is witnessed by the family $\Phi(x) := \{R(a, x, y), x = y\}$ for any fixed element $a$.

In this setting, the question of whether local VC-minimality is closed under reducts is unanswered by Adler’s [1] observation following remark 4, which is based upon the theory of a cyclic order. We will answer this question negatively in section 7 by showing that there are VC-minimal theories with non-VC-minimal reducts.

We say that $T, \Psi$ is unpackable if no ball is the finite union of finitely many other balls; this is a property of $\Psi$ and when invoking this definition, we will always (sometimes implicitly) be considering some fixed $\Psi$. When we say $T$ is unpackable, we mean $T$ with some given $\Psi$ is unpackable; for other $\Psi$ unpackability might not hold. We will call $T$ packable if $T$ is not unpackable for any generating family $\Psi$. We denote by $\mathcal{B}(A)$, the set of balls defined over a set $A$. The next proposition is due to Adler [1, Prop. 7],

**Proposition 2.2.** In a directed VC-minimal theory, every formula in one free variable is equivalent to the disjoint union of finitely many swiss cheeses. We will refer to this as the swiss cheese decomposition.

The next proposition is from the second author’s thesis [3].

**Proposition 2.3.** In a directed VC-minimal theory which is unpackable, swiss cheese decompositions are unique.

3. VC-minimality and (super) stability

Stability is easy to check in VC-minimal theories. We need only check the directed family $\Phi(x)$. Adler [1, Prop. 8] pointed out the following equivalencies, which are fairly straightforward to prove,

**Proposition 3.1.** The following are equivalent for a VC-minimal theory:

- There is a bound on the length of a chain of balls.
For every individual formula $\phi(x; \bar{y})$ in the distinguished family, there is a finite bound on the length of a chain of balls given by instances of $\phi(x; \bar{y})$.

- $T$ is stable.
- $T$ is equational.
- $T$ is simple.
- $T$ does not have the strict order property.

Adler [1] asks whether VC-minimality plus stability implies superstability. Our next goal will be to answer this question negatively in the case where $\Phi(x)$ is allowed to be an infinite family. We will also answer this question positively, in a stronger form, in the case that $T$ is finitely VC-minimal.

Example 3.2. We let $\mathcal{L} = \{E_i \mid i \in \mathbb{N}\}$ and let $T$ be the theory which says:

- $E_i$ is an equivalence relation.
- $E_i$ has infinitely many infinite classes.
- For $j > i$, $E_j$ refines $E_i$. That is, any equivalence class of $E_j$ lies entirely inside an equivalence class of $E_i$.
- For $j > i$, any equivalence class of $E_i$ contains infinitely many $E_j$ classes.

The theory is VC-minimal with balls defined as $\Phi(x) = \{xE_iy \mid i \in \mathbb{N}\} \cup \{x = y\}$.

Thus, in the case of an infinite family witnessing VC-minimality, we have a negative answer to Adler’s question. Along these lines, there are still interesting questions which one might ask. For instance, are there reasonably natural conditions which guarantee the superstability of a stable VC-minimal theory, which do not guarantee the superstability of an arbitrary stable theory? In other words, does VC-minimality really have anything to say about superstability?

We note in Example 3.2 below that $\text{Th}(\mathbb{Z}, +)$ is VC-minimal. Note that it is also strictly superstable.

4. Finitely VC-minimal theories

For this section, we assume that $T$ is a finitely VC-minimal theory. Thus we can assume that $\Phi(x)$ is given by a single directed VC codimension 1 formula, $\phi(x; \bar{y})$.

Proposition 4.1. If $T$ is stable, then $T$ is $\omega$-stable.

Proof. Assume that $T$ is stable. We show that over any countable model there are only countably many 1-types. From stability, there is
a finite upper bound on the length of a possible chain of containment of balls. Let \( M \) be a countable model of \( T \). Consider the following map 
\[
S(M) \to \mathcal{B}(M).
\]
given by sending a non-algebraic type 
\[
p(x) \mapsto B_p
\]
where \( B_p \) is the smallest ball \( \phi(x; \bar{c}) \in \mathcal{B}(M) \), such that \( p(x) \models \phi(x; \bar{c}) \).

