THE DEGREES OF BI-HYPERHYPERIMMUNE SETS

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Abstract. We study the degrees of bi-hyperhyperimmune (bi-hhi) sets. Our main result characterizes these degrees as those that compute a function that is not dominated by any \( \Delta^0_2 \) function, and equivalently, those that compute a weak 2-generic. These characterizations imply that the collection of bi-hhi Turing degrees is closed upwards.

1. Introduction

Csima and Kalimulin \[3\] gave an example of a structure whose degree spectrum is contained in the bi-hyperhyperimmune (bi-hhi) degrees. They suggested that its spectrum might be exactly the bi-hhi degrees, but they pointed out that it is not even known if bi-hyperhyperimmunity is closed upwards in the Turing degrees. It was this simple, if esoteric, question that motivated our paper. We prove that the collection of bi-hyperhyperimmune Turing degrees is closed upwards, and in fact, that it is a very natural degree class. Even so, it turns out that Csima and Kalimulin’s structure does not capture these degrees (see Corollary \[1.3\]).

To put into context the fact that the collection of bi-hyperhyperimmune degrees is closed upwards, note that the same is true of the bi-immune (Jockusch \[8\]) and bi-hyperimmune degrees. The latter follows from Kurtz \[12\], who showed that every hyperimmune degree contains a bi-hyperimmune set (hence as Jockusch noted, the hyperimmune and bi-hyperimmune degrees coincide), and the fact that the collection of hyperimmune degrees is closed upwards \[13\]. While it is certainly true that the collection of hyperimmuneimmune degrees is closed upwards \[7, 9\], they do not coincide with the bi-hyperhyperimmune degrees\[1\]. Similarly, Jockusch \[6\] proved that the immune and bi-immune degrees do not coincide. This might lead us to look for a parallel between the bi-hhi degrees and the bi-immune degrees, but as it turns out, the bi-hyperimmune degrees offer a much better guide to the behavior of the bi-hhi degrees.

Basic Definitions. A weak array is a uniformly c.e. sequence \( A = \{F_n\}_{n \in \omega} \) of finite sets. We say that \( A \) is disjoint if its members are pairwise disjoint. A set \( X \subseteq \omega \) contains \( F \subseteq \omega \) if \( F \subseteq X \). Similarly, \( X \) avoids \( F \subseteq \omega \) if \( F \cap X = \emptyset \). We say that \( X \) is bi-hyperhyperimmune (bi-hhi) if it both contains a member and avoids

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1Every \( \Delta^0_2 \) hyperhyperimmune set is strongly hyperhyperimmune (this is claimed by Cooper \[1\]; see Downey, Jockusch and Schupp \[1\] for a proof), implying that it is disjoint from an infinite c.e. set. Therefore, no \( \Delta^0_2 \) set is bi-hyperhyperimmune. But there are \( \Delta^0_2 \) hyperhyperimmune sets.
a member of every disjoint weak array. A degree \( d \) is bi-hyperhyperimmune if it contains a bi-hhi set.

It will be convenient to use a strong variant of bi-hyperhyperimmunity. If \( F \subseteq \omega \) is finite, let \( \hat{F} = \{ n \in \omega \mid \min(F) \leq n \leq \max(F) \} \). We say that \( X \subseteq \omega \) blockwise contains a finite set \( F \subseteq X \) if \( \hat{F} \subseteq X \). Similarly, \( X \) blockwise avoids a finite set \( F \subseteq \omega \) if \( \hat{F} \cap X = \emptyset \). We say that \( X \) is blockwise hyperhyperimmune if it is infinite and blockwise avoids a member of every disjoint weak array. Call \( X \) blockwise bi-hyperhyperimmune if it and its complement are both blockwise hyperhyperimmune, i.e., if \( X \) both blockwise contains a member and blockwise avoids a member of every disjoint weak array \( A \). A degree \( d \) is blockwise bi-hhi if it contains a blockwise bi-hhi set. Blockwise bi-hyperhyperimmunity is somewhat easier to work with than bi-hyperhyperimmunity. In particular, we will be able to show directly that the collection of blockwise bi-hhi Turing degrees is closed upwards. We will see that the blockwise bi-hhi degrees coincide with the bi-hhi degrees (Theorem 1.1), but that not every bi-hhi set is blockwise bi-hhi (Theorem 1.5).

We say that \( S \subseteq 2^{<\omega} \) is dense if every \( \sigma \in 2^{<\omega} \) has an extension in \( S \). A sequence \( X \in 2^\omega \) is weakly \( n \)-generic if for every dense \( \Sigma_0^n \) set \( S \subseteq 2^{<\omega} \), there is a prefix of \( X \) in \( S \). We will primarily be concerned with weak 2-genericity. Finally, a function \( f : \omega \to \omega \) is \( \Delta_0^2 \) escaping if it is not dominated by any \( \Delta_0^2 \) function.

