Instructions:
Do all six problems.¹

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

If you are unable to solve a problem completely, you may receive partial credit by weakening a conclusion or strengthening a hypothesis. In this case, include such information in your solution, so the graders know that you know that your solution is not complete.

If you want to ask a grader a question during the exam, write out your question on an 8½ by 11 sheet of paper. Give it to the proctor. The proctor will contact one of the logic graders who will retrieve your written question, write a response, copy the sheet of paper, and return it to the proctor.

E1. Let $M$ be a structure where $\phi(x, y)$ defines a linear order on an infinite set $X \subseteq M$. Given any linear order-type $\tau$, show that there are an $N \succeq M$ and an infinite set $Y \subset N$ of order-type $\tau$ defined by $\phi(x, y)$.

E2. Let $\oplus$ and $\otimes$ both be computable functions from $\omega \times \omega$ into $\omega$ such that $(\omega, \oplus)$ and $(\omega, \otimes)$ are both abelian groups with the property that every element other than the identity has order 5 or 7 or 35. Assume also that the two groups are isomorphic. Prove that there is a computable isomorphism between them.

E3. Prove that there is no family $\{A_\alpha : \alpha < \omega_1\} \subset \mathcal{P}(\omega)$ such that for all $\alpha < \beta$: $A_\beta \setminus A_\alpha$ is infinite and $|A_\alpha \setminus A_\beta| \leq 7$.

¹Note that this is different from exams up until a year ago.
C1. Consider the set $SD = \{ e : \varphi_e(0) \downarrow \text{ and } (\forall j < e) [\varphi_j(0) \neq \varphi_e(0)] \}$. Prove that SD is immune, i.e., contains no infinite c.e. set.

C2. Let $P$ be a nonempty $\Pi^0_1$-class. For any set $A$, show that there is a $B \in P$ such that the infimum of $\deg_T(A)$ and $\deg_T(B)$ is $0$.

C3. For any set $A$, construct a set $B \leq_T A''$ such that $\Sigma^0_1 = \Sigma^0_1[A] \cap \Sigma^0_1[B]$. 
E1 ans. Introduce new constant symbols $c_x$ for $x \in \tau$ and add to the theory all sentences of the form $\phi(c_x, c_y)$ for $x < y$ in $\tau$. Now use Compactness.

E2 ans. Using the group operation $\oplus$: Let $H_5, H_7 \subseteq \omega$ be the subgroups consisting of the identity together with all elements of order 5, 7 respectively. Likewise get $K_5, K_7 \subseteq \omega$ using the group operation $\otimes$. Then $(H_5, \oplus) \cong (K_5, \otimes)$, and let $f : H_5 \to K_5$ be a computable isomorphism. To get $f$: View $(H_5, \oplus)$ as a vector space over the 5 element field, and choose (by recursion) a computable basis $A_5 \subseteq H_5$. Likewise let $B_5 \subseteq K_5$ be a computable basis for $(K_5, \otimes)$. Note that $0 \leq |A_5| = |B_5| \leq \aleph_0$. Then a computable bijection from $A_5$ onto $B_5$ generates a computable isomorphism from $(H_5, \oplus)$ onto $(K_5, \otimes)$. Likewise $(H_7, \oplus) \cong (K_7, \otimes)$, and let $g : H_7 \to K_7$ be a computable isomorphism. Then $f, g$ generate a computable isomorphism from $(\omega, \oplus)$ onto $(\omega, \otimes)$; To see this, note that $(\omega, \oplus)$ is the direct sum of $(H_5, \oplus)$ and $(H_7, \oplus)$, and $(\omega, \otimes)$ is the direct sum of $(K_5, \otimes)$ and $(K_7, \otimes)$.

E3 ans. Assume that we had such $\{A_\alpha : \alpha < \omega_1\}$. For each $\xi$, choose $B_\xi \subset A_{\xi+1}\setminus A_\xi$ with $|B_\xi| = 8$. Since $|\omega|^8 = \aleph_0$, fix $\xi, \eta$ such that $\xi < \xi + 1 < \eta < \eta + 1$ and $B_\xi = B_\eta$. Let $B = B_\xi = B_\eta$. Then $B \subseteq A_{\xi+1}$ and $B \cap A_\eta = \emptyset$, so $B \subset A_{\xi+1}\setminus A_\eta$, so $|A_{\xi+1}\setminus A_\eta| \geq 8$, which is a contradiction (taking $\alpha = \xi + 1$ and $\beta = \eta$).

C1 ans. Towards a contradiction, fix an infinite c.e. subset $W$ of SD. By the Recursion Theorem, we can fix an index $e$ for which we can control $\varphi_e(0)$. Given $e$, wait for an index $i > e$ to enter $W$ and then set $\varphi_e(0) = \varphi_i(0)$, a contradiction.

C2 ans. For each pair $e, i$ of indices, try to force $\exists x [\Phi_e^A \neq \Phi_i^X]$ for all $X$ in a nonempty $\Pi^0_1$-subclass; otherwise the common value (if total) can be computed effectively.

C3 ans. Let $B$ be a 2-generic relative to $A$. Suppose for some $e$ and $i$, some condition $\sigma$ forces that $W_e^A = W_i^B$. Then consider the set $W$ of all $x$
so that for some $\tau \supseteq \sigma$, $x \in W^\gamma_i$. If there is some $x \in W \setminus W^A_\varepsilon$, then there is a $\tau \supseteq \sigma$ so that $\tau$ forces $x \in W^B_i$, which is a contradiction, since $\tau$ also forces that $W^A_\varepsilon = W^B_i$. If not, then $W^B_i \subseteq W \subseteq W^A_\varepsilon$. Thus if $W^A_\varepsilon = W^B_i$, then they are equal to the c.e. set $W$. 