We work on the level of definable sets (not particular formulas which cut out the set), thus the map is well-defined, since there are no infinite chains of balls, by stability of the theory. We claim the map is injective. Otherwise there would be two types \( q_1(x) \) and \( q_2(x) \) which map to the same smallest ball, \( \phi(x; \bar{c}) \). Then take any arbitrary formula \( \psi(x) \) with parameters in \( M \), and we shall see that \( \psi(x) \in q_1 \) if and only if \( \psi(x) \in q_2 \). By swiss cheese decomposition, we may assume that the formula is a ball (we know that the formula is a finite union of swiss cheeses which are each finite boolean combinations of balls, so analyzing the behaviors of the types with respect to the balls is enough to determine if they are the same type), \( \phi(x; \bar{c}_1) \). Of course, the ball given by \( \phi(x; \bar{c}_1) \) must be in one of the three configurations with \( \phi(x; \bar{c}) \):

- contained in \( \phi(x; \bar{c}) \)
- contain \( \phi(x; \bar{c}) \)
- disjoint from \( \phi(x; \bar{c}) \)

In the first and third configurations, by the definition of the above map, it must be the case that \( q_i(x) \models \neg \phi(x; \bar{c}_1) \). for both \( i = 1, 2 \). Similarly, in the second configuration, \( q_i(x) \models \phi(x; \bar{c}_1) \) for both \( i = 1, 2 \). So, since \( q_1(x) \) and \( q_2(x) \) agree about all balls, they must be the same type. Thus we have an injective map from \( S_1(M) \) into the set of balls defined over \( M \). There are clearly only countably many balls over any countable set, so \( T \) is \( \omega \)-stable.

\[\Box\]

Remark 4.2. We denote by \( p_{\phi(x; \bar{c})} \), the unique complete 1-type which is the preimage of a given (equivalence class) of \( \phi(x; \bar{c}) \). We will call \( p_{\phi(x; \bar{c})} \) the \textit{generic} type of \( \phi(x; \bar{c}) \). It follows from Cotter’s characterization of forking in the case that \( T \) is unpackable, that in this case \( p_{\phi(x; \bar{c})} \) is the unique type of maximal \( U \)-rank in the definable set \( \phi(x; \bar{c}) \). In the case that \( T \) is packable, this is not as clear.

5. VC-minimal Structures - More Examples

Example 5.1. \( \text{Th}(\langle \mathbb{Z}, + \rangle) \) is VC-minimal. Let \( \Phi(x) \) be the collection of formulae of the form \( x \equiv n \mod m! \) and \( x = y \). These sets form a directed family, and by Presburger’s quantifier-elimination result to
divisibility predicates \([9]\), all definable sets in one dimension are boolean combinations of these.

**Example 5.2.** The next example was originally given in \([4]\) and proved to be dp-minimal; their analysis, which we will review here, also goes a good portion of the way to showing VC-minimality, but we will need some additional analysis. Let \(\mathbb{R}[[x]]\) be the Hahn series field whose elements have coefficients and exponents in \(\mathbb{R}\) and well-ordered support. We will explain why the structure \((\mathbb{R}[[x]])\) as an expansion of an ordered group (which we describe below) is VC-minimal. For this, we use the language of \([4]\), but we would like to describe the setup here, in part because we need a slightly more detailed analysis.

For each \(q \in \mathbb{Q}\) and \(a \in \mathbb{R}[[x]]\), we interpret a unary function \(s_q(a) := q \cdot a\). For each \(a \in \mathbb{R}[[x]]\), let \(v(a)\) be its valuation. Then let \(P(\cdot)\) be the unary predicate such that
\[
P(y) := \{y \in \mathbb{R}[[x]] | v(y) \in \mathbb{Z}\} \cup \{y \in \mathbb{R}[[x]] | v(y) \geq 0\}.
\]
For each \(n \in \mathbb{N}\), we will have a binary predicate, \(R_n(\cdot, \cdot)\). Let \(R_0(z, y)\) hold if and only if \(z\) and \(y\) are in the same connected component of \(P\) or \(\mathbb{R}[[x]] \setminus P\). When \(n > 0\), \(R_n(z, y)\) holds if and only if \(z < y\) and between \(z\) and \(y\), there are \(n\) alternations of the relation \(P\) as one moves from \(z\) to \(y\) (with respect to the order). Note that because the element \(x\) is an infinitesimal with respect to the ordering, \(P\) is only interesting on the elements of \(\mathbb{R}[[x]]\) outside of the convex hull of \(\mathbb{R}\). Note also that each of the predicates \(R_i\) are definable from \(P\) and \(<\).