The main theorem. We are ready to state the various characterizations of the bi-hyperhyperimmune degrees.

**Theorem 1.1.** The following are equivalent for a Turing degree \( d \):

1. \( d \) computes a \( \Delta_0^2 \) escaping function.
2. \( d \) computes a weakly 2-generic sequence.
3. \( d \) contains a blockwise bi-hyperhyperimmune set.
4. \( d \) contains a blockwise hyperhyperimmune set.
5. \( d \) contains a bi-hyperhyperimmune set.

**Proof.** \( 1 \Rightarrow 2 \) is Theorem 3.3. \( 2 \Rightarrow 3 \) follows from the fact that the collection of blockwise bi-hhi Turing degrees is closed upwards (Lemma 2.2), and the easy fact that every weak 2-generic is a blockwise bi-hhi set (Lemma 2.3). \( 3 \Rightarrow 4 \) and \( 3 \Rightarrow 5 \) are trivial. Finally, \( 4 \Rightarrow 1 \) and \( 5 \Rightarrow 1 \) are Lemmas 3.1 and 3.2 respectively.

This result answers the question that motivated this work.

**Corollary 1.2.** The collection of bi-hyperhyperimmune Turing degrees is closed upwards.

Note that we have also shown that the blockwise hyperhyperimmune and blockwise bi-hyperhyperimmune degrees coincide.

The parallel between the bi-hyperimmune and bi-hhi degrees. Compare our main theorem to the following result.

**Theorem 1.3 (Kurtz [11]).** The following are equivalent for a Turing degree \( d \):

1. \( d \) computes a function that is not dominated by any computable function (i.e., \( d \) has hyperimmune degree).
2. \( d \) contains a weakly 1-generic sequence.
3. \( d \) contains a bi-hyperimmune set.
Some differences are clear. The collection of weakly 2-generic degrees is not closed upwards, so “computes” in part 2 of Theorem 1.1 cannot be replaced with “contains”. More importantly, hyperhyperimmunity is not strong enough to guarantee that a degree is $\Delta^0_2$ escaping, a fact that is closely related to the non-coincidence of the hyperhyperimmune and bi-hyperhyperimmune degrees.

Despite these differences, the similarity between the results is clear and it leads to a natural question: what is the right common generalization? No one has had the audacity to suggest a definition for bi-hyperhyperhyperimmunity, but perhaps we could prove that a degree is $\Delta^0_n$ escaping if and only if it computes a weak $n$-generic. This turns out not to be the case. In Theorem 4.6, we show that not every $\Delta^0_3$ escaping function computes a weak 3-generic.

Structure of the paper. The proof of the main theorem is contained in Sections 2 and 3. The former focuses on the blockwise versions of hhi and bi-hhi, while the latter focuses on $\Delta^0_2$ escaping functions. Section 4 starts with a collection of facts about the bi-hhi degrees. For example, we show that every bi-hyperhyperimmune degree is array non-computable and every $\Delta^0_3$ degree strictly above $\emptyset'$ is bi-hhi.

We finish the paper with three counterexamples: we prove that there is a bi-hhi degree that does not compute a presentation of Csima and Kalimulin’s structure (mentioned above); we show that there is a bi-hhi set that is not blockwise bi-hhi; and we prove that there is a $\Delta^0_3$ escaping function that does not compute a weak 3-generic. Related to the last example, in Theorem 3.4 we prove that every $\Delta^0_3$ escaping function computes a 2-generic.

2. Blockwise (bi-)hyperhyperimmune sets

The definition of blockwise bi-hyperhyperimmunity might seem contrived, but in fact, it appears to be a fairly natural and robust notion. The following lemma, which gives two nice characterizations of blockwise hyperhyperimmune sets, helps make this case. We say that an array $A$ is finitely intersecting if every $n \in \omega$ appears in at most finitely many members of $A$.

Lemma 2.1. The following are equivalent for an infinite set $X \subseteq \omega$:

1. $X$ is blockwise hyperhyperimmune.
2. $X$ avoids a member of every finitely intersecting weak array.
3. If $f: \omega \to \omega$ is any $\Delta^0_2$ function, then $(\exists n)[n, f(n)] \cap X = \emptyset$.

Proof. (1) $\Rightarrow$ (2) Let $A = \{F_n\}_{n \in \omega}$ be a finitely intersecting weak array. We construct a disjoint weak array $B = \{G_n\}_{n \in \omega}$ as follows. For each $n \in \omega$, we define $G_n$ in stages. Let $G_{n,0} = \{\langle n, 0 \rangle\}$. At a stage $s \in \omega$, take $m \in \omega$ to be least such that $F_m,s$ contains no element less than $\langle n, 0 \rangle$. If $F_{m,s} \subseteq \widehat{G}_{n,s}$, then do nothing. Otherwise, put the least $\langle n,k \rangle \geq \max(F_{m,s})$ into $G_{n,s+1}$.

Note that, for each $n$, there are only finitely many $m \in \omega$ such that $F_m$ contains an element less than $\langle n, 0 \rangle$. Hence, the choice of $m$ eventually stabilizes and $G_n$ is finite. Thus $B$ is a weak array. The definition of $B$ ensures that its members are pairwise disjoint. By assumption, $X$ blockwise avoids a member of $B$. But for every $G \in B$, there is an $F \in A$ such that $F \subseteq \widehat{G}$. So $X$ avoids a member of $A$.

(2) $\Rightarrow$ (3) Let $f: \omega \to \omega$ be a $\Delta^0_2$ function. Any total $\Delta^0_2$ function is majorized by a function that is computably approximable from below, so we may assume
that \( f \) is itself computably approximable from below. Consider the weak array \( A = \{ [n, f(n)] \}_{n \in \omega} \). Note that \( A \) is finitely intersecting because each \( n \in \omega \) is in at most \( n + 1 \) members of \( A \). So by assumption, there is an interval \( [n, f(n)] \in A \) such that \( [n, f(n)] \cap X = \emptyset \).

\( \emptyset \Rightarrow \{1 \} \) Let \( A = \{ F_n \}_{n \in \omega} \) be a disjoint weak array. Define a \( \Delta^0_2 \) function \( f: \omega \to \omega \) by \( f(n) = \max \left( \bigcup_{m < n} F_m \right) \). By assumption, there is an \( n \in \omega \) such that \( [n, f(n)] \cap X = \emptyset \). In cannot be the case that \( F_0, F_1, \ldots, F_n \) each contain an element less than \( n \), so there is an \( m \leq n \) such that \( F_m \subseteq [n, f(n)] \), hence \( F_m \cap X = \emptyset \). \( \square \)

**Lemma 2.2.** The collection of blockwise bi-hyperhyperimmune Turing degrees is closed upwards.

**Proof.** Assume that \( X \subseteq \omega \) is blockwise bi-hyperhyperimmune and \( D \supseteq T X \). We want to code \( D \) into a new blockwise bi-hhi set \( Z \subseteq \omega \). First, let

\[
Y = X \oplus X = \{ 2n \mid n \in X \} \cup \{ 2n + 1 \mid n \in X \}.
\]

We claim that \( Y \) is blockwise bi-hhi. Let \( f: \omega \to \omega \) be a \( \Delta^0_2 \) function. Applying the condition in part \( \emptyset \) of Lemma 2.1 to \( n \mapsto f(2n) \), there is an \( n \in \omega \) such that \( [n, f(2n)] \cap X = \emptyset \). Therefore, \( [2n, f(2n)] \cap Y = \emptyset \). So \( Y \) is blockwise hhi. Similarly, \( \omega \setminus Y \) is blockwise hhi too, so \( Y \) is blockwise bi-hhi.

The advantage of \( Y \) is that every maximal (under \( \subseteq \) ) interval contained in \( Y \) has even length. Let \( \{ [n_i, m_i] \}_{i \in \omega} \) be the maximal intervals contained in \( Y \), in the order that they appear. Let \( Z = \bigcup_{i \in \omega} [n_i, m_i - D(i)] \). It is not hard to see that \( Z \equiv_T D \). Clearly, \( Z \subseteq T D \), and \( i \in D \) if and only if the \( i \)th maximal interval contained in \( Z \) has odd length. We claim that \( Z \) is blockwise bi-hhi. Because \( Z \subseteq Y \), it is clearly blockwise hhi. Let \( f: \omega \to \omega \) be a \( \Delta^0_2 \) function. Since \( f(n) + 1 \) is also a \( \Delta^0_2 \) function and \( \omega \setminus Y \) is blockwise hhi, by Lemma 2.1 there is an \( n \in \omega \) such that \( [n, f(n) + 1] \subseteq Y \). This implies that \( [n, f(n)] \subseteq Z \), so \( Z \) is blockwise bi-hhi. \( \square \)

It is easy to see that blockwise bi-hyperhyperimmunity is a comeager property.