In the language \(L = \{\{s_q\}_{q \in \mathbb{Q}}, <, +, 0, P, R_n\}\), the theory \(Th(\mathbb{R}[[x]])\) has quantifier elimination \([4, 5.2]\). We note that if \(t(z, \bar{y})\) is some arbitrary term, then there is some \(q \in \mathbb{Q}\) and some term \(s\) such that \(T \vdash s_q(z) + s(\bar{y})\) \([4]\). Because \(s(\bar{y})\) is again an element of our structure, we may assume \(s(\bar{y}) = y_i\) for some \(i\). We denote by \([\cdot]\), the \(P\)-connected component of \(z\) union the \(P\)-connected component of \(-z\). For basic results about the behavior of the function \([-\cdot]\) from elements to components, under terms of our language, the reader is referred to \([4]\).

Every definable set is, by quantifier elimination, a boolean combination of sets of the form:

- \(z = y\)
- \(z < y \land P(s_p(z) + y_1), p \neq 0\).
- \(z < y \land \lnot P(s_p(z) + y_1), p \neq 0\).
- \(z < y \land R_i(s_r(z) + y_1, s_q(z) + y_2)\)

Unfortunately, the system made of these definable sets does not witness VC-minimality, because the intersections between these sets may be
quite complicated. One rather simple problem occurs when we consider the set $z < 0 \land P(z)$ and $z < 1 \land R_0(z, 0)$. The first set contains many nonstandard negative elements, but contains nothing positive. On the other hand, the second set contains nothing outside of the convex hull of the reals. We will modify the system in order to show that the structure is VC-minimal.

These sets will be balls in our system:

- $z = y$
- $z < y \land R_0(z, y)$
- $z < y \land R_0(z, y) \land P(z)$
- $z < y \land \neg R_0(z, y) \land \neg P(z)$

These sets do satisfy the directed intersection conditions of a VC-minimal system, and if we assume that $[z] > [y_1]$, then $P(s_p(z) + y_1)$ has the same truth value as $P(z)$, so we may write any set of the form

- $z < y \land P(s_p(z) + y_1), p \neq 0$.
- $z < y \land \neg P(s_p(z) + y_1), p \neq 0$.

as a boolean combination of sets of the form:

- $z < y \land R_0(z, y)$
- $z < y \land \neg R_0(z, y) \land P(z)$
- $z < y \land \neg R_0(z, y) \land \neg P(z)$

When $[z] < [y_1]$, the predicate $P(s_p(z) + y_1)$ does not depend on $z$, so we eliminate this possibility by noting that it is a union of sets listed above. The situation for sets of the form $P(s_p(z) + y_1)$ when $R_0(z, y_1)$ holds is more complicated. In this case, there are only finitely many $P$-components in which the sum might lie, since $[s_p(z) + y_1] \leq [y_1]$ (note that this fact only holds in this particular structure, not for any model of the theory), and this portion of the set may be expressed as a union of sets which use the $R_i$ predicates, which we analyze next.

So, to review, our system of balls (which is directed) contains (so far):

- $z = y$
- $z < y \land R_0(z, y)$
- $z < y \land \neg R_0(z, y) \land P(z)$
- $z < y \land \neg R_0(z, y) \land \neg P(z)$

Next we will consider sets of the general form $R_i(s_q(z) + y_1, s_r(z) + y_2)$. If $q, r \neq 0$ and $[z] > [y_j]$ for $j = 1, 2$ then $i = 0$, $R_i(s_q(z) + y_1, s_r(z) + y_2)$ holds automatically; we will not need to consider this case (it is a first order condition on the parameters and the input, so our final system of balls will simply stipulate that this does not happen).
If \( r = 0 \), the relation \( R_i(s_q(z) + y_1, y_2) \) then \([y_2] > [y_1] \), and if \( q \neq 0 \),
then we may assume \( q = 1 \), by changing the parameters appropriately.
We may assume that \([z] \geq [y_1] \), since otherwise we may remove \( z \)
from the predicate without changing the truth value (again, this is a
first order condition on the parameters and input, so our final system
of balls will stipulate that this arrangement does not occur). Thus
\( R_i(z + y_1, y_2) \) is equivalent to \( R_k(z + y_1, z) \) for some \( k \). In the case that
\([z] > [y_1] \), it must be that \( k = 0 \) for the predicate to hold; again, all of
this is first order and our final system of balls will rule this situation
out. The remaining case, that \([z] = [y_1] \) is subsumed in the Case 1 sets
below. The remaining cases for sets of the form \( R_i(s_q(z) + y_1, y_2) \) are
detailed in the following items.