**Lemma 2.3.** Weak 2-generic sets are blockwise bi-hyperhyperimmune.

**Proof.** Assume that \( X \subseteq \omega \) is weakly 2-generic. If \( A = \{ F_n \}_{n \in \omega} \) is a disjoint weak array, then \( \{ \sigma \in 2^{<\omega} \mid (\exists n, s)(\forall t \geq s)(\forall m \in F_n[t \sigma(m) = 1]) \} \) is a dense \( \Sigma^0_2 \) set of strings. So \( X \) must blockwise contain a member of \( A \). Because \( \omega \setminus X \) is weakly 2-generic too, \( X \) must also blockwise avoid a member of \( A \). Therefore, \( X \) is blockwise bi-hhi. \( \square \)

3. \( \Delta^0_2 \) Escaping Functions

In this section we finish the proof of the main theorem by proving the necessary facts about \( \Delta^0_2 \) escaping functions. There are \( \Delta^0_2 \) hyperhyperimmune sets, so it is not the case that every hyperhyperimmune set computes a \( \Delta^0_2 \) escaping function. However, it is easy to see that blockwise hyperhyperimmunity is sufficient.

**Lemma 3.1.** Every blockwise hyperhyperimmune set computes a \( \Delta^0_2 \) escaping function.

**Proof.** Assume that \( X = \{ x_0 < x_1 < x_2 < \cdots \} \) is blockwise hhi. Consider the \( X \)-computable function \( g: \omega \to \omega \) defined by \( g(n) = x_n \) for all \( n \in \omega \). We claim that \( g \) is \( \Delta^0_2 \) escaping. Let \( f: \omega \to \omega \) be a \( \Delta^0_2 \) function. There is an \( n \in \omega \) such that
We make a small assumption about its complement. Let \( \omega \) be the longest element in the chain of \( X \). For each \( n \in \omega \), let \( g(n) \) be the least \( m \in \omega \) such that \( \langle n, m \rangle \in X \). By assumption, \( g: \omega \to \omega \) is total, hence \( X \)-computable. We claim that \( g \) is \( \Delta^0_2 \) escaping. It is sufficient to show that no \( \Delta^0_2 \) function majorizes \( g \). Any total \( \Delta^0_2 \) function is majorized by a function that is computably approximable from below, so in fact, it is sufficient to show that no function that is computably approximable from below majorizes \( g \). Let \( f: \omega \to \omega \) be such a function. Consider the disjoint weak array \( \{(n, m) \mid m \leq f(n)\}_{n \in \omega} \). If \( X \) avoids the \( n \)-th set in this array, then \( g(n) > f(n) \). Thus, \( g \) is \( \Delta^0_2 \) escaping.

The second part follows because if \( X \subseteq \omega \) is bi-hhi, then its complement is immune. So \( \omega \setminus X \) does not contain the infinite c.e. set \( \omega^{[n]} \), for each \( n \in \omega \).

The next result is the last and most technically involved step in the proof of Theorem \[\Box\].

**Theorem 3.3.** Every \( \Delta^0_2 \) escaping function computes a weak 2-generic.

**Proof.** Let \( g: \omega \to \omega \) be a \( \Delta^0_2 \) escaping function. We may assume that \( g \) is increasing. Let \( \{U_k\}_{k \in \omega} \) be an effective list of \( \Sigma_2^0 \) sets of strings (not necessarily dense) and \( U_{k,s} \) a uniformly computable \( \Sigma_2 \) approximation, i.e., \( \sigma \in U_k \iff (\exists s)(\forall t \geq s)(\sigma \in U_{k,s}) \). The goal is to construct a weakly 2-generic sequence \( X \in 2^{\omega} \).

We define \( X(n) \) at stage \( n \in \omega \) of the construction. We may inductively assume that at the start of stage \( n \), for each \( k < n \) there is a distinguished string \( \sigma_k \) associated to \( U_k \). Furthermore, the distinguished strings form a chain comparable to \( X \setminus n \) (not necessarily ordered by their indices). If \( \sigma_k \notin U_{k,s} \) for any stage \( s \in (g(n - 1), g(n)] \), then declare \( \sigma_k \) to no longer be distinguished.

Now for each \( k \leq n \) with no distinguished string, we want to find it one. Let \( \tau \) be the longest element in the chain of \( X \setminus n \) and the currently distinguished strings. Find the \( k \leq n \) lacking a distinguished string and the \( \sigma \supseteq \tau \) that minimize \( \max\{|\sigma|, s(\sigma, k)\} \), where \( s(\sigma, k) \) is the last stage \( s \leq g(n) \) such that \( \sigma \notin U_{k,s} \). In other words, we are looking for a short extension of \( \tau \) that has looked like it is in \( U_k \) for a long time. Make \( \sigma_k = \sigma \) the distinguished string for \( U_k \). Let \( \tau = \sigma_k \) and repeat this process until every \( k \leq n \) has a distinguished string.

Note that the distinguished strings still form a chain comparable to \( X \setminus n \). Define \( X(n) \) to preserve comparability with this chain. This completes the construction.

**Verification.** Assume that \( U_k \) is dense. We define a \( \Delta^0_2 \) function \( f: \omega \to \omega \) such that if \( n \in \omega \) is the least number greater than or equal to \( k \) such that \( g(n) \geq f(n) \), then at the end of stage \( n \) of the construction above, the distinguished string \( \sigma_k \) for \( U_k \) is contained in \( U_{k,s} \) for all \( s \geq g(n) \). Therefore, it will remain distinguished and will be a prefix of \( X \).

Fix \( n \geq k \). Assume that we know \( g \setminus n \); we will remove this assumption below. We want to define \( f(n) \) to be large enough to lock in the distinguished string for \( U_k \). By assumption, we know the sequence of distinguished strings at the start of
stage \( n \). Let \( s_0 = g(n-1) \) (or 0, if \( n = 0 \)). Take \( s_1 \) large enough that for every \( j < n \), if \( \sigma_j \) has not permanently entered \( U_j \) by stage \( s_0 \), then it will leave before stage \( s_1 \). So, if \( g(n) \geq f(n) \geq s_1 \), the only distinguished strings that remain at the start of stage \( n \) are those that will never be canceled.

Let \( \tau \) be the longest element in the chain of \( X \upharpoonright n \) and the remaining distinguished strings. Pick the shortest \( \sigma \in U_k \) that extends \( \tau \) and let \( s_2 \geq \max \{ s_1, |\sigma| \} \) be large enough that \( \sigma \) has permanently entered \( U_k \). Pick \( s_3 \geq s_2 \) large enough that, for all \( j \leq n \), all elements of \( U_j \cup s_2 \) of length at most \( s_2 \) that have not permanently entered \( U_j \) by stage \( s_2 \) will leave by stage \( s_3 \). Now if \( g(n) \geq s_3 \), then the only thing that stops \( \sigma \) from becoming the distinguished string for \( U_k \) is the presence of a better candidate \( \sigma_j \) for \( U_j \), with \( j \leq n \). Note that \( s(\sigma_j, j) \leq \max \{ |\sigma|, s(\sigma, k) \} \leq s_2 \), so \( \sigma_j \) must have appeared by \( s_2 \). The fact that \( \sigma_j \) did not leave \( U_j \) by \( s_3 \) tells us that it is locked in as the distinguished string for \( U_j \). If \( j \neq k \), then repeat the process just described starting with \( s_1 \) equal to \( s_3 \). This process can only repeat finitely many times. After the final repetition, \( s_3 \) will be large enough to guarantee that if \( g(n) \geq s_3 \), then the distinguished string \( \sigma \) chosen for \( U_k \) is permanent. Let \( f(n) = s_3 \). Note that this whole process can be carried out with a \( \emptyset' \) oracle.

We must remove the assumption that we know \( g \upharpoonright n \). Take \( g \upharpoonright k \) as given. For each \( n \geq k \), assume that \( g \upharpoonright (k, n) \) has not yet exceeded \( f \upharpoonright (k, n) \). This gives us a finite number of possibilities for \( g \upharpoonright n \); apply the process above to each case to find a sufficiently large value of \( f(n) \), and let \( f(n) \) be the maximum of these. So if \( n \geq k \) is least such that \( g(n) \geq f(n) \), then at stage \( n \) of the construction of \( X \), we lock in a distinguished string \( \sigma_k \in U_k \). This guarantees that \( \sigma_k \prec X \). \( \square \)

We will show in Theorem 4.6 that there is a \( \Delta^0_3 \) escaping function that does not compute a weak 3-generic. Compare that to the following result.

**Theorem 3.4.** Every \( \Delta^0_3 \) escaping function computes a 2-generic.

**Proof.** Given a \( \Delta^0_3 \) escaping function \( g: \omega \to \omega \), the construction is the same as above. The verification is almost the same, except that we define a \( \Delta^0_3 \) function \( f: \omega \to \omega \) for every \( U_k \), whether or not it is dense. At the point in the definition of \( f(n) \) that we want the shortest \( \sigma \in U_k \) that extends \( \tau \), there is no longer a guarantee that such a \( \sigma \) exists. We ask \( \emptyset'' \) if there is a \( \sigma \in U_k \) that extends \( \tau \). If so, we proceed as before, trying to ensure that there is a \( \sigma_k \in U_k \) such that \( \sigma_k \prec X \). If not, then finish with the definition of \( f(n) \). In the latter case, if \( g(n) \geq f(n) \), then \( U_k \) is not dense along \( X \). \( \square \)

4. CONSEQUENCES AND COUNTEREXAMPLES

The next result collects several observations about the bi-hyperhyperimmune degrees, all of which follow easily from the work above.