- **(Case 1)** Assume \( i > 0 \), \( R_0(z, y_1) \land R_i(z, y_2) \land R_k(z + y_1, y_2) \). In
  this case, the \( y_2 \) is superfluous, and we can replace the formula
  with \( R_0(z, y_1) \land R_j(z + y_1, y_2) \) for appropriate \( j \).

- **(Case 2)** Assume \( i > 0 \), \( R_i(y_1, z) \land R_0(z, y_2) \land R_k(z, z + y_2) \). In
  this case, \( k \) must be 0, and \( y_1 \) is superfluous. So, the formula
  becomes \( R_0(z, y_2) \land R_0(z, z + y_2) \). This is equivalent to a Type
  A ball with \( j = 0 \).

- **(Case 3)** \( R_0(z, y_1) \land R_0(z, y_2) \land R_i(z + y_1, z + y_2) \land R_j(z + y_1, y_1) \land
  R_k(z + y_2, y_2) \). We can see that \( j \geq k \). The whole formula is
equivalent to \( R_0(z, y_1) \land R_j(z + y_1, y_1) \land R_k(z + y_2, y_2) \), which is
of type A.

Now, we need only consider Case 1 sets, which we will further split
up. Suppose that we are given a Case 1 set of the form \( R_0(z, y_1) \land
R_j(z + y_1, y_1) \). Suppose that \( y_2 \) is the parameter which gives the \((j+1)^{st}
leading component of \( y_1 \) (that is, the monomials which lie in the same
\( P \)-component as \( z + y_1 \)). We will consider sets of the form \( R_0(z, y_1) \land
R_j(z + y_1, y_1) \land z + y_1 < y_2 \) if \( y_2 \) is nontrivial and \( R_0(z, y_1) \land R_j(z + y_1, y_1)
\) in the case that it is. The general Case 1 sets are, of course given
(for appropriate choice of parameters) by our modified Case 1 sets for
appropriate choice of the parameters.

Now, our system of balls is given by:

1. \( z = y \)
2. \( z < y \land R_0(z, y) \)
3. \( z < y \land \neg R_0(z, y) \land P(z) \)
4. \( z < y \land \neg R_0(z, y) \land \neg P(z) \)
5. \( R_0(z, y_1) \land R_j(z + y_1, y_1) \land z + y_1 < y_2 \) in case the \((j+1)^{st}
component of \( y_1 \) is given by \( y_2 \).
6. \( R_0(z, y_1) \land R_j(z + y_1, y_1) \) in case the \((j+1)^{st} \) component of \( y_1 \)
is empty.
Let us briefly discuss intersections of the balls in this system. Sets of the form $z < y \land R_0(z, y)$ consist of subintervals of $P$-components, so with sets of the form

$$z < y \land \neg R_0(z, y) \land P(z)$$

, they clearly satisfy the intersection requirements for being directed.

Also, $z < y \land R_0(z, y)$ contains the set

$$R_0(z, y_1) \land R_j(z + y_1, y_1) \land z + y_1 < y_2$$

just in the case that $y_1$ satisfies $y_1 < y \land R_0(y_1, y)$. Otherwise the intersection is empty. The situation is similar for intersections of sets of the form $z < y \land R_0(z, y)$ with

$$R_0(z, y_1) \land R_j(z + y_1, y_1).$$

Sets of the form

$$z < y \land \neg R_0(z, y) \land P(z)$$

and

$$z < y \land \neg R_0(z, y) \land \neg P(z)$$

are the only sets of our system which contain elements from multiple $P$-components; further, they are unions of $P$-components, so they intersect nicely with all other sets in our system.

Finally, sets of the form

$$R_0(z, y_1) \land R_j(z + y_1, y_1) \land z + y_1 < y_2$$

intersect

$$R_0(z, y_3) \land R_j(z + y_3, y_3) \land z + y_4 < y_4$$

precisely when they have the same monomials which lie in their $j$ leading $P$-components. In that case, without loss of generality, $y_2 < y_4$, so the first set is contained in the second. Intersections with sets of the final form are similar, so our system is directed.

The sets do not witness the VC-minimality of the theory; the main part of our argument which depends on the particular structure $\mathbb{R}[[x^\mathbb{R}]]$ occurred when we considered the predicate $P(z + y_1)$. We argued that when $[z] = [y_1]$, there are only finitely many $P$-components into which the sum might land. Then, we converted this sort of formula to a finite union of $R_i$-formulae. However, in elementary extensions of $\mathbb{R}[[x^\mathbb{R}]]$, there are not finitely $P$ components into which the sum might land; of course, this occurs any time the value group is non-Archimedean.
6. A natural dp-minimal theory which is not VC-minimal

The goal of this section is to give an example of a dp-minimal theory which is not VC-minimal. For the definition of dp-minimality, see [1]. For the previously known example of such a theory, see [4, Prop. 3.7]. That example is less natural, and the proof involves cardinality reasons rather than structural reasons. To make the dividing line between these two conditions clearer, we give a more natural example which separates the classes.