**Proposition 4.1.**

1. Every bi-hyperhyperimmune degree is array non-computable. (Hence no array computable degree computes a weak 2-generic.)
2. No bi-hyperhyperimmune degree has minimal Turing degree.
3. Every \( \Delta^0_3 \) degree strictly above \( \emptyset' \) is bi-hyperhyperimmune.
4. Not every \( \Sigma^0_3 \prec \Delta^0_3 \) degree is bi-hyperhyperimmune.
5. Every degree strictly above \( \emptyset' \) computes a non-\( \Delta^0_3 \) degree that is not bi-hyperhyperimmune.
There are downward cones of bi-hyperhyperimmune degrees.

Proof.  
1. We have shown that every bi-hhi computes a $\Delta^0_2$ escaping function. On the other hand, Downey, Jockusch and Stob [5] proved that all functions of array computable degree are dominated by the modulus function of $\emptyset'$.
2. No weak 2-generic (indeed, no 1-generic) is minimal.
3. Miller and Martin [13] showed that every non-computable $\Delta^0_2$ degree is hyperimmune. Relativizing to $\emptyset'$, every $\Delta^0_2$-degree strictly above $\emptyset'$ is hyperimmune relative to $\emptyset'$, hence $\Delta^0_2$ escaping.
4. Shore [14] and Cooper, Lewis and Yang [2] independently proved the existence of minimal degrees in $\Sigma^0_2 \setminus \Delta^0_2$.
5. There is a $\Delta^0_2$ function tree $T : 2^{<\omega} \to 2^{<\omega}$ such that every path through $T$ has minimal Turing degree. Let $D > T \emptyset'$ and consider $T[D]$, which is minimal, hence does not have bi-hhi degree. Note that $T[D] \leq_T T \oplus D \leq_T \emptyset' \oplus D \leq_T D$. But $D \leq_T T[D] \oplus T$ and $D$ is not $\Delta^0_2$, so $T[D]$ is not $\Delta^0_2$.
6. Based on a result of Martin, Jockusch [10] proved that the 2-generic degrees are downwards dense, meaning that every non-computable degree below a 2-generic computes a 2-generic. This implies that every non-computable degree below a 2-generic is bi-hyperhyperimmune. □

We finish the paper with three counterexamples. The first gives a negative answer to Question 6.6 in Csima and Kalimullin [3]. If $n \in \omega$ and $F \subseteq \omega$, they defined $\{n\} \oplus F$ to be the following infinite graph. It consists of an $\omega$-chain with an $(n+5)$-cycle linked to 0. For each $m \in F$, there is a 3-cycle linked to $m$, and for each $m \notin F$, there is a 4-cycle linked to $m$. Consider the graph $H$ that is the disjoint union of all $\{n\} \oplus F$ such that
- $n \in \omega$,
- $F \subseteq \omega$ is finite, and
- if $\{W_{\varphi_n(m)}\}_{m \in \omega}$ is a disjoint weak array, then $\exists m \{W_{\varphi_n(m)}\} \subseteq F$.

Csima and Kalimullin proved that every degree that computes a copy of $H$ is bi-hhi. They suggested that the spectrum of $H$, i.e., the collection of degrees computing a copy of $H$, might be exactly the bi-hyperhyperimmune degrees. This is not the case.

**Proposition 4.2.** If $a$ is in the spectrum of $H$ then $a$ is high_2 (i.e., $a'' \geq 0'''$).

Proof. It is routine to check that
$$U = \{n \in \omega \mid \{W_{\varphi_n(m)}\}_{m \in \omega} \text{ is a disjoint weak array with no empty members}\}$$
is $\Pi^0_3$ complete. (The real complexity comes from the fact that every member of the array is finite, not from the completeness of $\varphi_n$, the disjointness of the array, or the nonemptiness of the members.) Note that $\{n\} \oplus \emptyset$ is a component of $H$ if and only if $n \notin U$. But if $a$ computes a copy of $H$, then $a''$ can determine if $\{n\} \oplus \emptyset$ is a component of $H$. Therefore, $a'' \geq 0'''$.

**Corollary 4.3.** There is a bi-hyperhyperimmune degree that does not compute a copy of $H$.

Proof. It is not hard to see that there is a bi-hyperhyperimmune degree that is not high_2. For example, if $A \leq_T \emptyset'$ is 2-generic, then $A'' \equiv_T A \oplus \emptyset' \equiv_T \emptyset'' [10]$. Such an $A$ has bi-hhi degree, but does not compute a copy of $H$. □
Question 4.4. Is the collection of bi-hyperhyperimmune Turing degrees the spectrum of a structure?

There is a \( \Delta^0_2 \) hyperhyperimmune set. On the other hand, we have proved that every blockwise hyperhyperimmune computes a \( \Delta^0_2 \) escaping function, hence cannot be \( \Delta^0_2 \). Therefore, there is a hhi set that is not blockwise hhi. This simple argument does not work to prove that there is a bi-hhi set that is not blockwise bi-hhi, as we know that the two properties agree degree-wise.

Theorem 4.5. There is a bi-hyperhyperimmune set \( X \subseteq \omega \) such that neither \( X \) nor \( \omega \setminus X \) is blockwise hyperhyperimmune.

Proof. Let \( \{ \{ V_{k,j} \}_{j \in \omega} \mid k \in \omega \} \) be an effective list of all uniform sequences of disjoint c.e. sets. Define a \( \Delta^0_2 \) function \( f : \omega \to \omega \) as follows. Fix \( n \in \omega \). For each \( k < n \), let \( j(k) \) be least such that \( V_{k,j(k)} \cap [0,n] = \emptyset \). Note that \( j(k) \leq n + 1 \), so \( \emptyset' \) can find it. If \( V_{k,j(k)} = \emptyset \), then let \( g(k) = 0 \). Otherwise, let \( g(k) \) be an element of \( V_{k,j(k)} \). It should be clear that \( \emptyset' \) can compute \( g(k) \).

For each \( k, m < n \), define \( i(k,m) \) such that \( m \in V_{k,i(k,m)} \), if such an index exists. Let \( h(k,m) \) be a member of \( V_{k,i(k,m)} \) larger than \( n \). If \( h(k,m) \) is otherwise undefined, let \( h(k,m) = 0 \). Again, \( \emptyset' \) can compute \( h(k,m) \).

Let \( f(n) = \max(n + 2, g(k), h(k,m)) \)

Our goal is to construct a bi-hyperhyperimmune set \( X \subseteq \omega \) that neither contains nor avoids an interval of the form \( [n,f(n)] \). By Lemma 2.1, this ensures that neither \( X \) nor its complement is blockwise hyperhyperimmune. We construct \( X \) by initial segments. Let \( \tau_0 = 01 \). Assume that, at the beginning of stage \( k \in \omega \), we have a string \( \tau_k \in 2^{\omega} \) that neither contains nor avoids an interval of the form \( [n,f(n)] \). If \( \mathcal{A} = \{ V_{k,j} \}_{j \in \omega} \) has an infinite member, then it is not a weak array, so let \( \tau_{k+1} = \tau_k \).

If \( \mathcal{A} \) has an empty member, then \( X \) automatically contains and avoids this member, so again let \( \tau_{k+1} = \tau_k \). Otherwise, we want to extend \( \tau_k \) to \( \sigma \) so that \( \sigma \) contains a member of \( \mathcal{A} \). The definition of \( f \) forces \( [n,f(n)] \) to have length at least three, meaning that it is always safe to extend \( \tau_k \) by adding alternating ones and zeros, starting with whichever differs from the last bit of \( \tau_k \). So we may assume that \( |\tau_k| > k \).

Let \( p = |\tau_k| \). Choose \( F \in \mathcal{A} \) such that \( q = \min(F) > p \). Extend \( \tau_k \) to a string \( \rho \) of length \( q - 1 \) by adding alternating ones and zeros. Define \( \sigma \) extending \( \rho \) such that \( \sigma \) only adds zeros except at the positions in \( F \), where it adds ones, and \( |\sigma| = \max(F) + 1 \). We claim that \( \sigma \) does not contain an interval of the form \( [n,f(n)] \). If it does, then we must have \( n \geq q \) and \( n \in F \). But \( \mathcal{A} \) has no empty member, so \( g(k) \) is in a member of \( \mathcal{A} \) disjoint from \( [0,n] \). Therefore, \( n < g(k) \) and \( g(k) \notin F \), so \( f(n) \geq g(k) \) implies that \( [n,f(n)] \not\subseteq \sigma \). We also claim that \( \sigma \) does not avoid an interval of the form \( [n,f(n)] \). If it does, then it must be the case that \( n = (q,\max(F)) \). But \( q \in F \) and \( k,q < n \), so we define \( h(k,q) \) to be an element of \( F \) larger than \( n \), if possible. This is possible because \( n < \max(F) \), so \( f(n) \geq h(k,q) \) implies that \( [n,f(n)] \not\subseteq \sigma \).