**Theorem 6.1.** $T = \text{Th}(\mathbb{Z}, +, <)$ is not VC-minimal.

**Proof.** Towards a contradiction, let $\Phi(x)$ be a potential system of balls witnessing VC-minimality. Let $M \models T$ be an $|\Phi(x)|^+$-saturated model.

Any definable set $\phi(x, \bar{y})$ is a finite union of distinct periodic sets (with finite periods), $P_1, \ldots, P_n$ restricted to particular intervals which depend on the parameters (and might be empty). We denote, by $P_i(\bar{a})$, the intervals upon which the periodic set $P_i$ agrees with $\phi(x, \bar{a})$. By convention, we will always demand the the length of such an interval is longer than the period of the periodic set. Of course, depending on the sets in question, $P_i(\bar{a})$ might not be disjoint from $P_j(\bar{a})$. The system of $\Phi$ consists of families of definable sets $\phi(x, \bar{y})$ where the parameters $\bar{y}$ are allowed to vary over $M$. $P_i$ is a right-eventual periodic set of $\phi(x, \bar{a})$ if the interval $(\beta, \infty)$ is in $P_i(\bar{a})$ for some $\beta$. A priori, we may have selected a system $\Phi$ such that for some $\phi(x, \bar{y}) \in \Phi$, for various choices of the parameters, the right-eventual periodic set varies. Since there are only finitely many possibilities for the right-eventual periodic set as we vary the parameters, we will assume that $\Phi(x)$ consists only of families $\phi(x, \bar{y})$ with a particular unique right-eventual set. It is clear that if we started with some system of balls witnessing VC-minimality, then this new system with unique right-eventual periodic sets also witnesses VC-minimality.

We say that a periodic set $P_i$ in a given ball with parameters $\bar{y}$ is **right-movable** if for any $\beta$ there is $\bar{a} \in M \models T$ such that $P_i(\bar{a})$ contains an infinite interval with right endpoint larger than $\beta$. For example, any periodic set $P(\bar{y})$ which is the right-eventual periodic set of $\phi(x, \bar{y})$ is right-movable for the family $\phi(x, \bar{y})$.

**Claim 6.2.** Some family of balls $\phi(x, \bar{z})$ in $\Phi$ has two right-movable sets $P_1 \neq P_2$.

**Proof.** (Lemma 6.2) Assume that every family $\phi(x, \bar{y})$ has only one right movable set. Then, for each family $\phi(x, \bar{y})$, only the right-eventual set is right-movable. Let $Q$ be the right-eventual set of $\phi(x, \bar{y})$. Then there is some $\beta \in M$ so that for every $\bar{y}$, $Q(\bar{y}) \supseteq (\beta, \infty)$. Then, by saturation...
of \( M \), there is some \( \gamma \in M \) such that any boolean combination of balls is periodic beyond \( \gamma + n \) for some \( n \in \mathbb{N} \). Since the collection of definable sets does not have this property, we have derived a contradiction. \( \square \)

Let \( P = P_i \) for some \( i \in \{1, 2\} \) be the right-eventual set of \( \phi(x, \bar{y}) \).

**Claim 6.3.** Let \( P_1, P_2 \), be right-moveable for \( \phi \in \Phi \). Then \( P_1 \subseteq P_2 \) or \( P_2 \subseteq P_1 \).

**Proof.** (Lemma 6.3) The lemma follows from 6.2 because by varying the parameters appropriately (for some choices \( \bar{a}_1, \bar{a}_2 \)) \( \text{int}(P_1(\bar{a}_1)) \cap \text{int}(P_2(\bar{a}_2)) \) is an infinite interval. Then since \( \Phi(x) \) is a directed system, and \( P(\bar{a}_1) \cap P(\bar{a}_2) \) is infinite, one of the containments must hold. \( \square \)

Without loss of generality, we will assume that \( P_1(\bar{z}) \subseteq P_2(\bar{z}) \).