The process to extend \( \sigma \) to \( \tau_{k+1} \) to avoid a member of \( \mathcal{A} \), while preserving the property that it neither contains nor avoids an interval of the form \( [n,f(n)] \), is completely symmetric. Let \( X = \bigcup_{k \in \omega} \tau_k \). The construction ensures that \( X \) is bi-hyperhyperimmune but that neither \( X \) nor its complement is blockwise hyperhyperimmune.

As mentioned in the introduction, Kurtz [11] proved that every function that is not dominated by a computable function, i.e., every \( \Delta^0_1 \) escaping function, computes...
a weak 1-generic. We have shown that every $\Delta^0_2$ escaping function computes a weak 2-generic. Our final counterexample shows that this pattern does not extend, at least not in the most naïve way. Recall that, by Theorem 3.4, every $\Delta^0_3$ escaping function does compute a 2-generic.

**Theorem 4.6.** There is a $\Delta^0_3$ escaping function that does not compute a weak 3-generic.

**Proof.** We build a $\Delta^0_3$ function tree $f : \omega^\omega \to \omega^\omega$ such that if $h \in \omega^\omega$, then $f[h]$ does not compute a weak 3-generic. Furthermore, for all $\tau \in \omega^\omega$ and $n \in \omega$, it will be the case that $f(\tau n)$ extends $f(\tau)m$ for some $m \geq n$. So, there is an $h \in \omega^\omega$ such that $f[h]$ is $\Delta^0_3$ escaping.

Let $\{\tau_s\}_{s \in \omega}$ be an effective enumeration of $\omega^\omega$ such that each string is enumerated only after its proper prefixes have been enumerated. In particular, $\tau_0$ is the empty string. The construction of $f$ is done relative to $\emptyset''$. We begin stage $s \in \omega$ of the construction with $f$ defined on $\tau_0, \ldots, \tau_s$. For each $t \leq s$, we have an associated infinite c.e. set of strings $V_{t,s}$ extending $f(\tau_t)$. When we define $f(\tau_n)$, it will be an extension of an element of $V_{t,s}$. During the construction, we build a sequence of dense $\Sigma^0_1[\emptyset'']$ sets of binary strings $\{U_e\}_{e \in \omega}$. We ensure that if $h \in \omega^\omega$ and $\varphi_e^{f[h]} \in 2^\omega$, then no prefix of $\varphi_e^{f[h]}$ is in $U_e$. So $f[h]$ does not compute a weak 3-generic by $\varphi_e$, for each $e \in \omega$.

Let $f(\tau_0)$ be the empty string and let $V_{0,0} = \{m\}_{m \in \omega}$. At stage $s =< e, i >$, we make sure that every string of length $i$ has an extension in $U_e$. Let $k = i + s + 1$. For each $t \leq s$, use $\emptyset''$ to determine if there is a binary string $\rho_t$ of length $k$ such that infinitely many $\sigma \in V_{t,s}$ can be extended to binary strings $\sigma'$ such that $\varphi_e^{\sigma'} | k = \rho_t$. If so, let $V_{t,s+1}$ be an infinite set of such extensions (each extending a different element of $V_{t,s}$). If not, let $\rho_t$ be undefined and let $V_{t,s+1}$ be $V_{t,s}$ without the finitely many $\sigma$ that can be extended to $\sigma'$ to make $\varphi_e^{\sigma'} | k$ converge to a binary string. Put every binary string of length $k$ into $U_e$ except for $\rho_0, \ldots, \rho_s$. This ensures that every binary string of length $i$ has an extension in $U_e$. We have done this while not adding a prefix of any possible $\varphi_e^{f[h]}$. Assume that $\tau_{s+1} = \tau_n$. Choose $\sigma \in V_{t,s+1}$ such that $\sigma$ extends $f(\tau_t)m$ for some $m \geq n$. Remove $\sigma$ from $V_{t,s+1}$ and let $f(\tau_{s+1}) = \sigma$. Let $V_{s+1,s+1} = \{\sigma m\}_{m \in \omega}$. This completes stage $s$.

The construction ensures that if $h \in \omega^\omega$, then $f[h]$ does not compute a weak 3-generic. As noted, $h$ can be chosen so that $f[h]$ is $\Delta^0_3$ escaping. 

The construction actually shows that no amount of “non-domination strength” is enough to compute a weak 3-generic. In particular, if $\{g_i\}_{i \in \omega}$ is any countable collection of functions, we can choose $h$ so that $f[h]$ is not dominated by any $g_i$. It is still the case that $f[h]$ does not compute a weak 3-generic.

**References**


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