**Claim 6.4.** For any family of balls \( \psi(x, \bar{y}) \) in \( \Phi \), let \( Q(\bar{y}) \) be the right-eventual periodic set \( Q(\bar{y}) \). Then either

\[ • Q \cap P_2 = \emptyset. \]
\[ • Q \subseteq P_1. \]
\[ • Q \supseteq P_2. \]

**Proof.** (Lemma 6.4) We are assuming, without loss of generality, that \( P_1 \subseteq P_2 \). Thus, the only other possibility which might occurs (besides the three listed) is that \( P_1 \subseteq Q \subseteq P_2 \).

Fix a tuple of \( \psi(x, \bar{b}) \), and let \( Q(\bar{b}) = (\beta, \infty) \). We can choose a tuple \( \bar{a} \) so that both \( P_1(\bar{a}) \cap (\beta, \infty) \) is infinite and \( P_2(\bar{a}) \cap (\beta, \infty) \) is infinite. If it were the case that \( P_1 \subseteq Q \subseteq P_2 \), then the two balls \( \phi(x, \bar{a}) \) and \( \psi(x, \bar{b}) \) would non-trivially intersect, which would contradict \( \Phi(x) \) being a directed system. \( \square \)

Fix some \( C \), a periodic set such that \( P_1 \subseteq C \subseteq P_2 \) and consider the family of sets \( \{ x > \alpha \mid x \in C \} \). By 6.4 \( S_\alpha \) is not a boolean combination of families of balls. \( \square \)

### 7. VC-minimality and reducts

VC-minimality is not preserved in reducts: the theory of dense cyclic orders is a non-VC-minimal reduct of the theory of a dense cyclic order with one constant, which is VC-minimal \([1]\). A dense cyclic order is a structure for the signature \( S := \{[, ,] \} \) with one ternary relation satisfying

- for all \( x, y, z \), if \([x, y, z] \), then \([y, z, x] \). (cyclic)
- for all \( x, y, z \), if \([x, y, z] \), then not \([z, y, x] \). (asymmetric)
- for all \( x, y, z \), if \([x, y, z] \) and \([x, z, w] \), then \([x, y, w] \). (transitive)
• for all \(x, y, z\), either \([x, y, z]\) or \([z, y, x]\). (total)

Though satisfying in answering that VC-minimality is not closed under reducts, we can still ask if local VC-minimality is closed under reducts. We present here a counterexample to that question.

**Theorem 7.1.** Local VC-minimality is not preserved under reducts.

**Proof.** The idea of the example is to take an infinite family of disjoint dense cyclic orders, so that we must name a point on each of the unboundedly many dense cyclic orders, that is a section, to obtain VC-minimality. The VC-minimal theory will be named \(T_m\), in signature \(S_m\); its not even locally VC-minimal reduct will be named \(T_0\), in signature \(S_0\).

Both signatures contain a binary relation symbol \(E\); both theories assert that \(E\) is an equivalence relation with infinitely many infinite classes. The signature \(S_0\) also contains a ternary relation symbol \([, , ]\) for the cyclic orders on equivalence classes of \(E\). The signature \(S_m\) will also contain a unary function symbol \(f\) for the section and a binary relation symbol \(<\) for the dense linear orders of equivalence classes of \(E\).

Here are a complete list of axioms for \(T_m\):

1. \(E\) is reflexive, symmetric, and transitive.
2. For all \(x, y\), if \(x < y\) then \(xEy\).
3. \(<\) is antisymmetric and transitive. (order)
4. For all \(x, y\), if \(xEy\), then \(x < y\) or \(y < x\) or \(x = y\). (total on \(E\)-classes)
5. For all \(x, y\), if \(xEy\) then \(fx = fy\); and for all \(x\), \(fx \leq x\). (\(f\) picks out the least element in each class.)
6. For all \(x\) there is some \(y\) such that \(x < y\). (no greatest element in any class)
7. \(E\) has infinitely many infinite classes and \(<\) is dense.

The VC-minimality of \(T_m\) is witnessed by the following directed family:

\[\{x = x, xEy, f(y) < x < y, x = y, x \neq x\}\]

(Here, \(x\) is the variable and \(y\) is the parameter.)

We obtain \(T_0\) by taking the reduct to \(S_0\) of the definitional expansion of \(T_m\) to \(S_0 \cup S_m\) given by

\([x, y, z] \leftrightarrow (x < y < z \lor y < z < x \lor z < x < y)\).

The proof of non-VC-minimality of \(T_0\) now follows similarly to the proof that the circle order is not VC-minimal. \(\square\)
Corollary 7.2. The theory $T_0$ given above is convexly orderable, but is not locally VC-minimal.

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